Asymptotic Stability of Small Oseen Type Navier-Stokes Flow under 3-D Large Perturbation

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1 Introduction

Let Ω be $\mathbb{R}^3$ or $\mathbb{R}^2 \times \mathbb{T}^1$, where $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ is one dimensional flat torus. We consider the incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in} \quad \Omega \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in} \quad \Omega \times (0, \infty), \\ u(0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p(x,t)$ respectively stand for an unknown velocity field and a pressure. The functions u_0 denote a given initial velocity. ∂_t, Δ denotes partial derivative in time and Laplace operator on the Euclidean space respectively. The differential operator $u \cdot \nabla$ denotes $\sum_{1 \le j \le 3} u_j \partial_j$.

Let us recall a special self-similar solution called the three dimensional Oseen vortex or Lamb-Oseen vortex:

$$\mathrm{Os}(x_h, x_v, t) = \frac{\Gamma}{2\pi} \frac{(-x_2, x_1, 0)}{|x_h|^2} (1 - \mathrm{e}^{-\frac{|x_h|^2}{4t}}), \quad x_h = (x_1, x_2), \; x_v = x_3, \tag{1.2}$$

where Γ is the total circulations. The two-dimensional Oseen vortex is the Navier-Stokes flow whose initial vorticity is a Dirac measure supported at the origin, and it stands for one of the simplest vortex. The three-dimensional Oseen vortex is an extension of two-dimensional one. In this paper, we discuss L^2 asymptotic stability to somewhat generalized Oseen vortex (Oseen type Navier-Stoke flow) under large three-dimensional perturbation in $\mathbb{R}^2_h \times \mathbb{T}^1_v$.

We will introduce some results on solvability of the Navier-Stokes equations. There are many results on the existence of the solution to (1.1). It is well known that Leray [18] showed the existence of a global-in-time weak solution u in $\mathbb{R}^n$ to (1.1) satisfying the following energy estimate:

$$\|u(\tau)\|_{L^2}^2 + \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \le \|u_0\|_{L^2}^2$$

for initial data $u_0 \in L^2$. Unfortunately, the Oseen vortex is not a Leray's weak solution since the energy of the Oseen vortex is infinite.

For non-L^2-initial data, Kato [12] proved that (1.1) is globally well-posed for small L^m-initial data in $\mathbb{R}^m$ with $m \geq 2$ by using iteration to the integral formulation of (1.1):

$$u(t) = \mathrm{e}^{t\Delta} u_0 - \int_0^t \mathrm{e}^{(t-\tau)\Delta} P(u(\tau) \cdot \nabla u(\tau)) d\tau, \tag{1.3}$$

where $\mathrm{e}^{t\Delta}$ and P are the heat kernel and the Helmholtz projection respectively. The choice of function space is related to the scaling transformation:

$$v(x,t) \to \lambda v(\lambda x, \lambda^2 t), \quad p(x,t) \to \lambda^2 p(\lambda x, \lambda^2 t),$$

which dose not change the equation. Scale-invariant function spaces are critical ones that iteration method works. In this case $L^m(\mathbb{R}^m)$ and $L_t^\infty L_x^m(\mathbb{R}^m \times (0,\infty))$ are scale-invariant function space under the above scaling transformation. Independently, Giga and Miyakawa [7] proved the existence of the solutions in $L^r(\mathbb{R}^r)$ in bounded domains with the Dirichlet boundary condition. The result of this paper was obtained even before [12] but it took long time to be published after the paper was accepted.

In three-dimensional case, $L^3(\mathbb{R}^3)$ is the critical Lebesgue space, but it does not include homogeneous functions like $\frac{1}{|x|}$. This means that $L^3(\mathbb{R}^3)$ is too restrictive to construct a self-similar solution. In this direction, Giga and Miyakawa [6] proved that the vorticity equations is well-posed for small initial data and there is a unique self-similar solution by taking initial vorticity in the Morrey space $M^{\frac{3}{2}}(\mathbb{R}^3)$. The Morrey space is scale-invariant under natural the above natural scaling and include homogeneous functions. Moreover, since $\mathrm{rot}\mathrm{Os}(\cdot, 0) \in M^{\frac{3}{2}}$, the result of [6] provides generalized Navier-Stokes flows that contain the three dimensional Oseen vortex provided that Γ is sufficiently small. However, in [6], smoothness for initial data is needed to define $\mathrm{rot} u_0$. For instance, for a bounded function $\Theta(x)$ on the two dimensional unit sphere whose derivative is not a Radon measure, $\mathrm{rot}(\Theta(\frac{x}{|x|})\mathrm{Os}(\mathrm{x}, 0))$ is not in $M^{\frac{3}{2}}$. On the other hand, Kozono and Yamazaki [15] proved well-posedness for small initial data in weak-L^2 space in two-dimensional exterior domains. Since the two-dimensional Oseen vortex is in weak-L^2 space, the results of [15] provide its generalization. There is no restriction on smoothness of initial data in [15]. In Cannone [2] and Koch and Tataru [13], it was showed that (1.1) is globally well-posed for small initial data in the Besov spaces $B_{p,\infty}^{-1+\frac{n}{p}}(\mathbb{R}^n)$ $(1 < p < \infty)$ and $BMO^{-1}(\mathbb{R}^n)$ space respectively. The result of [13] is the most general on the well-posedness to (1.1).

Our aim is to show asymptotic stability to the solution that is constructed in the first aim under large three-dimensional perturbation. Asymptotic stability for the Navier-Stokes equations has been widely studied. However, there are few the results on the asymptotic stability under large perturbation. In three-dimensional case, Schonbek [21] proved that 0 is asymptotically stable for $L^2 \cap L^1$-perturbation on $\mathbb{R}^3$. Subsequently, Miyakawa and Schonbek [20] study optimal decay rate. On the other hand, Kozono [14] proved asymptotic stability for the Leray's weak solution $u \in L^p_t L^q_x$ satisfying Serrin's condition [22] ($\frac{2}{p} + \frac{3}{q} = 1$ for $2 \le p < \infty$ and $3 < q \le \infty$) on uniformly C^3 domains. This result allows unbounded domains such as a exterior domain or a domain with non-compact boundbary. Karch, Pilarczyk and Schonbek [11] proved L^2-asymptotic stability for small mild solution $V \in \mathcal{X}_\sigma$, where $\mathcal{X}_\sigma$ is a function space of solenoidal vector fields satisfying $|\langle v \cdot \nabla V, w\rangle| \le C(\sup_{t>0}\|V(t)\|_{\mathcal{X}_\sigma})\|\nabla v\|_{L^2}\|\nabla w\|_{L^2}$ for all $v, w \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$. This result allows many function spaces. For instance, weak L^3 space satisfies above estimate, and then it is a subspace of $\mathcal{X}_\sigma$. The decay rate to $L^{3,\infty}$-mild solutions was also studied by [8]. Although [11] is the most comprehensive result for the asymptotic stability of small mild solutions to (1.1), the three dimensional Oseen vortex is not included in this result.

In the two-dimensional case, Iftimie, Karch and Lacave [10] show that, for initial perturbation $v_0 \in L^2$, there exists a positive constant $\delta = \delta(\|v_0\|_{L^2})$, if the total circulation is smaller that δ, the Oseen vortex is asymptotically stable in exterior domain. In this result, size of the total circulation need to be smaller as initial perturbation beocome to be larger. Gallay and Maekawa [5] inproved this point. They show the asymptotic stability of the small Oseen vortex for$L^q \cap L^2$-initial perturbation ($1 < q < 2$). In this result, smallness of initial perturbation is independent of size of the total circulation. Maekawa [19] proved asymptotic stability for the solutions obtained by [15] under $\overline{C_0^\infty}^{L^{2,\infty}}$-large perturbation in the whole space and the exterior domain. This result give us asymptotic stability to the small two-dimensional Oseen vortex.

Let us consider our problem in more detail. We will first generalize three dimensional Oseen vortex. For the this point, since the two-dimensional Oseen vortex is in $L^{2,\infty}$ and three dimensional Oseen vortex is independent of x_v variable, it is good idea to construct mild solution in an anisotropic function space $Y^2 := L^\infty_v L^{2,\infty}_h$ with the norm $\|f\|_{Y^2} = \|\|f(x_h, x_v)\|_{L^{2,\infty}_h}\|_{L^\infty_v}$. Note the three dimensional Oseen vortex is in Y^2 at fixed time. Moreover, Y^2 is scale-invariant under the natural scaling. Therefore we can construct a mild solution to (1.1) by Fujita-Kato principle.

Our aim is to show asymptotic stability of Oseen type Navier-Stokes flow under arbitrarily large perturbation $v_0 \in L^\infty_v \overline{C^\infty_{0,h}}(\mathbb{R}^2_h \times \mathbb{T}^1_v)$. We call the mild solution constructed in the above procedure the basic flow with initial data b_0. To prove asymptotic stability, there are several step. For simplicity, we assume $v_0 \in L^\infty_v C^\infty_{0,h}$.

We first have to show the existence of a weak solution to the perturbed Navier-Stokes equations:

$$\begin{cases} \partial_t v - \Delta v + v \cdot \nabla v + vb \cdot \nabla v + v \cdot \nabla b + \nabla q = 0, \quad \text{in} \quad \mathbb{R}^2_{\mathrm{h}} \times \mathbb{T}^1_{\mathrm{v}} \times (0,\infty), \\ \operatorname{div} v = 0, \quad \text{in} \quad \mathbb{R}^2_{\mathrm{h}} \times \mathbb{T}^1_{\mathrm{v}} \times (0,\infty), \\ v(0) = v_0, \quad \text{on} \quad \mathbb{R}^2_{\mathrm{h}} \times \mathbb{T}^1_{\mathrm{v}}. \end{cases} \tag{1.4}$$

For the vertor field v satisfying above eqations, we find that $v + \tilde{b}$ satisfies (1.1) with initial data $v_0 + b_0$. Since the fifth term of the left-hand side of the above equation $v \cdot \nabla \tilde{b}$ has singularity at $t = 0$, it is difficult to get the energy inequality by integrating on $\mathbb{R}^2_h \times \mathbb{T}^1_v \times (0, t)$ and show the existece of a weak solution to (1.4) directly. To avoid this, we construct a unique local-in-time mild solution v to (1.4) on $(0, T]$ for some $T > 0$ with initial data $\tilde{v}_0$ in a subspace of $L^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)$, after that, we show the existence of global-in-time weak solution with initial data $v(T)$. The local-in-time mild solution is constructed as in [19] for two-dimensional case. we follow his approach. To show the existence of a weak solution with initial data $v(T)$, we first construct a unique solution to approximated equations to (1.4) with energy inequality that is independent of approximation parameter. Next, taking limit to the approximated solution, we obtain a weak solution to (1.4).

Finally, we prove the decay of $\|v(t)\|_{L^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)}$ as $t \to \infty$. To this end, since the domain is vertically periodic, we can apply the Fourier expansion to v with respect to x_v variable:

$$
\begin{aligned}
v(x_h, x_v, t) &= v^0(x_h, t) + \sum_{j \neq 0} v^j(x_h, t)\ \mathrm{e}^{2\pi i j} \\
&=: v^0 + v_{os}.
\end{aligned}
$$

Using orthogonality of the Fourier series, we can derive the equation that v^0 satisfies. Since the averaged term v^0 is independent of x_v, we can apply two-dimensional argument as in [19] to get the decay of $\|v_0(t)\|_{L^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)}$ as $t \to \infty$. Unfortunately, because of the non-linearity of (1.4) and dependence of v_{os} on x_v variable, it is difficult to show the decay to the oscillating term by using same way as the averaged term. However, we can avoid this difficulty the Poincáre inequality and get the decay of $\|v_{os}\|_{L^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)}$. It is worth to mention that there was no result on asymptotic stability to the three-dimensional Oseen vortex under three-dimensional perturbation, even if basic flows or initial perturbation are small, and domain has no boundary. Our result is somewhat restrictive in terms of domain.

2 Main results

In this section, we firstly define some notations and notions to state our two main theorem. Secondly, we mention them.

We define vertically anisotropic function spaces to define the mild solutions to (1.4) that include the three dimensional Oseen vortex.

Definition 2.1. Let $\Omega = \mathbb{R}^3$ *or* $\mathbb{R}^2_h \times \mathbb{T}^1_v$. We define vertically anisotropic spaces $X^p(\Omega) (1 \leq p \leq \infty)$ and $Y^q((\Omega))$ $(1 < q < \infty)$ by

$$
X^p := \left\{ f = (f_1, f_2, f_3) \in L^1_{loc}(\Omega) : \mathrm{div}\ f = 0,\ ||f||_{X^p} < \infty \right\}, \tag{2.1}
$$

$$
Y^q := \left\{ f = (f_1, f_2, f_3) \in L^1_{loc}(\Omega) : \mathrm{div}\ f = 0,\ ||f||_{Y^q} < \infty \right\}, \tag{2.2}
$$

where

$$\|f\|_{X^p} := \sup_{x_v\in\mathbb{R}}(\int_{\mathbb{R}^2_h}|f(x_h,x_v)|^p dx_h)^{\frac{1}{p}} < \infty,$$
$$\|f\|_{Y^q} := \sup_{x_v\in\mathbb{R}}\sup_{\lambda>0}\lambda(|\{x_h\in\mathbb{R}^2 : |f(x_h,x_v)|>\lambda\}|)^{\frac{1}{q}} < \infty$$

respectively, where $|S|$ denotes the Lebesgue measure of S.

We note our main theorem. First one is a existence of the Oseen type solutions;

Theorem 2.2. *Let $\Omega = \mathbb{R}^3$ or $\mathbb{R}^2\times\mathbb{T}^1$. Let $u_0 \in Y^2(\Omega)$. Then there exists a positive number δ such that, if $\|u_0\|_{Y^2(\Omega)} \le \delta$, there exists a unique mild solutions $u \in C_tY^2_x(\Omega\times(0,\infty))$ of (1.1):*

$$u(t) = \mathrm{e}^{t\Delta}u_0 - \int_0^t \mathrm{e}^{(t-\tau)\Delta}P\mathrm{div}u(\tau)\otimes u(\tau))d\tau \quad \text{in} \quad Y^2(\Omega), \tag{2.3}$$

for all $t\in(0,T)$, where $\mathrm{e}^{t\Delta}$ and P are the heat kernel and the Helmholtz projection respectively, such that

$$\sup_{0<t<T}\|u(t)\|_{Y^2(\Omega)} + \sup_{0<t<T}t^{\frac{1}{4}}\|u(t)\|_{X^4(\Omega)} \le C\|u_0\|_{Y^2(\Omega)}, \tag{2.4}$$
$$u(t)\to u_0 \quad weakly* \quad \text{in} \quad Y^2(\Omega)+X^p(\Omega) \quad as\ t\to 0 \tag{2.5}$$

where $\frac{1}{p}=\frac{1}{r}+\frac{1}{4}$ for all $\frac{1}{2}<\frac{1}{r}<\frac{3}{4}$.

Proof of this theorem based on Fujita-Kato iteration scheme. We omit details here. Second our theorem is a asymptotic stability result of the Oseen type solution in vertically periodic domain;

Theorem 2.3. *Let $\Omega = \mathbb{R}^2_h\times\mathbb{T}_v$, $\delta>0$ be sufficiently small and $b(x,t)$ (basic flow) be a solution to (NS) in Theorem 2.2 with initial data $b_0\in Y^2(\Omega)$ with $\|b_0\|_{Y^2}<\delta$. Then, for $v_0 \in L^\infty_v\overline{C^\infty_{0,h_\sigma}}^{L^{2,\infty}}(\Omega)$ (initial perturbation), there exists a weak solution $w(x,t)$ to (1.1) with initial data $w_0=v_0+b_0$, which satisfies such that (1.1) in the sense of distribution, such that*

$$\lim_{t\to\infty}\|w(t)-b(t)-\mathrm{e}^{t\Delta}v_0\|_{L^2(\Omega)}=0$$

In this paper, we give a outline of the proof this Theorem 2.3 when $v_0\in L^\infty_vC^\infty_{0,h}$

3 Out line of the proof of Theorem 2.3

There are two step to show Theorem 2.3 :

Step1 Proof of the existence of a weak solution to the perturbed equation (1.4) with logarithmic energy estimate.

Step2 Proof of energy decay of the solution to the perturbed equation.

On step1, we first construct a unique solution to a approximate equation of perturbed equation, and taking its limit, we can get a weak solution to the perturbed equation with logarithmic energy estimate as in Maekawa [19]. Summing up this procedure, we get the following proposition.

Proposition 3.1. *Let $T > 0$, $v_0 \in L^\infty_v C^\infty_{0,h}(\Omega)$ with $\operatorname{div} v_0 = 0$ and b be a solution to (1.1) in Theorem 2.3. Then there exists a weak solution $v \in L^\infty_t L^2_x(\Omega \times (0,T))$ to (1.4) such that*

$$||v(t)||^2_{L^2} + \int_1^t ||\nabla v(s)||^2_{L^2}\, ds \leq C_1 + C_2 ||b_0||^2_{Y^2} \log(1+t) \tag{3.1}$$

for all $t \in (1, T)$, where $C_1 = C_1(v_0)$ and C_2 is independent of T.

Applying the Fourier expansion to v with respect to x_v, we can decompose v into averaged part v_a and oscillating part v_{os};

$$\begin{aligned} v(x_h, x_v, t) &= \sum_{k \in \mathbb{Z}} v_k(x_h, t) \mathrm{e}^{2\pi i x_v k} = v_0(x_h, t) + \sum_{k \neq 0} v_k(x_h, t) \mathrm{e}^{2\pi x_v k} \\ &=: v_a(x_h, t) + v_{os}(x_h, x_v, t). \end{aligned}$$

Because of orthogonality of the Fourier series, it follows from (3.1) that

$$\|v_a(t)\|^2_{L^2(\mathbb{R}^2)} + \int_1^t \|\nabla_h v_a\|^2_{L^2(\Omega)} \leq C + C\delta^2 \log(1+t) \tag{3.2}$$

$$\|v_{os}(t)\|^2_{L^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)} + \int_1^t \|\nabla v_{os}\|^2_{L^2(\Omega)} \leq C + C\delta^2 \log(1+t), \tag{3.3}$$

where $\delta > 0$ is a constant in Theorem 2.3. Since we can apply the Poincaré inequality to the oscillating part, we can derive the decay of v_{os} directly from (3.3). Therefore, it is essential to show the decay of v_a. We first show the following proposition to show this.

Proposition 3.2. *Let $T > 0$. Put $w_a := (-\Delta_h)^{-\frac{1}{4}} v_a$, where $(-\Delta_h)^{s/2} f = \mathcal{F}^{-1}(|\xi_h|^s \mathcal{F} f)$ for $s \in \mathbb{R}$. Then there exist constants $C > 0$ and $M > 0$ such that*

$$\begin{aligned} &\|w_a(t)\|^2_{L^2(\mathbb{R}^2)} + \int_1^t \|\nabla_h w_a(t)\|^2_{L^2(\mathbb{R}^2)} d\tau \\ &\leq C(1+t)^{M\delta^2} \left(1 + \log(1+t) + \sup_{1 \leq \tau \leq t} \|v_{os}(\tau)\|_{L^2(\mathbb{R}^2_h \times \mathbb{T}^1_v)} \log(1+t)\right) \end{aligned} \tag{3.4}$$

for all $1 < t \leq T$.

Proof. Integrate (1.4) with respect to x_v, then we get

$$\partial_t v_a^1 - \Delta_h v_a^1 + \operatorname{div} \int_{\mathbb{T}^1} (v^1 v + b^1 v + v^1 b) dx_v + \partial_1 q = 0 \tag{3.5}$$

$$\partial_t v_a^2 - \Delta_h v_a^2 + \operatorname{div} \int_{\mathbb{T}^1} (v^2 v + b^2 v + v^2 b) dx_v + \partial_2 q = 0 \tag{3.6}$$

$$\partial_t v_a^3 - \Delta_h v_a^3 + \operatorname{div} \int_{\mathbb{T}^1} (v^3 v + b^3 v + v^3 b) dx_v = 0. \tag{3.7}$$

(3.5) (3.6) are the two dimensional perturbed Navier-Stokes system and (3.7) is two dimensional heat equation respectively. It follows from integration by parts that

$$\begin{aligned}
&\frac{1}{2}\partial_t \|w_a\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla w_a\|_{L^2(\mathbb{R}^2)}^2 \\
&\leq |\int_{\mathbb{R}^2} \int_{\mathbb{T}^1} (v \otimes v + b \otimes v + v \otimes b) dx_v : \nabla_h (-\Delta_h)^{-\frac{1}{4}} w_a dx_h| \\
&= |\int_{\mathbb{R}^2} \int_{\mathbb{T}^1} ((v_a + v_{os}) \otimes (v_a + v_{os}) + b \otimes (v_a + v_{os}) \\
&+ (v_a + v_{os}) \otimes b) dx_v : \nabla (-\Delta_h)^{-\frac{1}{4}} w_a dx_h| \\
&= |\int_{\mathbb{R}^2} \int_{\mathbb{T}^1} (v_a \otimes v_a + v_{os} \otimes v_{os} + b \otimes v_a + b \otimes v_{os} + v_a \otimes b \\
&+ v_{os} \otimes b) dx_v : \nabla (-\Delta_h)^{-\frac{1}{4}} w_a dx_h| \\
&=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned} \tag{3.8}$$

Estimate for I_1 The Sobolev embedding

$$\|v_a\|_{L^4(\mathbb{R}^2)} \leq C \|(-\Delta_h)^{\frac{1}{4}} v_a\|_{L^2(\mathbb{R}^2)} \tag{3.9}$$

and the interpolation inequality

$$\|(-\Delta_h)^{\frac{1}{4}} v_a\|_{L^2(\mathbb{R}^2)} \leq C \|v_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla_h v_a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \tag{3.10}$$

yield

$$\begin{aligned}
|I_1| &\leq C \|v_a\|_{L^4(\mathbb{R}^2)}^2 \|(-\Delta_h)^{\frac{1}{4}} w_a\|_{L^2(\mathbb{R}^2)} \\
&\leq C \|(-\Delta_h)^{\frac{1}{4}} v_a\|_{L^2(\mathbb{R}^2)}^2 \|(-\Delta_h)^{\frac{1}{4}} w_a\|_{L^2(\mathbb{R}^2)} \\
&\leq C \|v_a\|_{L^2(\mathbb{R}^2)} \|(-\Delta_h)^{\frac{1}{2}} v_a\|_{L^2(\mathbb{R}^2)} \|(-\Delta_h)^{\frac{1}{4}} w_a\|_{L^2(\mathbb{R}^2)} \\
&\leq C \|\nabla_h v_a\|_{L^2(\mathbb{R}^2)} \|(-\Delta_h)^{\frac{1}{4}} w_a\|_{L^2(\mathbb{R}^2)}^2 \\
&\leq C \|\nabla_h v_a\|_{L^2(\mathbb{R}^2)} \|w_a\|_{L^2(\mathbb{R}^2)} \|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Applying the Young inequality to the last inequality, we find

$$|I_1| \leq C \|\nabla_h v_a\|_{L^2(\mathbb{R}^2)}^2 \|w_a\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{16} \|\nabla_h w_a\|_{L^2(\mathbb{R}^2)}^2$$

Estimate for I_2 Using the Schwarz inequality, (3.9), (3.10) and the Young inequality, we find

$$
\begin{aligned}
|I_2| &\le C\left\|\int_{\mathbb{T}^1} v_{os}\otimes v_{os}dx_v\right\|_{L^2(\Omega)}\|(-\Delta_h)^{\frac14}w_a\|_{L^2(\mathbb{R}^2)}\\
&\le C\int_{\mathbb{T}}\|v_{os}\|^2_{L^4_h(\mathbb{R}^2)}dx_v\|w_a\|^{\frac12}_{L^2(\mathbb{R}^2)}\|\nabla_h w_a\|^{\frac12}_{L^2(\mathbb{R}^2)},\\
&\le C\int_{\mathbb{T}}\|(-\Delta_h)^{\frac14}v_{os}\|^2_{L^2_h(\mathbb{R}^2)}dx_v\|w_a\|^{\frac12}_{L^2(\mathbb{R}^2)}\|\nabla_h w_a\|^{\frac12}_{L^2(\mathbb{R}^2)}\\
&\le C\|v_{os}\|_{L^2(\Omega)}\|\nabla v_{os}\|_{L^2(\Omega)}\|w_a\|^{\frac12}_{L^2(\mathbb{R}^2)}\|\nabla_h w_a\|^{\frac12}_{L^2(\mathbb{R}^2)}\\
&\le C_1\|v_{os}\|^2_{L^2(\Omega)}\|\nabla v_{os}\|_{L^2(\Omega)}\\
&\quad + C_2\|\nabla v_{os}\|^2_{L^2(\Omega)}\|w_a\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{16}\|\nabla_h w_a\|^2_{L^2(\mathbb{R}^2)}\\
&\le C_1\|v_{os}\|_{L^2(\Omega)}\|\nabla v\|^2_{L^2(\Omega)}\\
&\quad + C_2\|\nabla v\|^2_{L^2(\Omega)}\|w_a\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{16}\|\nabla_h w_a\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}
$$

In the last inequality, we used the Poincaré inequality.

Estimate for I_3 and I_5. Using the Hölder inequality,(3.9), (3.10) and the Young inequality, we find

$$
\begin{aligned}
|I_3|+|I_5| &\le C\int_{\mathbb{T}^1}\|b\|_{L^4_h(\mathbb{R}^2)}dx_v\|v_a\|_{L^4_h(\mathbb{R}^2)}\|(\Delta_h)^{\frac14}w_a\|_{L^2(\mathbb{R}^2)}\\
&\le C\|b\|_{X^4(\Omega)}\|(-\Delta_h)^{\frac14}v_a\|_{L^2(\mathbb{R}^2)}\|w_a\|^{\frac12}_{L^2(\mathbb{R}^2)}\|\nabla_h w_a\|^{\frac12}_{L^2(\mathbb{R}^2)}\\
&\le C\|b\|^4_{X^4(\Omega)}\|w_a\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{16}\|\nabla_h w_a\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}
$$

Estimate for I_4 and I_6. Using the Hölder inequality, (3.9), (3.10) and the Pincaré inequality, we find

$$
\begin{aligned}
|I_4|+|I_6| &\le C\int_{\mathbb{T}^1}\|b\|_{L^4_h(\mathbb{R}^2)}\|v_{os}\|_{L^4_h(\mathbb{R}^2)}dx_v\|(-\Delta_h)^{\frac14}w_a\|_{L^2(\mathbb{R}^2)}\\
&\le C\|b\|_{X^4(\Omega)}\int_{\mathbb{T}^1}\|v_{os}\|^{\frac12}_{L^2_h(\mathbb{R}^2)}\|\nabla_h v_{os}\|^{\frac12}_{L^2_h(\mathbb{R}^2)}dx_v\|(-\Delta_h)^{\frac14}w_a\|_{L^2(\mathbb{R}^2)}\\
&\le C\|b\|_{X^4(\Omega)}\|v_{os}\|^{\frac12}_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|\nabla v_{os}\|^{\frac12}_{L^2(\Omega)}\|w_a\|^{\frac12}_{L^2(\mathbb{R}^2)}\|\nabla_h w_a\|b^{\frac12}_{L^2(\mathbb{R}^2)}\\
&\le C_1\|b\|^2_{X^4(\Omega)}\|v_{os}\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\\
&\quad + C_2\|\nabla v_{os}\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|w_a\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{16}\|\nabla_h w_a\|^2_{L^2(\mathbb{R}^2)}\\
&\le C_1\|b\|^2_{X^4(\Omega)}\|\nabla v_{os}\|_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\\
&\quad + C_2\|\nabla v_{os}\|^2_{L^2(\mathbb{R}^2_h\times\mathbb{T}^1_v)}\|w_a\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{16}\|\nabla_h w_a\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}
$$

Thus, from (3.8), above estimates and the Gronwall inequality, we get

$$\|w_a(t)\|_{L_h^2}^2 + \int_1^t \|\nabla w_a(\tau)\|_{L_h^2}^2 d\tau \le \exp(\Phi(t))\|w_a(1)\|_{L_h^2}^2 + \int_1^t \Psi(\tau)d\tau \tag{3.11}$$

where

$$\Phi(t) = C_1 \int_1^t (\|\nabla v(\tau)\|_{L^2}^2 + \|b(\tau)\|_{X^4}^4)d\tau$$
$$\Psi(t) = C_2 \exp(\int_\tau^t \Phi(s)ds)(\|v_{os}(t)\|_{L^2}\|\nabla v_{os}(t)\|_{L^2}^2 + \|b(t)\|_{X^4}^2\|\nabla v_{os}(t)\|_{L^2}).$$

Using (3.3) and (2.4), we find

$$\Phi(t) \le C_1(1+\delta^2\log(1+t)).$$

and

$$\begin{aligned}
&\int_1^t \Psi(t)d\tau \\
&\le C_2(1+t)^{C_1\delta^2}(\sup_{1\le\tau\le t}\|v_{os}(\tau)\|_{L^2}\int_1^t \|\nabla v_{os}(\tau)\|^2 d\tau + \int_1^t \|b(\tau)\|_{X^4}^2\|\nabla v_{os}(\tau)\|_{L^2}d\tau) \\
&\le C_2(1+t)^{C_1\delta^2}(\sup_{1\le\tau\le t}\|v_{os}(\tau)\|_{L^2}\int_1^t \|\nabla v_{os}(\tau)\|^2 d\tau \\
&\quad + (\int_1^t \|b(\tau)\|_{X^4}^4 d\tau)^{\frac{1}{2}}(\int_1^t \|\nabla v_{os}(\tau)\|_{L^2}^2 d\tau)^{\frac{1}{2}}) \\
&\le C_2(1+t)^{C_1\delta^2}(1+\log(1+t) + \sup_{1\le\tau\le t}\|v_{os}(\tau)\|_{L^2}\log(1+t)).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\|w_a(t)\|_{L^2(\mathbb{R}^2)}^2 + \int_1^t \|\nabla w_a(t)\|_{L^2(\mathbb{R}^2)}^2 d\tau \\
&\le C(1+t)^{M\delta^2}\left(1+\log(1+t) + \sup_{1\le\tau\le t}\|v_{os}(\tau)\|_{L^2(\mathbb{R}_h^2\times\mathbb{T}_v^1)}\log(1+t)\right).
\end{aligned} \tag{3.12}$$

□

Let $t>1$. Using Proposition 3.2 and (3.3), we find

$$\begin{aligned}
&||w_a(t)||_{L^2}^2 + ||v_{os}(t)||_{L^2}^2 + \int_1^t ||\nabla w_a(\tau)||_{L^2}^2 \, d\tau + \int_1^t ||\nabla v_{os}(\tau)||_{L^2}^2 \, d\tau \\
&\le C(1+t)^{M\delta^2}\log^{\frac{3}{2}}(1+t) + (\text{lower order}).
\end{aligned} \tag{3.13}$$

We see from (3.13) that there exists $t_0 \in [t/2, t]$ such that

$$\begin{aligned}
&||w_a(t_0)||_{L^2}^2 + ||v_{os}(t_0)||_{L^2}^2 + t_0||\nabla w_a(t_0)||_{L^2}^2 + t_0||\nabla v_{os}(t_0)||_{L^2}^2 \\
&\le C(1+t_0)^{M\delta^2}\log^{\frac{3}{2}}(1+t_0) + (\text{lower order}).
\end{aligned} \tag{3.14}$$

Therefore, we find from the Poincaré inequality and the above inequality that

$$
\begin{aligned}
&\|v(t_0)\|_{L^2}^2 \\
&\le 2(\|v_a(t_0)\|_{L^2}^2 + \|v_{os}(t_0)\|_{L^2}^2) \\
&\le C(\|w_a(t_0)\|_{L^2}\|\nabla w_a(t_0)\|_{L^2} + \|v_{os}(t_0)\|_{L^2}\|\nabla v_{os}(t_0)\|_{L^2}) \\
&\le C(1+t_0)^{-\frac{1}{2}+M\delta^2}\log^{\frac{3}{2}}(1+t_0) + \text{(lower order)} \\
&\le C(1+t)^{-\frac{1}{2}+M\delta^2}\log^{\frac{3}{2}}(1+t) + \text{(lower order)}.
\end{aligned}
\tag{3.15}
$$

Now we know that v satisfies

$$
\begin{aligned}
&\partial_t v - \Delta v + \mathrm{div}(v\otimes v + b\otimes v + v\otimes b) + \nabla q = 0 \ in\ \Omega\times(0,\infty), \\
&\mathrm{div}\ v = 0\ in\ \Omega\times(0,\infty),
\end{aligned}
$$

then, applying integration by part and the Gronwall inequality to the perturbed equation, we find

$$
\|v(t)\|_{L^2}^2 + \int_{t_0}^{t}\|\nabla v(\tau)\|_{L^2}^2 d\tau \le \mathrm{e}^{\int_{t_0}^{t}\|b(\tau)\|_{X^4}^4 d\tau}\|v(t_0)\|_{L^2}^2, \tag{3.16}
$$

for $t_0\in[\frac{t}{2},t]$. Since $\int_{t_0}^{t}\|b(\tau)\|_{X^4}^4 d\tau \le C\log\frac{t}{t_0} < \infty$ and (3.15) , we obtain

$$
\begin{aligned}
\text{RHS (3.16)} &\le C\|v(t_0)\|_{L^2}^2 \\
&\le C(1+t)^{-\frac{1}{2}+M\delta^2}\log^{\frac{3}{2}}(1+t) + \text{(lower order)}
\end{aligned}
$$

If we take $\delta>0$ so small that $-\frac{1}{2}+M\delta^2<0$, we get the desired decay of v.

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