Realizations of ADE type logarithmic principal W-algebras

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1 Introduction

Definition 1. Let V be a vertex operator algebra (VOA) and the vertex operator of $a \in V$ is denoted by

$$Y(a,z) = a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \in \operatorname{End} V[[z^{\pm}]].$$

1. The VOA V is C_2-cofinite if $\dim R_V < \infty$. Here, $R_V = V/C_2(V)$ and

$$C_2(V) = \operatorname{span}_{\mathbb{C}}\{a_{-2}b \mid a,b \in V\}.$$

2. The VOA V is rational if the representation category $\operatorname{Rep} V$ of V is semisimple, i.e. ${}^{\forall}V$-module $M \in \operatorname{Rep} V$ is decomposed into the direct sum of irreducible V-modules.

The theory of C_2-cofinite BUT irrational VOAs is less complete than that of C_2-cofinite and rational VOAs, because only a few examples of C_2-cofinite but irrational VOAs are found. Our aim is to construct many C_2-cofinite but irrational VOAs, and the strategy is to generalize the well-known example of such kinds of VOAs. One of the most well-known examples of C_2-cofinite but irrational VOAs is the triplet W-algebra. This VOA is defined as the kernel of the narrow screening operator on the rescaled root lattice of A_1 type, and studied by many people (e.g. [FGST1]-[FGST3], [AM1]-[AM3], [NT], [TW]). The definition of the triplet W-algebra is generalized in ADE types immediately, and we call these VOAs the logarithmic principal W-algebras (In this report, we omit "principal" and call these VOAs logarithmic W-algebras for simplicity). However, despite of the importance of logarithmic W-algebras in studies of C_2-cofinite but irrational VOAs, there are not much known except for the case of A_1 type (triplet W-algebras) because of the complicated structures.

On the other hands, B.L.Feigin and I.Yu.Tipunin introduced a geometric approach to the studies of logarithmic W-algebras and their characters [FT]. They introduced Feigin-Tipunin algebras as sheaf cohomologies on the flag varieties and conjectured the following (we call Feigin-Tipunin conjecture):

1. The Feigin-Tipunin algebras are geometric realizations of the logarithmic W-algebras.
2. The character formulas of the logarithmic W-algebras.
3. The W-algebra module structures the logarithmic W-algebras.

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The author proved this conjecture in his master thesis. In this report, we want to introduce this result.

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2 Some results on the triplet W-algebras

In order to avoid duplication, we give the setting in general ADE types all along.

2.1 Setting

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be an ADE type simple Lie algebra of rank l and its triangular decomposition, $\mathfrak{h}$ and $\mathfrak{b} = \mathfrak{n}_- \oplus \mathfrak{h}$ its Cartan and Borel subalgebras respectively, G, H, and B the semisimple, simply-connected, complex algebraic groups corresponding to $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{b}$ respectively. The labelling of the Dynkin diagrams are the one in [B]. Let Q be the root lattice of $\mathfrak{g}$, Q' the weight lattice of $\mathfrak{g}$, Q'_+ the set of dominant integral weights of $\mathfrak{g}$, $\alpha_1, \ldots, \alpha_l$ the simple roots of $\mathfrak{g}$, Π the set of simple roots of $\mathfrak{g}$, $\omega_1, \ldots, \omega_l$ the fundamental weights of $\mathfrak{g}$. We denote by $(\cdot, \cdot)$ the standard invariant form of $\mathfrak{g}$, W the Weyl group of $\mathfrak{g}$, (c_{ij}) the Cartan matrix of $\mathfrak{g}$ and (c^{ij}) the inverse matrix to (c_{ij}), ρ the half sum of positive roots, h the (dual) Coxeter number of $\mathfrak{g}$, Ω the abelian group Q'/Q. We choose the representatives of elements from Ω in Q' in the following way: for A_l, D_l, E_6, E_7, E_8, we choose $\{0, \omega_1, \ldots, \omega_l\}$, $\{0, \omega_1, \omega_{l-1}, \omega_l\}$, $\{0, \omega_1, \omega_3\}$, $\{0, \omega_2\}$, $\{0\}$ respectively. For $X = A$ or D or E, X_l means that $\mathfrak{g}$ is the X type simple Lie algebra of rank l. We fix an integer $p \in \mathbb{Z}_{\geq 2}$.

2.2 Lattice VOA and the irreducible modules

Let $\mathcal{L}$ be the lattice VOA associated to the rescaled root lattice $\sqrt{p}Q$. In a manner of speaking, lattice VOAs is "lattice $\otimes$ Fock spaces". We explain this: Let $\mathcal{F}_0$ be the rank l Heisenberg vertex algebra. By abuse of notation, we sometimes use $\mathcal{F}_0$ as the rank l Heisenberg algebra. By using this notation, we denote by $\mathcal{F}_\alpha = \mathcal{F}_0|\alpha\rangle$ the Fock space corresponding to α. Then, as a vector space,

$$\mathcal{L} \simeq \bigoplus_{\alpha \in \sqrt{p}Q} \mathcal{F}_\alpha \tag{1}$$

and the basis is given by

$$\left\{ (\alpha_{i_1})_{-n_1} \cdots (\alpha_{i_k})_{-n_k} |\sqrt{p}\beta\rangle \mid 1 \leq i_1 \leq \cdots \leq i_k \leq l, \alpha_{i_j} \in \Pi, n_j \in \mathbb{N}, \beta \in Q \right\}. \tag{2}$$

Irreducible modules of $\mathcal{L}$ are classified by elements of abelian group $\Lambda = \frac{1}{\sqrt{p}}Q'/\sqrt{p}Q$ ([D]). We choose the basis elements $\{\lambda_j = \frac{1}{\sqrt{p}}\omega_j | 1 \leq j \leq l\}$ of $\frac{1}{\sqrt{p}}Q'$. For each equivalence class $\langle\lambda\rangle \in \Lambda$, we choose a unique representative $\lambda \in \frac{1}{\sqrt{p}}Q'$ of $\langle\lambda\rangle \in \Lambda$ as

$$\lambda = -\sqrt{p}\hat{\lambda} + \bar{\lambda} = -\sqrt{p}\hat{\lambda} + \sum_{j=1}^{l} s_j \lambda_j, \tag{3}$$

where $\hat{\lambda} \in Q'_+/Q \cap Q'_+$ and $s_j = 0, \cdots, p-1$.

For $\lambda \in \frac{1}{\sqrt{p}}Q'$, we denote the irreducible module of $\mathcal{L}$ corresponding to $\langle\lambda\rangle \in \Lambda$ by

$$\mathcal{L}_{\langle\lambda\rangle} = \bigoplus_{\alpha\in\sqrt{p}Q} \mathcal{F}_{\lambda+\alpha}. \tag{4}$$

We often denote by $\mathcal{L}_\lambda$ the $\mathcal{L}_{\langle\lambda\rangle}$.

2.3 Shifted conformal vector

We choose the (shifted) energy momentum field of $\mathcal{L}$ in the form

$$T(z) = \frac{1}{2}\sum_{1\leq i,j\leq l} c^{ij} : \alpha_i(z)\alpha_j(z) : +Q_0\partial\rho(z) \tag{5}$$

where

$$Q_0 = \sqrt{p} - \frac{1}{\sqrt{p}}. \tag{6}$$

In other words, the conformal vector $\omega \in \mathcal{F}_0 \subseteq \mathcal{L}$ is defined by

$$\omega = \frac{1}{2}\sum_{1\leq i,j\leq l} c^{ij}(\alpha_i)_{-1}\alpha_j + Q_0(\rho)_{-2}|0\rangle$$

and

$$\omega(z) = T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}.$$

The central charge c of $T(z)$ is

$$c = l + 12(\rho,\rho)(2 - p - \frac{1}{p}) = l + h\dim\mathfrak{g}(2 - p - \frac{1}{p}) \tag{7}$$

and for $\lambda \in \frac{1}{\sqrt{p}}Q'$, the conformal weight Δ_λ of $|\lambda\rangle$ is

$$\Delta_\lambda = \frac{1}{2}|\lambda - Q_0\rho|^2 + \frac{c-l}{24} = \frac{1}{2}|\lambda|^2 - Q_0(\lambda,\rho). \tag{8}$$

2.4 Screening operators and narrow screening operators

We consider the screening operators f_i and narrow screening operators F_i on $\mathcal{L}_\lambda$ defined by

$$f_i = |\sqrt{p}\alpha_i\rangle_0, \tag{9}$$

$$F_i = |-\frac{1}{\sqrt{p}}\alpha_i\rangle_0, \tag{10}$$

for $i = 1,\cdots,l$.

Remark 1. Strictly speaking, the definitions of F_i are different for each $\mathcal{L}_\lambda$ and our definition of F_i is that in the case of $\lambda \in \sqrt{p}Q'$ (see [CRW]), thus we have to denote not F_i but $F_{i,\lambda}$ on $\mathcal{L}_\lambda$. However, the differences of definitions do not effect on the proof of our main results, and thus, we denote F_i by $F_{i,\lambda}$ on $\mathcal{L}_\lambda$ for simplicity.

A straightforward calculation gives the relations

$$[f_i, T(z)] = [F_i, T(z)] = 0 \tag{11}$$

and

$$[f_i, F_j] = 0. \tag{12}$$

In particular, (11) means that f_i and F_i preserve the conformal grading.

2.5 Definition of the logarithmic W-algebras and results on the triplet W-algebras

Definition 2. The sub VOA of $\mathcal{L}$

$$W(p)_Q = \bigcap_{i=1}^{l} \ker F_i|_{\mathcal{L}} \subseteq \mathcal{L}.$$

called the logarithmic W-algebra associated to p and Q. In the case of A_1 type, $W(p)_Q$ is the triplet W-algebra (e.g. [AM1]).

If $\mathfrak{g}$ is of A_1 type, the following results are obtained.

Theorem 1 (Adamovic-Milas, etc.)**.**

1. *Let L_k be the Virasoro VOA at level $k = p - 2$. Then,* $\ker f|_{\mathcal{F}_0} \simeq L_k$.
2. *Let $\mathcal{R}_m$ be the* $\dim = m$ *irreducible $\mathfrak{sl}_2$-module and $L_{k,\mu}$ the irreducible L_k-module. Then,*

$$\ker F|_{\mathcal{L}_\lambda} \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{R}_{2n+1+\hat{\lambda}} \otimes L_{k,-n\sqrt{p}\alpha_1+\lambda} \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \bigoplus_{i=0}^{2n+\hat{\lambda}} f^i L_k | - n\sqrt{p}\alpha_1 + \lambda\rangle.$$

3. *The triplet W-algebra* $\ker F|_{\mathcal{L}}$ *is simple C_2-cofinite and irrational VOA.*
4. $\{\ker F|_{\mathcal{L}_\lambda}\}$ *is the complete set of the irreducible modules of the triplet W-algebra* $\ker F|_{\mathcal{L}}$.

The author generalized Theorem 1.(1),(2) by using the geometric method introduced in [FT]. We give the setting and main results in the next section.

3 Feigin-Tipunin algebras and main results

3.1 Feigin-Tipunin algebras

For $1 \leq i \leq l$, we consider the following operators h_i acting on $\mathcal{L}_\lambda$:

$$h_i = -\frac{1}{\sqrt{p}}(\alpha_i)_0 + \frac{1}{\sqrt{p}}(\alpha_i, \bar{\lambda}), \tag{13}$$

where $\bar{\lambda}$ is as in (3).

The operator h_i does not commute with $T(z)$, but sometimes we also call h_i the screening operator in this paper for simplicity.

Theorem 2 ([FT, Theorem 4.1])**.**

1. *The above operators $\{f_i\}_{i=1}^{l}$ and $\{h_i\}_{i=1}^{l}$ induces the action of $\mathfrak{b}$ on $\mathcal{L}_\lambda$.*
2. *The action of $\mathfrak{b}$ in (1) is integrable.*

Theorem 2.(1) is proved in [FT] and the author proved Theorem 2.(2) in his master thesis.

For $\lambda \in \Lambda$, we consider the homogeneous vector bundle

$$\xi_\lambda = G \times_B \mathcal{L}_\lambda \tag{14}$$

on the flag variety G/B. The action of B on G is given by the right multiplication and on $\mathcal{L}_\lambda$ by the one given in Theorem 1.

Lemma 1. *$H^0(\xi_0)$ has the structure of vertex operator algebra and $H^0(\xi_\lambda)$ is a $H^0(\xi_0)$-module.*

Definition 3. The VOA $H^0(\xi_0)$ is called Feigin-Tipunin algebra associated to $p \in \mathbb{Z}_{\geq 2}$ and Q.

Now we are ready to describe the main theorem.

3.2 Main results

Theorem 3. *(Main theorem)*

1. *We have the isomorphism*

$$H^n(\xi_\lambda) \simeq \begin{cases} 0 & (n \geq 1), \\ \bigcap_{i=1}^{l} \ker F_i|_{\mathcal{L}_\lambda} & (n = 0), \end{cases}$$

of modules of the vertex operator algebra $\mathcal{WLX}_l(p) \simeq \bigcap_{i=1}^{l} \ker F_i|_{\mathcal{L}}$ (in the case of $\lambda = 0$).

2. *For $\mu \in Q'_+$, we denote by $\mathcal{R}_\mu$ the finite dimensional highest weight irreducible $\mathfrak{g}$-module with highest weight μ and $\mathrm{Weight}(\mathcal{R}_\mu)$ by the set of weights of $\mathcal{R}_\mu$. Let $\mathcal{W}^k(\mathfrak{g})$ be the principal universal affine W-algebra ([Ar]) of level $k = p - h$.*
Then, We have the free field realization of the W-algebra

$$\mathcal{W}^k(\mathfrak{g}) = \bigcap_{i=1}^{l} \ker f_i|_{\mathcal{F}_0}$$

and the G-module isomorphism

$$\begin{aligned} H^0(\xi_\lambda) &\simeq \bigoplus_{\alpha \in Q'_+ \cap Q} \mathcal{R}_{\alpha + \hat{\lambda}} \otimes \mathcal{W}^k(\mathfrak{g}) | -\sqrt{p}\alpha + \lambda \rangle \\ &= \bigoplus_{\alpha \in Q'_+ \cap Q} \bigoplus_{\substack{1 \leq i_1, \cdots, i_N \leq l \\ \alpha + \hat{\lambda} - \sum_{j=1}^{N} \alpha_{i_j} \in \mathrm{Weight}\, \mathcal{R}_{\alpha+\hat{\lambda}}}} f_{i_1} \cdots f_{i_N} \mathcal{W}^k(\mathfrak{g}) | -\sqrt{p}\alpha + \lambda \rangle \subseteq \mathcal{L}_\lambda. \end{aligned}$$

In particular, we obtain the $\mathcal{W}^k(\mathfrak{g})$-module extension

$$\bigcap_{i=1}^{l} \ker F_i|_{\mathcal{L}_\lambda} = \bigoplus_{\alpha \in Q'_+ \cap Q} \bigoplus_{\substack{1 \leq i_1, \cdots, i_N \leq l \\ \alpha + \hat{\lambda} - \sum_{j=1}^{N} \alpha_{i_j} \in \mathrm{Weight}\, \mathcal{R}_{\alpha+\hat{\lambda}}}} f_{i_1} \cdots f_{i_N} \mathcal{W}^k(\mathfrak{g}) | -\sqrt{p}\alpha + \lambda \rangle.$$

3. *We have the full character formulas*

$$\mathrm{Tr}_{\bigcap_{i=1}^{l} \ker F_i|_{\mathcal{L}_\lambda}} q^{L_0 - \frac{c}{24}} z_1^{\alpha_1} \cdots z_l^{\alpha_l} = \sum_{\alpha \in Q'_+ \cap Q} \chi^{\mathfrak{g}}_{\alpha+\hat{\lambda}}(z) \Big(\sum_{\sigma \in W} (-1)^{L(\sigma)} \frac{q^{\frac{1}{2}|\sqrt{p}\sigma(\alpha+\rho+\hat{\lambda}) - \bar{\lambda} - \frac{1}{\sqrt{p}}\rho|^2}}{\eta(q)^l} \Big),$$

where $\chi^{\mathfrak{g}}_\beta(z)$ is the Weyl character formula of $\mathcal{R}_\beta$ and $L(\sigma)$ the length of $\sigma \in W$, $\eta(q) = (q;q)_\infty = \Pi_{n \geq 1}(1 - q^n)$.

Remark 2. Theorem 3.(1) claims that the Feigin-Tipunin algebras are geometric realizations of the logarithmic W-algebras. Theorem 3.(2) is generalization of Theorem 1.(1), (2). However, strictly speaking, we have to show that each $\mathcal{W}^k(\mathfrak{g})$-modules that appear in the modules of the logarithmic W-algebras $\bigcap_{i=1}^{l} \ker F_i|_{\mathcal{L}_\lambda}$ is irreducible. At least in the case of $p \geq h$ and $\alpha = \lambda = 0$, this claim is shown [CrM]. In other words, under the assumption of $p \geq h$, $W^k(\mathfrak{g})$ is simple W-algebra.

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