Note on Symmetric Group and Classical Invariant Theory

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Abstract

In this paper, we construct the analogue theory of Eisenstein series in classical invariant theory. The groups appearing from the construction are also investigated.

1 Introduction

Eisenstein series can give very concrete example of modular forms. By corresponding combinatorics and modular forms, we introduced the concept of E-polynomials.

On the other hand, classical invariant theory plays important roles in many branches of mathematics. Here we show a construction of the analogue theory of Eisenstein series. We give an investigation of graded and centralizer ring that appear.

2 Classical Invariant Theory

We begin the discussion by a ground form of degree m

$$f = \sum_{i=0}^{m} u_i \binom{m}{i} \xi_1^{m-i} \xi_2^{i}$$

for a positive number m. While $\xi_1,\ \xi_2$ are transformed according to

$$(\xi_1\ \xi_2) = (\xi_1'\ \xi_2')A \text{ (”contragrediently”)},$$

f changes into a form of the new variables $\xi_1',\ \xi_2'$ with the coefficients $u_0',\ u_1',\ \ldots,\ u_m'$ where

$$\begin{pmatrix} u_0' \\ u_1' \\ \vdots \\ u_m' \end{pmatrix} = \big(A\big)_m \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_m \end{pmatrix}.$$

To shorten, we write $u' = \big(A\big)_m u$.

We operate $SL(2, \mathbf{C})$ on $\mathbf{C}[u] = \mathbf{C}[u_0, u_1, \ldots, u_m]$ by the above representation and consider the invariant subring $S(2, m)$ defined by:

$$S(2, m) := \{J \in \mathbf{C}[u] :\ J(u') = J(u),\ {}^{\forall}A \in SL(2, \mathbf{C})\}.$$

It is known that $S(2,m)$ is of finite type over $\mathbf{C}$ and here we consider only invariants of even degree and denote it by $S(2,m)^e$.

In order to obtain the useful construction of invariants, we shall interpret the ground form as

$$f = u_0 \prod_{i=1}^{m} (\xi_1 - \varepsilon_i \xi_2).$$

The fundamental theorem of symmetric functions gives the explicit relations between u_is and ε_is. At any rate, the following lemma is a construction of invariants we expected (*cf.* [2]).

Lemma 1 *An expression of the form*

$$u_0^r \sum (\varepsilon_i - \varepsilon_j)(\varepsilon_k - \varepsilon_l) \ldots,$$

in which every ε_i appears r times in each product and which is symmetric in $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ can be considered as an invariant of degree r.

3 Preliminaries

Let g be a positive integer. We start with a ground form of degree $2g+2$

$$\begin{aligned} f &= \sum_{i=0}^{2g+2} u_i \binom{2g+2}{i} \xi_1^{(2g+2)-i} \xi_2^i \\ &= u_0 \prod_{i=1}^{2g+2} (\xi_1 - \varepsilon_i \xi_2). \end{aligned}$$

We would like to concentrate on one type of invariants we shall define now. We fix the following polynomial

$$\varphi_{2n} = u_0^{2n} (\varepsilon_1 - \varepsilon_2)^{2n} (\varepsilon_3 - \varepsilon_4)^{2n} \ldots (\varepsilon_{2g+1} - \varepsilon_{2g+2})^{2n}.$$

We denote by G the symmetric group of degree $2g+2$. The group G acts on the polynomial ring $\mathbf{C}[\varepsilon_1, \ldots, \varepsilon_{2g+2}]$ as $F(\ldots, \varepsilon_i, \ldots)^\sigma = F(\ldots, \varepsilon_{i^\sigma}, \ldots)$. The stabilizer $G_{\varphi_{2n}}$ of φ_{2n} is defined by the elements of G that do not move φ_{2n}.

Proposition 2 *The group $G_{\varphi_{2n}}$ can be generated by the $(g+1)+2$ elements*

$$\begin{gathered} (1\ 2), (3\ 4), \ldots, (2g+1\ \ 2g+2), \\ (1\ 3)(2\ 4), (1\ 3\ 5\ \ldots\ 2g+1)(2\ 4\ \ldots\ 2g+2) \end{gathered}$$

and is isomorphic to $C_2^{g+1} \rtimes S_{g+1}$. In particular, $G_{\varphi_{2n}}$ does not depend on n.

We denote by K the group $G_{\varphi_{2n}}$ and by κ the cardinality of $K \backslash G$.

4 Result

In this section, we investigate the subring of $S(2, 2g+2)$.

We set

$$\psi_{2n} = \sum_{K\backslash G \ni \sigma} \varphi_{2n}^{\sigma},$$

which is actually an element of degree $2n$ in $S(2, 2g+2)$ by Lemma 1. We shall denote by A_g the ring generated by ψ_{2n} $(n = 1, 2, \dots)$ over $\mathbf{C}$. The ring A_g is a subring of the invariant ring $S(2, 2g+2)$.

Theorem 3 *The ring A_g is finitely generated over $\mathbf{C}$ and generated by $\psi_2, \psi_4, \dots, \psi_{2\kappa}$.*

Theorem 4 *(1) A_1 is generated by ψ_2, ψ_6 and coincides with $S(2, 4)^e$.*

(2) A_2 is generated by $\psi_2, \psi_4, \psi_6, \psi_{10}$ and coincides with $S(2, 6)^e$.

(3) A_3 is strictly smaller than $S(2, 8)^e$.

Now we explore combinatorial properties of the permutation group arising from the action of G on $\Omega = K\backslash G$. Define a permutation group $\mathcal{G}$ as a representation of G on Ω. Let G_1 be the stabilizer of a point 1. For each orbit Δ, we define an adjacency matrix $\mathfrak{P}(\Delta) = (v)_{\alpha,\beta}^{\Delta}$ by

$$v_{\alpha,\beta}^{\Delta} = \begin{cases} 1 & \exists g \text{ such that } 1^g = \beta \text{ and } \alpha^{g^{-1}} \in \Delta \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Denote the matrices $\mathfrak{P}(\Delta)$ by $A_0 = I, A_1, ..., A_d$ where d is class of association scheme. It is known that the matrices $A_0 = I, A_1, ..., A_d$ generate an algebra, called *bose-Mesner* algebra $\mathfrak{A}$ of the association scheme $\mathfrak{X}$.

Denote $\mathfrak{A}^{(k)}$ be the centralizer of algebra of the $k-$th tensor representation of G. We have the following theorem.

Theorem 5 *For $g = 2$, we have that*

$$\mathfrak{A}^{(k)} \cong \begin{cases} 3\mathcal{M}_1 & k = 1 \\ \bigoplus_{i \in I} \mathcal{M}_i & k \geq 2 \end{cases}$$

$$\begin{aligned} \overrightarrow{d^{(k)}} &= \overrightarrow{d^{(k-1)}} A \\ &= (a_k, b_k, c_k, d_k, e_k, f_k, g_k, h_k, i_k, j_k, k_k), \end{aligned}$$

where $I = \{a_k, b_k, c_k, d_k, e_k, f_k, g_k, h_k, i_k, j_k, l_k\}$ and

$$
\begin{aligned}
a_k &= \frac{1}{48}(15^{k-1} - 7^k + 7.3^{k-2} - 8) & b_k &= \frac{1}{48}(5.15^{k-1} + 7^k - 24.3^{k-2} - 4)\\
c_k &= \frac{1}{16}(3.15^{k-1} - 7^k + 4) & d_k &= \frac{1}{48}(5.15^{k-1} + 3.7^k + 66.3^{k-2})\\
e_k &= \frac{1}{24}(5.15^{k-1} + 7^k - 24.3^{k-2} - 4) & f_k &= \frac{1}{3}(15^{k-1} - 3^{k-1})\\
g_k &= \frac{1}{48}(5.15^{k-1} - 3.7^k + 84.3^{k-2} - 12 & h_k &= \frac{1}{24}(5.15^{k-1} - 7^k - 6.3^{k-2} + 4)\\
i_k &= \frac{1}{16}(3.15^{k-1} + 7^k + 2.3^k) & j_k &= \frac{1}{48}(5.15^{k-1} - 7^k + 30.3^{k-2} - 8)\\
k_k &= \frac{1}{48}(15^{k-1} + 7^k + 60.3^{k-2} + 20)
\end{aligned}
$$

Corollary 6 *We have that*

1. $\mathfrak{A}^{(k)}$ *is commutative if and only if* $k = 1$.
2. *The dimension of* $\mathfrak{A}^{(k)}$ *can be obtained by*

$$\dim \mathfrak{A}^{(k)} = \frac{1}{720}(15^{2k} + 15.7^{2k} + 100.3^{2k} + 300).$$

We apply the Corollary 6 for $k = 1, .., 5$. The following table is the result.

k	1	2	3	4	5
$\dim \mathfrak{A}^{(k)}$	3	132	18373	3680582	806796423

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