SIEGEL MODULAR FORMS AND SPECIAL POLYNOMIALS

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This is a resume of my talk at RIMS workshop. Although the title contains *Siegel modular forms*, it expresses just original motivation. Here we explain only a very simple part of results on relations with Siegel modular forms, and give mostly results on new special polynomials, since that part of the theory is interesting as itself. For applications to Siegel modular forms such as automorphic differential operators and special values of the standard L functions of Siegel modular forms, see reference [1], [7], [8] given at the end of this report.

This report is given in a survey style without any proofs. Almost no prerequisite is necessary. The contents of the claims are partly published in papers but some of them are still written only in preprints.

1. Classical theory of Gegenbauer polynomials

1.1. **Definition.** As a prototype of our theory, we first explain some well known classical theory. For a *generic* complex number λ, we define Gegenbauer polynomials $C_a^\lambda(t)$ of degree a by the following generating function

$$\frac{1}{(1-2tz+z^2)^\lambda} = \sum_{a=0}^{\infty} C_a^\lambda(t) z^a.$$

More explicitly, we have

$$C_a^\lambda(t) = \sum_{0\leq s\leq a/2} (-1)^s \binom{a-s+\lambda-1}{a-s}\binom{a-s}{s}(2t)^{a-2s}.$$

If we put $y = C_a^\lambda(t)$, then we have the Gegenbauer differential equation

$$(1-t^2)y'' - (2\lambda+1)ty' + a(a+2\lambda)y = 0.$$

We have the following orthogonality between these polynomials.

$$\int_{-1}^{1} C_a^\lambda(t)C_b^\lambda(t)(1-t^2)^{\lambda-1/2} = 0 \qquad \text{if } a \neq b.$$

1.2. **Interpretation of Gegenbauer polynomials.** For a so-called Riemannian symmetric pair (G,K) of Lie groups, a representation of G which has K-fixed vectors is called *class one*. For example, if we take $(G,K) = (SO(d), SO(d-1))$, we take the space $Harm_a^d$ of harmonic polynomials $P(x)$ in d variables $x \in \mathbb{R}^d$ of degree a. Then $SO(d)$ acts on $Harm_a^d$ irreducibly. This is called the spherical representation of degree

a. This has a $SO(d-1)$ fixed vector. We have $SO(d)/SO(d-1) \cong S^{d-1}$ (i.e. $d-1$ dimensional sphere), and a $SO(d-1)$ fixed polynomial is written as

$$P(x) = n(x)^{a/2} C_a^{(d-2)/2}\left(\frac{x_1}{\sqrt{n(x)}}\right),$$

where we put $x = (x_1, \ldots, x_d)$ and $n(x) = x_1^2 + \cdots + x_d^2$. Since $C_a(x)$ is of degree a and $C_a(-t) = (-1)^a C_a(t)$, the above P is a polynomial in x. The harmonicity of P is proved as follows. We put $t = x_1/\sqrt{n(x)}$. Then by the chain rule of derivatives, we can show

$$\sum_{i=1}^{d} \frac{\partial^2 P}{\partial x_i^2} =$$

$$n(x)^{a/2-1}\left((1-t^2)\frac{d^2 C_a}{dx^2}(t) - (d-1)t\frac{dC_a}{dx}(t) + a(d+a-2)C_a(t)\right).$$

This is 0 by the Gegenbauer differential equation. This is the usual explanation written in many standard books.

But we give another interpretation which seems more intrinsic (and of course also well known.) We consider polynomials $\widetilde{P}(x,y)$ in $2d$ variables where x, $y \in \mathbb{R}^d$. We assume that $\widetilde{P}(x,y)$ is harmonic for each x and y. We also assume that $\widetilde{P}(xh,yh) = \widetilde{P}(x,y)$ for any $h \in O(d)$, where $O(d)$ is the usual real orthogonal group of matrix size d. Then, if $d \geq 2$, by the fundamental theorem on invatiants, there exists a polynomial P in three variables $n(x) = (x,x)$, $n(y) = (y,y)$ and (x,y) such that

$$\widetilde{P}(x,y) = P\begin{pmatrix} n(x) & (x,y) \\ (x,y) & n(y) \end{pmatrix}.$$

When $\widetilde{P}(x,y)$ is of degree a for each x or y, up to constant we have

$$P\begin{pmatrix} n(x) & (x,y) \\ (x,y) & n(y) \end{pmatrix} = (n(x)n(y))^{a/2} C_a^{(d-2)/2}\left(\frac{(x,y)}{\sqrt{n(x)n(y)}}\right).$$

We can apply this polynomial to *automorphic* differential operators on Siegel modular forms. For a real symmetric matrix Y, we write $Y > 0$ if Y is positive definite. We denote by H_n the Siegel upper half space of degree n as follows.

$$H_n = \{Z = Z^t \in M_n(\mathbb{C}); Z = X + iY, X, Y \in M_n(\mathbb{R}), Y > 0\}$$

For $Z = \begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix} \in H_2$ and for the above P, we put

$$\mathbb{D} = P\begin{pmatrix} \dfrac{\partial}{\partial \tau_1} & \dfrac{1}{2}\dfrac{\partial}{\partial z} \\ \dfrac{1}{2}\dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial \tau_2} \end{pmatrix}.$$

Then for a Siegel modular form $F(Z)$ on H_2 of weight $k = d/2$, the function

$$(\mathbb{D}F)\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix}$$

is a modular form of weight $k+a$ for each τ_1 and τ_2.
Our first motivation was to ask what happens for more general F and domains. But apart from such motivations, it turns out that the theory of polynomials itself is very interesting and fruitful. We explain this in the rest of this report.

2. Simple generalization(Joint work with D. Zagier)

For the content of this section, see [9].
We assume that n is an integer such that $n \geq 2$. Consider polynomials $\widetilde{P}(x_1, \ldots, x_n)$ in $x_i \in \mathbb{R}^d$ satisfying the following conditions.

(1) $\widetilde{P}(x_1, \ldots, x_n)$ is harmonic for each x_i.

(2) For some $\mathbf{a} = (a_1, \ldots, a_n) \in (\mathbb{Z}_{\geq 0})^n$, the polynomial $\widetilde{P}$ is of degree a_i for the variable x_i for each i.

(3) We have $\widetilde{P}(x_1h, \ldots, x_nh) = \widetilde{P}(x_1, \ldots, x_n)$ for all $h \in O(d)$.

We consider an $n \times n$ symmetric matrix T of variable components t_{ij}. By Condition (3), we have a polynomial $P(T)$ in t_{ij} for $T = (t_{ij})$ such that $P((x_i, x_j)_{1\leq i,j\leq n}) = \widetilde{P}(x_1, \ldots, x_n)$ if $d \geq n$. As in the case of Gegenbauer polynomials, it is better to consider P instead of $\widetilde{P}$. We consider the coordinate change from (x_i) to T for the condition (1). We write the (mixed) Laplacians Δ_{ij} by the coordinate of T. For

$$\Delta_{ij} = \sum_{\nu=1}^{d} \frac{\partial^2}{\partial x_{i\nu}\partial x_{j\nu}},$$

the corresponding Laplacian in the coordinate of T is given by

$$D_{ij} = d\partial_{ij} + \sum_{k,l=1}^{n} t_{kl}\partial_{ik}\partial_{jl}$$

where we put $\partial_{ij} = (1+\delta_{ij})\frac{\partial}{\partial t_{ij}}$. So the condition (1) is written as $D_{ii}P = 0$ for $i = 1, \ldots, n$. The operators D_{ij} for $i \neq j$ play important roles later.

We denote by $\mathcal{P}_{\mathbf{a}}(d)$ the space of polynomials $P(T)$ such that $D_{ii}P(T) = 0$ and $P((c_ic_jt_{ij}) = (\prod_{i=1}^{n} c_i^{a_i})P(T)$. We call this a higher sphecial polynomial of multidegree $\mathbf{a} = (a_1, \ldots, a_n)$. We put $\mathcal{P}(d) = \oplus_{\mathbf{a}}\mathcal{P}_{\mathbf{a}}(d)$.

For any P, $Q \in \mathcal{P}(d)$, we have an inner product

$$(P, Q)_d = c_n(d)\int_{T>0} P(T)\overline{Q(T)}\det(T)^{(d-n-1)/2}dT$$

where $dT = \prod_{i\le j} dt_{ij}$ and

$$c_n(d) = 2^{-nd/2}\pi^{-n(n-1)/4}\prod_{i=1}^{n-1}\Gamma\left(\frac{d-i}{2}\right)^{-1}.$$

Here $c_n(d)$ is adjusted so that $(1,1)_d = 1$. This integral converges for $d > n-1$ but is actually meromorphically continued for whole $d \in \mathbb{C}$ and for most of d, this is holomorphic. (We omit the details.)

If $\mathbf{a} \neq \mathbf{b}$ and $P \in \mathcal{P}_{\mathbf{a}}(d)$ and $Q \in \mathcal{P}_{\mathbf{b}}(d)$, then we have

$$(P,Q)_d = 0.$$

For dimensions of $\mathcal{P}_{\mathbf{a}}(d)$, we have the following results. When $n = 2$ and d is generic, we have

$$\dim \mathcal{P}_{\mathbf{a}}(d) = \begin{cases} 1 & \text{if } a_1 = a_2, \\ 0 & \text{otherwise.} \end{cases}$$

When $n = 3$ and d is generic, we have

$$\dim \mathcal{P}_{\mathbf{a}}(d) = \begin{cases} 1 & \text{if } a_i \le a_j + a_k \text{ for all } \{i,j,k\} = \{1,2,3\}, \\ 0 & \text{otherwise.} \end{cases}$$

Here the word *generic* can be explicitly defined, but omitted here.

When $n \ge 4$, we have $\dim \mathcal{P}_{\mathbf{a}}(d) \ge 2$ in general. More precisely, the generating function of the dimensions can be given by

$$\sum_{i=1}^{n}\sum_{a_i=0}^{\infty}(\dim \mathcal{P}_{\mathbf{a}}(d)) z_1^{a_1}\cdots z_n^{a_n} = \prod_{1\le i<j\le n}\frac{1}{1-z_iz_j}.$$

The above dimension formula means that for $n \le 3$, then non-zero polynomials of $\mathcal{P}_{\mathbf{a}}(d)$ is automatically an orthogonal basis of $\mathcal{P}(d)$ for $n = 2$ and 3. For $n \ge 4$, we do not know if there are any natural orthogonal basis of $\mathcal{P}_{\mathbf{a}}(d)$ for all $\mathbf{a}$ since the dimension is not one. Actually, by experiments, we see that it seems there are no natural otrhogonal basis. So we have the following natural question.

Problem: Is there any natural basis of $\mathcal{P}(d)$ and $\mathcal{P}_{\mathbf{a}}(d)$?

Answer: There are two canonical bases of $\mathcal{P}(d)$ dual to each other, though they are not orthogonal bases for $n \ge 4$.

2.1. **Monomial basis.** For $\mathbf{a} \in \mathbb{Z}_{\ge 0}$, we put

$$N_0(\mathbf{a}) = \{\boldsymbol{\nu} = \boldsymbol{\nu}^t = (\nu_{ij}) \in M_n(\mathbb{Z}); \nu_{ii} = 0, \nu_{ij} \ge 0, \boldsymbol{\nu}\cdot\mathbf{1} = \mathbf{a}\},$$

where $\mathbf{1} = (1,1,\ldots,1)^t$. We write $N_0 = \bigcup_{\mathbf{a}} N_0(\mathbf{a})$. By some abstract argument, we can show that

$$\dim \mathcal{P}_{\mathbf{a}}(d) = \#(N_0(\mathbf{a})).$$

So it is natural to expect that there is a basis of $\mathcal{P}_{\mathbf{a}}(d)$ indexed by $N_0(\mathbf{a})$. We denote by $\mathbb{C}[T]$ the polynomial ring over $\mathbb{C}$ of all the variables t_{ij} for $T = (t_{ij})$.

Theorem 2.1. *Assume that d is a complex number such that $d \notin \mathbb{Z}_{\leq 0}$. Then for each $\boldsymbol{\nu} = (\nu_{ij}) \in N_0(\mathbf{a})$, there exists the unique polynomial $P^M_{\boldsymbol{\nu}}(T) \in \mathcal{P}_{\mathbf{a}}(d)$ such that*

$$P^M_{\boldsymbol{\nu}}(T) = T^{\boldsymbol{\nu}} + Q(T)$$

where $T^{\boldsymbol{\nu}} = \prod_{i<j} t_{ij}^{\nu_{ij}}$ and $Q(T) \in (t_{11}, t_{22}, \ldots, t_{nn})\mathbb{C}[T]$ (i.e. those taking value 0 under the restriction to $t_{11} = t_{22} = \cdots = t_{nn} = 0$). These polynomials $P^M_{\boldsymbol{\nu}}(T)$ for $\boldsymbol{\nu} \in N_0$ are a basis of $\mathcal{P}(d)$.

We can construct $P^M_{\boldsymbol{\nu}}(T)$ explicitly in the following way. For a vector $v = (v_1, \ldots, v_n) \in (\mathbb{Z}_{\geq 0})^n$, we put

$$\delta(T)^v = \prod_{i=1}^n t_{ii}^{v_i}.$$

For any $P \in \mathcal{P}_{\mathbf{a}}(d)$, we define an operator $R_{ij}(\mathbf{a})$ by

$$R_{ij}(\mathbf{a})P = \delta(T)^{\mathbf{a}+e_i+e_j-(2-d)\mathbf{1}/2} D_{ij} \delta(T)^{(2-d)\mathbf{1}/2-\mathbf{a}} P(T).$$

Here e_k is the unit vector whose k component is 1 and the other components are 0. We can define R_{ij} as an element of an algebra of operators on $\mathbb{C}[T]$ which gives $R_{ij}(\mathbf{a})$ on polynomials of multidegree $\mathbf{a}$. This R_{ij} is independent of $\mathbf{a}$ and maps $\mathcal{P}_{\mathbf{a}}(d)$ to $\mathcal{P}_{\mathbf{a}+e_i+e_j}(d)$. Here we can show that the actions of R_{ij} are commutative for all i, j on $\mathcal{P}(d)$. Because of this, for any $\boldsymbol{\nu} = (\nu_{ij}) \in N_0$, we may write

$$\mathbf{R}^{\boldsymbol{\nu}} = \prod_{i<j} R_{ij}^{\nu_{ij}}.$$

Proposition 2.2. *We have*

$$P^M_{\boldsymbol{\nu}}(T) = \frac{1}{\epsilon_{2\mathbf{a}}(d-2)} \mathbf{R}^{\boldsymbol{\nu}}(1),$$

where 1 is the constant function taking the value 1 and

$$\epsilon_{2\mathbf{a}}(d-2) = \prod_{i=1}^n d(d+2)\cdots(d+2a_i-2).$$

2.2. **Descending basis.** We explain another canonical basis. Firstly it is obvious that D_{ij} are commutative for any i, j since they came from Δ_{ij} originally. In particular $D_{kk}D_{ij} = D_{ij}D_{kk}$ for $1 \leq i \neq j \leq n$ and $1 \leq k \leq n$, so we have

$$D_{ij}\mathcal{P}_{\mathbf{a}}(d) \subset \mathcal{P}_{\mathbf{a}-e_i-e_j}(d).$$

Theorem 2.3. *For generic d, there exists a set of non-zero polynomials $P^D_\nu(T) \in \mathcal{P}(d)$ indexed by $\boldsymbol{\nu} \in N_0$ such that*

(1) $P^D_{\mathbf{0}}(T) = 1$,

(2) $D_{ij}P^D_\nu(T) = P^D_{\nu-\mathbf{e_{ij}}-\mathbf{e_{ji}}}(T)$,

where $\mathbf{0}$ is the zero matrix and $\mathbf{e_{kl}}$ is the $n \times n$ matrix whose (k,l) components is one and all the other components are zero. Here we *regard $P^D_{\boldsymbol{\nu}}(T) = 0$ if any component of $\boldsymbol{\nu}$ is negative. Besides, the set $\{P^D_\nu(T); \boldsymbol{\nu} \in N_0\}$ is a basis of $\mathcal{P}(d)$ dual to the monomial basis $\{P^M_\nu(T); \boldsymbol{\nu} \in N_0\}$, that is, we have*

$$(P^D_{\boldsymbol{\nu}}(T), P^M_{\boldsymbol{\mu}}(T)) = \delta_{\boldsymbol{\nu}\boldsymbol{\mu}},$$

where $\delta_{\boldsymbol{\nu}\boldsymbol{\mu}}$ is the Kronecker delta.

We call $\{P^D_{\boldsymbol{\nu}}(T)\}$ a descending basis. We will explain a construction of $P^D_\nu(T)$ below. The proof of this construction is very long and not easy at all. Of course we have good technical reasons that we can prove this, but the process itself is still very mysterious for us, and we do not know any intrinsic reason why this works.

We consider an $n \times n$ symmetric matrix $X = (x_{ij}) = X^t$ consisting of dummy variables x_{ij}, where we assume $x_{11} = x_{22} = \cdots = x_{nn} = 0$. We introduce new variables σ_i $(i = 1, \ldots, n)$ defined by

$$\det(\lambda 1_n - TX) = \sum_{i=0}^{n} (-1)^i \sigma_i(T, X) \lambda^{n-i}.$$

Here $\sigma_i(T, X)$ are of course polynomials in the coordinates of T and X, but we regard σ_i themselves as new variables for a while. For any $\boldsymbol{\nu} = (\nu_{ij}) \in N_0$, we put

$$X^{\boldsymbol{\nu}} = \prod_{i<j} x_{ij}^{\nu_{ij}}.$$

Our aim is to describe a generating function of $P^D_{\boldsymbol{\nu}}(T)$. In other words, we want to write down a series $G^{(n)}(T, X)$ such that

$$G^{(n)}(T, X) = \sum_{\boldsymbol{\nu} \in N_0} P_{\boldsymbol{\nu}}(T) X^{\boldsymbol{\nu}}$$

where each $P_{\boldsymbol{\nu}}(T)$ is non-zero and proportional to $P^D_{\boldsymbol{\nu}}(T)$. This series can be explicitly obtained in the following way.

For any $i \in \mathbb{C}$ such that $i \notin \{-1, -2, \ldots, \}$, we put

$$J_i(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!(i+1)_r} = 1 + \frac{x}{i+1} + \frac{x^2}{2(i+1)(i+2)} + \cdots$$

For any integer $k \geq 2$, we put

$$\mathcal{M}_k = \sum_{\substack{0<a,b<k \\ k \leq a+b}} \sigma_{a+b-k} \partial_a \partial_b$$

where we put $\partial_a = \frac{\partial}{\partial \sigma_a}$.

We put

$$G^{(1)}(\sigma_1) = \left(1 - \frac{\sigma_1}{2}\right)^{2-d}$$

and for $d > n-1$, we define $G^{(n)}$ by

$$G^{(n)}(\sigma_1, \dots, \sigma_n) = J_{(d-n-1)/2}(\sigma_n \mathcal{M}_n) J_{(d-n)/2}(\sigma_{n-1}\mathcal{M}_{n-1}) \\ \cdots\cdots J_{(d-3)/2}(\sigma_2 \mathcal{M}_2)(G^{(1)}(\sigma_1))$$

Writing σ_i by X and T in the final stage, this is a formal power series in components of X with polynomial coefficients in t_{ij}, and this gives the generating function we want. Since coefficients of polynomials $P_{\boldsymbol{\nu}}(T)$ are rational functions of d, the restriction $d > n-1$ is replaced by much weaker condition, but we omit the details here.

Example. When $n = 2$, we have

$$G^{(2)}(\sigma_1, \sigma_2) = \frac{1}{(1-\sigma_1/2)^2 - \sigma_2)^{(d-2)/2}}.$$

Here $\sigma_1 = 2t_{12}x_{12}$, $\sigma_2 = -(t_{11}t_{22} - t_{12}^2)x_{12}^2$, so we have

$$G^{(2)} = \frac{1}{(1 - 2t_{12}x_{12} + t_{11}t_{22}x_{12}^2)^{(d-2)/2}}$$

Here in the coordinate of $(x, y) \in (\mathbb{R}^d)^2$, we have $t_{12} = (x, y)$, $t_{11} = n(x)$, $t_{22} = n(y)$. So this is nothing but the usual generating function of (homogenous) Gegenbauer polynomials.

When $n = 3$, we have

$$G^{(3)}(\sigma_1, \sigma_2, \sigma_3) = \frac{1}{\sqrt{\Delta_0^2 - 8\sigma_3}} \left(\frac{\Delta_0 + \sqrt{\Delta_0^2 - 8\sigma_3}}{2} \right)^{-(d-4)/2},$$

where $\Delta_0 = (1 - \sigma_1/2)^2 - \sigma_2$. This is a new generating function.

For $n \geq 4$, the series $G^{(n)}$ is not algebraic in general. But for example, when $n = d = 4$, we have

$$G^{(4)} = \sqrt{\frac{\Delta_1 + \sqrt{\Delta_2}}{2\Delta_2}}$$

where

$$\Delta_1 = \Delta_0^2 - 8\sigma_3 + 4\sigma_4$$
$$\Delta_2 = \Delta_1^2 - 16\sigma_4(4 - 2\sigma_1 - \Delta_0)^2.$$

When $n = 4$ and d is an even integer such that $d \geq 4$, then we can show that $G^{(n)}$ is always an algebraic generating function. For other d, this is not algebraic.

One interesting point of this description is that the expression of $G^{(n)}$ has a recursive structure on n, in spite of the fact that for different n, the meanings of σ_i are different. Practically, we can continue calculation of the generating furnctions recursively starting from $n = 1$ to general n as far as we want, and we can calculate the descending basis $P_{\boldsymbol{\nu}}^D(T)$ for small $\mathbf{a}$ by replacing σ_i by polynomials in T and X by definition and seeing the coefficient of $X^{\boldsymbol{\nu}}$.

3. Most general case

For this section, see [5].

3.1. **Two bases.** We fix two natural integers $n \geq 2$ and $r \geq 2$. We say that a polynomial $\widetilde{P}(Y)$ in components of an $n \times d$ matrix $Y = (y_{i\nu})$ is pluri-harmonic if

$$\Delta_{ij} P = 0 \qquad \text{for all } \Delta_{ij} = \sum_{\nu=1}^{d} \frac{\partial^2}{\partial y_{i\nu} \partial y_{j\nu}}.$$

Throughout section 3, we fix an ordered partition $\mathbf{n} = (n_1, n_2, \ldots, n_r)$ of n with $n = n_1 + \cdots + n_r$ and $n_i \geq 1$. (We changed the notation from $x_i \in \mathbb{R}^d$ to y_{ij} since x woud be confusing with the matrix X of dummy variables.) We take an $n \times d$ matrix Y of variable components and write

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_r \end{pmatrix},$$

where $Y_i \in M_{n_i,d}$. We denote by $GL(n_i)$ the general linear group of matrix size n_i over $\mathbb{C}$. We put

$$GL(\mathbf{n}) = GL(n_1) \times \cdots \times GL(n_r).$$

For each fixed irreducible polynomial representation (ρ, V) of $GL(\mathbf{n})$, we consider following V-valued polynomials $\widetilde{P}(Y)$.

(1) We have

$$\widetilde{P} \begin{pmatrix} A_1 Y_1 \\ \vdots \\ A_r Y_r \end{pmatrix} = \rho(A) \widetilde{P}(Y),$$

where $A = (A_1, \ldots, A_r) \in GL(\mathbf{n}) = GL(n_1) \times \cdots \times GL(n_r)$.

(2) Each component of $\widetilde{P}(Y)$ is pluri-harmonic for each Y_i.

(3) $\widetilde{P}(Yh) = \widetilde{P}(Y)$ for all $h \in O(d)$.

As before, by condition (3), for $d \geq n$, we have a V-valued polynomial $P(T)$ in t_{ij} for $\widetilde{P}(Y)$ such that $\widetilde{P}(Y) = P(YY^t)$. We will interpret the condition (1) and (2) by a condition on T. We write T by matrix blocks as $T = (T_{pq})_{1 \leq p,q \leq r}$ where T_{pq} are $n_p \times n_q$ matrices. We put

$$I(\mathbf{n}) = \{(k,l); n_1 + \cdots + n_{i-1} + 1 \leq k, l \leq n_1 + \cdots + n_i \text{ for some } 1 \leq i \leq r\}.$$

In other words, we have $(k,l) \in I(\mathbf{n})$ if and only if t_{kl} is a component of some diagonal block $T_{pp} = Y_p Y_p^t$. In T variable, the conditions (1) and (2) means

(3) $D_{ij} P(T) = 0$ for all $(i,j) \in I(\mathbf{n})$.

(4) $P(ATA^t) = \rho(A)P(T)$, where

$$A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_r \end{pmatrix}.$$

Now we regard these conditions (3) and (4) as conditions on general V-valued polynomials $P(T)$, forgetting Y. So now d can be any complex number. We denote the vector space of V-valued polynomials in components of T satisfying (3) and (4) by $\mathcal{P}^{\mathbf{n}}_{\rho}(d)$. From this space we can construct automorphic differential operators on Siegel modular forms. That is, for $P(T) \in \mathcal{P}^{\mathbf{n}}_{\rho}(d)$, put

$$\mathbb{D} = P\left(\frac{\partial}{\partial Z}\right) \qquad \text{where } \frac{\partial}{\partial Z} = \left(\frac{1+\delta_{ij}}{2}\frac{\partial}{\partial z_{ij}}\right).$$

We write $Z \in H$ by block matrices as $Z = (Z_{pq})$ in the same way as T and write $Z_p = Z_{pp}$. If F is a Siegel modular forms of degree n of weight $k = d/2$, then the function

$$(\mathbb{D}F)\begin{pmatrix} Z_1 & 0 & \cdots & 0 \\ 0 & Z_2 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & Z_r \end{pmatrix}$$

is a Siegel modular form of weight $det^k \otimes \rho_p$ for each Z_p, where $\rho = \otimes_{p=1}^r \rho_p$ for irreducible representations ρ_p of $GL(n_i)$.

But before considering the representation ρ, it is useful to define the following space of scalar valued polynomials.

$$\mathcal{P}^{\mathbf{n}}(d) = \{P(T) \in \mathbb{C}[T]; D_{ij}P = 0 \text{ for all } (i,j) \in I(\mathbf{n})\}.$$

We embed $GL(\mathbf{n})$ to the space of $n \times n$ matrices by mapping each component of $A = (A_1, \ldots, A_r) \in GL(\mathbf{n})$ to the diagonal blocks of an $n \times n$ matrix as in the condition (4).

Proposition 3.1. *The space $\mathcal{P}^{\mathbf{n}}(d)$ is invariant by the action of $GL(\mathbf{n})$ given by $P(T) \to P(AT\,{}^tA)$ for $A \in GL(\mathbf{n})$.*

So it is reasonable to see the space $\mathcal{P}^{\mathbf{n}}(d)$ first and consider the irreducible decomposition of $GL(\mathbf{n})$ later.

3.2. **Two canonical bases for partition n.** We put

$\mathcal{P}^{\mathbf{n}}_{\mathbf{a}}(d) = \{$polynomials in $\mathcal{P}^{\mathbf{n}}(d)$ of multidegree $\mathbf{a}\} = \mathcal{P}^{\mathbf{n}}(d) \cap \mathcal{P}_{\mathbf{a}}(d)$.

$N^{\mathbf{n}}_0(\mathbf{a}) = \{\nu \in N_0(\mathbf{a}); \nu_{ij} = 0$ for all $(i,j) \in I(\mathbf{n})\}$.

We put $N^{\mathbf{n}}_0 = \cup_{\mathbf{a}} N^{\mathbf{n}}_0(\mathbf{a})$.

Then for generic d, we can prove that

$$\dim \mathcal{P}^{\mathbf{n}}_{\mathbf{a}}(d) = \#(N^{\mathbf{n}}_0(\mathbf{a})).$$

We write block matrices of T as before by

$$T = (T_{pq}),$$

where T_{pq} are $n_p \times n_q$ matrices.

Theorem 3.2. *(Monomial basis). Assume that $d \in \mathbb{R}$ and $d > n-1$. For each $\boldsymbol{\nu} \in N_0^{\mathbf{n}}$, there exists the unique polynomial $P_{\boldsymbol{\nu}}^{M,\mathbf{n}}(T) \in \mathcal{P}^{\mathbf{n}}$ such that*

$$P_\nu^{M,\mathbf{n}}(T) = T^{\boldsymbol{\nu}} + Q(T)$$

for some polynomial $Q(T)$ with $Q(T)|_{T_{11}=T_{22}=\cdots=T_{rr}} = 0$, where we write

$$T^{\boldsymbol{\nu}} = \prod_{i<j} t_{ij}^{\nu_{ij}}.$$

The set of $\{P_\nu^{M,\mathbf{n}}(T); \boldsymbol{\nu} \in N_0\}$ is a basis of $\mathcal{P}^{\mathbf{n}}(d)$.

We call these polynomials *monomial basis for the partition* $\mathbf{n}$. The notion of monomial basis depends on the partition $\mathbf{n}$. Although $\mathcal{P}^{\mathbf{n}}(d) \subset \mathcal{P}(d)$, we note that $\{P_{\boldsymbol{\nu}}^{M,\mathbf{n}}(T)\}$ is not a part of monomial basis $\{P_{\boldsymbol{\nu}}^{M}(T)\}$ in the sense of section 2.

Theorem 3.3 (Descending basis)**.** *For an $n \times n$ symmetric matrix X, write $X = (X_{pq})$ by matrix blocks where X_{pq} are $n_p \times n_q$ matrices. Defining $\sigma_i = \sigma_i(T, X)$ as before, we put*

$$G^{\mathbf{n}}(\sigma_1, \ldots, \sigma_n) = G^{(n)}(\sigma_1, \ldots, \sigma_n)|_{X_{11}=X_{22}=\cdots=X_{rr}=0}$$

and write

$$G^{\mathbf{n}}(\sigma_1, \ldots, \sigma_r) = \sum_{\nu \in N_0^{\mathbf{n}}} P_{\boldsymbol{\nu}}^{\mathbf{n}}(T) X^{\boldsymbol{\nu}}.$$

Then the set of $P_{\boldsymbol{\nu}}^{\mathbf{n}}(T)$ is a basis of $\mathcal{P}^{\mathbf{n}}(d)$.

3.3. **Construction.** First we give a concrete construction of the space $\mathcal{P}_\rho(d)$ by using the monomial basis. We fix a multidegree $\mathbf{a}$. We put

$$\begin{aligned} W_1 = {} & \text{the subspace of } \mathbb{C}[T] \text{ spanned by polynomials} \\ & \text{in the set } \{P_\nu^M(T); \boldsymbol{\nu} \in N_0(\mathbf{a}), \boldsymbol{\nu} \notin N_0(\mathbf{a})^{\mathbf{n}}\}. \end{aligned}$$

Proposition 3.4. *We have*

$$\mathcal{P}_{\mathbf{a}}^{\mathbf{n}}(d) = W_1^{\perp}.$$

By using this, we can explicitly write down $P_\nu^{M,\mathbf{n}}(T)$ in the following way. Any polynomial in $\mathcal{P}(d)$ is a linear combination of the monomial basis $P_{\boldsymbol{\nu}}^M(T)$ in the sense of section 2. By the fact that $\mathcal{P}^{\mathbf{n}}(d) \subset \mathcal{P}(d)$ and by definition, the monomial basis $\mathcal{P}_{\boldsymbol{\nu}}^{M,\mathbf{n}}$ for a partition $\mathbf{n}$ for $\boldsymbol{\nu} \in N_0^{\mathbf{n}}(\mathbf{a})$ can be written as

$$P_{\boldsymbol{\nu}}^{M,\mathbf{n}}(T) = P_{\boldsymbol{\nu}}^M(T) + \sum_{\substack{\boldsymbol{\mu} \in N_0(\mathbf{a}) \\ \boldsymbol{\mu} \notin N_0^{\mathbf{n}}(\mathbf{a})}} c_{\boldsymbol{\mu}} P_{\boldsymbol{\mu}}^M(T)$$

for some constants $c_{\boldsymbol{\mu}} \in \mathbb{C}$. So the problem is how to write down $c_{\boldsymbol{\mu}}$.

We put $c_{\boldsymbol{\mu},\boldsymbol{\kappa}} = (P^M_{\boldsymbol{\mu}}(T), P^M_{\boldsymbol{\kappa}}(T))$. Then by the previous Proposition, the coefficients $c_{\boldsymbol{\mu}}$ are solutions of the following simultaneous equation.

$$c_{\boldsymbol{\nu}\boldsymbol{\kappa}} + \sum_{\substack{\boldsymbol{\mu}\in N_0(\mathbf{a}) \\ \boldsymbol{\mu}\notin N_0^{\mathbf{n}}(\mathbf{a})}} c_{\boldsymbol{\mu}} c_{\boldsymbol{\mu},\boldsymbol{\kappa}} = 0.$$

But before solving this, we need concrete values of $c_{\boldsymbol{\mu},\boldsymbol{\kappa}}$. We explain how to obtain these values. Any polynomial $P(T) \in \mathcal{P}(d)$ is written as a linear combination of the descending basis $P^D_{\boldsymbol{\nu}}(T)$ of $\mathcal{P}(d)$. So write

$$P^M_{\boldsymbol{\mu}}(T) = \sum_{\boldsymbol{\nu}\in N_0} d_{\boldsymbol{\nu}} P^D_{\boldsymbol{\nu}}(T)$$

for some constants $d_{\boldsymbol{\nu}} \in \mathbb{C}$. Since $P^M_{\boldsymbol{\nu}}(T)$ and $P^D_{\boldsymbol{\kappa}}(T)$ are dual with each other, we have

$$c_{\boldsymbol{\mu},\boldsymbol{\kappa}} = d_{\boldsymbol{\kappa}}(P^D_{\boldsymbol{\kappa}}(T), P^M_{\boldsymbol{\kappa}}(T)) = d_{\boldsymbol{\kappa}}.$$

But by the relation $D_{ij}P^D_{\boldsymbol{\nu}}(T) = P^D_{\boldsymbol{\nu}-e_{ij}-e_{ji}}(T)$ and $P^D_{\mathbf{0}}(T) = 1$, we have

$$\left(\prod_{i<j} D_{ij}^{\kappa_{ij}}\right) P^M_{\boldsymbol{\mu}}(T) = \sum_{\boldsymbol{\nu}\in N_0} d_{\boldsymbol{\nu}} \left(\prod_{i<j} D_{ij}^{\kappa_{ij}}\right) P^D_{\boldsymbol{\nu}}(T) = d_{\boldsymbol{\kappa}} \in \mathbb{C}.$$

We already explained how to calculate $P^M_{\boldsymbol{\mu}}(T)$ in section 2, starting from constant 1 by simple differential operators. So by clear concrete algorithm, solving linear equations, we can calculate $P^{M,\mathbf{n}}_{\nu}(T)$. (By the way, Theorem 3.2 assures that the linear equation above can be solved uniquely under the assumption that $d > n-1$.)

3.4. **Algorithm to calculate $\mathcal{P}_\rho(d)$.** Next we explain how to calculate basis of $P^{\mathbf{n}}_{\rho}(d)$ by using monomial basis for a partition $\mathbf{n}$. It is well known that for $i \neq j$, the representations ρ_{ij} of $GL(n_i) \times GL(n_j)$ on the polynomial rings $\mathbb{C}[T_{ij}]$ in components of $n_i \times n_j$ matrix T_{ij} giving by $P(T_{ij}) \to P(A_i T_{ij} A_j^t)$ is decomposed to

$$\rho_{ij} = \bigoplus_{depth(\lambda_{ij}) \leq min(n_i,n_j)} \rho_{\lambda_{ij},n_i} \otimes \rho_{\lambda_{ij},n_j},$$

where λ_{ij} runs over dominant integral weights (or equivalently the Young diagrams) and ρ_{λ_{ij},n_i} is the irreducible representation of $GL(n_i)$ corresponding to λ_{ij}. Here ρ_{λ_{ij},n_i} and ρ_{λ_{ij},n_j} correspond to the same λ_{ij} but the groups are different. We note that

$$\mathbb{C}[T] = \otimes_{i<j}\mathbb{C}[T_{ij}].$$

Then the natural representation $\widetilde{\rho}$ of $GL(\mathbf{n})$ on $\mathbb{C}[T]$ is the tensor product representations of ρ_{ij}, so the restriction to $GL(n_i)$ of ρ on $\mathbb{C}[T]$ is a subspace in the sum of tensor products $\otimes_{j\neq i}\rho_{\lambda_{ij},n_i}$ of representations of $GL(n_i)$ for various λ_{ij}. We note that $\otimes_{j\neq i}\rho_{\lambda_{ij},n_i}$ is not irreducible at all in general.

Anyway, we fix an irreducible representation ρ of $GL(\mathbf{n})$ realized on $\mathbb{C}[T]$ as a part of the above representation. Multiplicity of ρ is not one in general. We put $d_\rho = \dim \rho$ and take a vector of polynomials

$$F(T) = \begin{pmatrix} f_1(T) \\ \vdots \\ f_{d_\rho}(T) \end{pmatrix}$$

with $f_i(T) \in$
$C[T]$ such that

$$F(ATA^t) = \rho(A)F(T).$$

It seems that there are no natural standard way to write down the above F, but it would be usually possible to write this down for any concretely given ρ.

Theorem 3.5. *Assumption and notation being as above, for*

$$f_i(T) = \sum_{\nu \in N_0^{\mathbf{n}}} c_{i\boldsymbol{\nu}} T^{\boldsymbol{\nu}},$$

we replace monomials $T^{\boldsymbol{\nu}}$ by the corresponding monomial basis $P^M_{\boldsymbol{\nu}}(T)$ and put

$$P_i(T) = \sum_{\nu \in N_0^{\mathbf{n}}} c_{i\nu} P_\nu^{M,\mathbf{n}}(T)$$

and

$$P(T) = \begin{pmatrix} P_1(T) \\ \vdots \\ P_{d_\rho}(T) \end{pmatrix}.$$

Then $P(T) \in P_\rho^{\mathbf{n}}(d)$. All the elements of $\mathcal{P}_\rho^{\mathbf{n}}(d)$ are obtained in this way.

If the multiplicity of ρ is one, then the above $P(T)$ is unique up to constant, but in general, there are several linearly independent $P(T)$ corresponding to the number of multiplicity.

Next we consider a theoretical characterization of our polynomials using the descending basis and the generating series. We write a block decomposition of X as $X = X^t = (X_{pq})$ where X_{pq} is an $n_p \times n_q$ matrix and we assume that $X_{pp} = 0$ for $p = 1, \ldots, r$. We regard the polynomial ring $\mathbb{C}[X]$ in components of X as $GL(\mathbf{n})$ module by the action $P(X) \to P(A^t XA)$ for $A \in GL(\mathbf{n})$.

Theorem 3.6. *We have*

$$\mathrm{Hom}_{GL(n_1)\times\cdots\times GL(n_r)}(\mathbb{C}[X], V) \cong \mathcal{P}_\rho^{\mathbf{n}}(d),$$

where the map is given by $c \to c(G^{(\mathbf{n})}(X,T))$.

In this sense, our generating series $G^{(\mathbf{n})}(X,T)$ is *universal.* If we write

$$\mathbb{D}_U = G^{(n)}(X, \frac{\partial}{\partial Z})$$

and denote by $\mathbb{C}[[X]]$ the ring of formal power series in components of X, then this is a $\mathbb{C}[[X]]$ valued differential operators which preserves automorphy for the restriction to diagonal blocks $(Z_1,\ldots,Z_r)$, changing automorphy factor from $det^{d/2}$ to $det^{d/2}\otimes\tilde{\rho}$. So it would be natural to called this the universal automorphic differential operator.

4. Further remarks

(1) There exists a general formula for polynomials in $\mathcal{P}_\rho(d)$ in the case $n_1 = n_2 = m$, $n = 2m$ written in one-line. (see [6]).

(2) There exists a theory of holonomic systems in two cases,
(i) The case when $n_1 = n_2 = m$, $n = 2m$ (see [3]),
(ii) The case when $n = 3$ and $n_1 = n_2 = n_3 = 1$ ([10]). In the latter case, we have a theory of non-polynomial solutions. We have some candidate of holonomic system for general n with all $n_i = 1$.
(3) The case $r = 1$ is not included in the above consideration. When $r = 1$, numbers d and representations ρ such that $\mathcal{P}_\rho(d) \neq 0$ is very restricted. As for the space of polynomials $\mathcal{P}_\rho(d)$ in our setting in this report, we know which d and ρ appears and how to write them. But our problem in this case is that we have no proof at moment that this really gives differential operators on Siegel modular forms. The proof of this open problem should be quite different from the case $r \geq 2$.

References

[1] N. Dummigan, T. Ibukiyama and H. Katsurada, Some Siegel modular standard L-values, and Shafarevich-Tate groups, J. Number Theory **131** (2011), 1296–1330.
[2] T. Ibukiyama, On differential operators on automorphic forms and invariant pluri-harmonic polynomials, Comment. Math. Univ. St. Pauli, **48** (1999), 103–118.
[3] T. Ibukiyama, T. Kuzumaki and H. Ochiai, Holonomic systems of Gegenbauer type polynomials of matrix arguments related with Siegel modular forms, J. Math. Soc. Japan **64**(2012), 273–316.
[4] T. Ibukiyama, Composition of automorphic differential operators on Siegel modular forms, Comment. Math. Univ. St. Pauli. **65** (2016), 97–109.
[5] T. Ibukiyama, Universal automorphic differential operators on Siegel modular forms, Preprint.
[6] T. Ibukiyama, One-line formula for automorphic differential operators on Siegel modular forms, preprint.
[7] T. Ibukiyama and H. Katsurada, Exact critical values of the symmetric fourth L functions and vector valued Siegel modular forms, J. Math. Soc. Japan **66**(2014), 139–160.
[8] T. Ibukiyama, H. Katsurada, C. Poor and D. Yuen, Congruences to Ikeda-Miyawaki lifts and triple L-values of elliptic modular forms, J. Number Theory **134**(2014), 142–180

[9] T. Ibukiyama and D. Zagier, Higher Spherical polynomials, Max Planck Institute for mathematics Preprint series, 2014-41.
[10] T. Ibukiyama and D. Zagier, Higher spherical polynomials and higher spherical functions, in preparation.

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