The effects of discretization on Hyers–Ulam stability

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1 Introduction: Summary of the Continuous Case

Let $\varepsilon > 0$ be given. Consider the simple differential equation

$$x'(t) = \lambda x(t), \quad \lambda \in \mathbb{C}, \quad t \in \mathbb{R}, \tag{1.1}$$

and the related perturbed equation

$$\phi'(t) = \lambda \phi(t) + q(t), \quad |q(t)| \le \varepsilon, \quad t \in \mathbb{R}. \tag{1.2}$$

If such a function ϕ as in (1.2) exists, is there a solution x of (1.1) and a constant $K > 0$ such that

$$|\phi(t) - x(t)| \le K\varepsilon, \quad \forall t \in \mathbb{R}?$$

If so, then (1.1) is said to be Hyers–Ulam stable (HUS), with HUS constant K. For what value(s) of $\lambda \in \mathbb{C}$ does (1.1) have HUS? If there a best (minimal) constant K in that case?

Let $\mathrm{Re}(\lambda) > 0$, and set

$$\phi(t) = \phi_0 e^{\lambda t} + e^{\lambda t} \int_0^t q(s) e^{-\lambda s} ds, \quad |q(s)| \le \varepsilon. \tag{1.3}$$

Then ϕ satisfies the perturbed equation (1.2), and

$$\int_0^\infty q(s) e^{-\lambda s} ds$$

exists. Note that

$$x(t) = x_0 e^{\lambda t}, \qquad x_0 = \phi_0 + \int_0^\infty q(s) e^{-\lambda s} ds$$

is a well-defined solution of (1.1), and

$$|\phi(t) - x(t)| = e^{\mathrm{Re}(\lambda) t} \left| -\int_t^\infty q(s) e^{-\lambda s} ds \right| \le \frac{\varepsilon}{\mathrm{Re}(\lambda)}.$$

Therefore (1.1) has HUS for $\mathrm{Re}(\lambda) > 0$, with HUS constant $\frac{1}{\mathrm{Re}(\lambda)}$. Continuing with $\mathrm{Re}(\lambda) > 0$, let

$$\phi(t) = \frac{\varepsilon}{\mathrm{Re}(\lambda)} e^{\lambda t} - \frac{\varepsilon}{\mathrm{Re}(\lambda)} e^{i\,\mathrm{Im}(\lambda) t}.$$

Then

$$|\phi'(t) - \lambda\phi(t)| = \varepsilon|e^{i\,\mathrm{Im}(\lambda)t}| = \varepsilon,$$

so ϕ satisfies (1.2). Since $x(t) = \frac{\varepsilon}{\mathrm{Re}(\lambda)}e^{\lambda t}$ solves (1.1), we have

$$|\phi(t) - x(t)| = \left|-\frac{\varepsilon}{\mathrm{Re}(\lambda)}e^{i\,\mathrm{Im}(\lambda)t}\right| = \frac{\varepsilon}{\mathrm{Re}(\lambda)},$$

which means the best (minimal) HUS constant is at least $\frac{1}{\mathrm{Re}(\lambda)}$. Altogether, the constant $K = \frac{1}{\mathrm{Re}(\lambda)}$ is the best HUS constant in this case. Likewise, if $\mathrm{Re}(\lambda) < 0$, then (1.1) has HUS with best HUS constant given by $\frac{1}{|\mathrm{Re}(\lambda)|}$.

If $\mathrm{Re}(\lambda) = 0$, then $\lambda = i\beta$. For $\phi(t) = \varepsilon t e^{i\beta t}$,

$$|\phi'(t) - \lambda\phi(t)| = |\varepsilon e^{i\beta t}| = \varepsilon$$

implies that ϕ satisfies (1.2), and

$$|\phi(t) - x(t)| = |\varepsilon t e^{i\beta t} - c e^{i\beta t}| = |\varepsilon t - c| \to \infty,$$

meaning (1.1) is not HUS for $\mathrm{Re}(\lambda) = 0$.

SUMMARY: Equation (1.1) has HUS iff $\mathrm{Re}(\lambda) \neq 0$, with best HUS constant $K = \frac{1}{|\mathrm{Re}(\lambda)|}$.

QUESTION: If we discretize the derivative operator in (1.1), how should we do this? What happens to the HUS discussion above for such a discretization? Two common discrete approximations to $\frac{d}{dt}x(t)$ are the forward/backward h-difference operators for step size $h > 0$:

$$\Delta_h x(t) = \frac{x(t+h) - x(t)}{h}, \qquad \nabla_h x(t) = \frac{x(t) - x(t-h)}{h}.$$

One could proportionally combine these using the diamond-alpha discrete operator

$$\Diamond_\alpha x(t) := \alpha\Delta_h x(t) + (1-\alpha)\nabla_h x(t), \quad \alpha \in [0,1]. \tag{1.4}$$

Instead, one could discretize (1.1) via the so-called discrete Cayley equation

$$\Delta_h x(t) = \lambda\left[\mu x(t+h) + (1-\mu)x(t)\right], \quad \mu \in [0,1]. \tag{1.5}$$

For (1.4), we introduce the imaginary diamond-alpha ellipse, which unifies and extends the left Hilger imaginary circle (forward, Delta case) and the right Hilger imaginary circle (backward, nabla case), for the discrete diamond-alpha derivative with constant step size. We then establish the Hyers–Ulam stability (HUS) of the first-order linear complex constant coefficient discrete diamond–alpha derivative equation, proving that the imaginary diamond-alpha ellipse fails to have HUS, while inside and outside the ellipse the equation has HUS. In particular, for each parameter value not on the diamond-alpha ellipse, we determine explicitly the best (minimum) HUS constant in terms of the elliptical

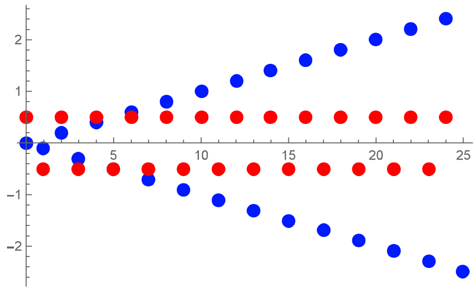

Figure 1: No HUS for $\lambda = \frac{-2}{h}$: red$= x$, blue$= \phi$

real part of the coefficient. For (1.5), we establish that the equation exhibits instability along a certain circle, but is Hyers–Ulam stable inside and outside that circle; the circle becomes infinite and coincides with the vertical imaginary axis when the proportionality is equal. In the case of Hyers–Ulam stability, the best constant is found explicitly. The theory is also explained in terms of radial solutions, and an example is illustrated in the next section.

2 Main Results: Summary of the Discrete Cases

Recall from the introductory section that (1.1) is approximated by

$$\Delta_h x(t) = \lambda x(t),$$

with the perturbation equation

$$|\Delta_h \phi(t) - \lambda \phi(t)| \leq \varepsilon.$$

- For $\lambda = \frac{-1}{h}$, there's no first-order difference equation.
- For $\lambda = 0$, the forward difference equation has no HUS, just like the continuous case.

For example, consider the forward difference operator with $\lambda = \frac{-2}{h}$,

$$\Delta_h x(t) = \lambda x(t), \qquad |\Delta_h \phi(t) - \lambda \phi(t)| \leq \varepsilon$$

- For $\lambda = \frac{-2}{h}$, take $\phi(t) = \varepsilon t(-1)^{\frac{t}{h}}$ and $x(t) = c(-1)^{\frac{t}{h}}$ to see there's no HUS.

See Figure 1.

Theorem 2.1 (Onitsuka 2017)**.** *Let* $\lambda < \frac{-2}{h}$*. Given* $\varepsilon > 0$*, suppose* $\phi : \mathbb{Z} \to \mathbb{C}$ *satisfies*

$$|\Delta_h \phi(t) - \lambda \phi(t)| \leq \varepsilon, \qquad t \in \{0, h, 2h, 3h, \cdots\}.$$

Then

$$\lim_{t\to\infty} \phi(t)(1 + \lambda h)^{\frac{-t}{h}}$$

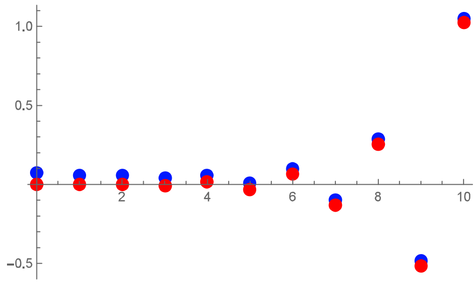

Figure 2: HUS for $\lambda < \frac{-2}{h}$: red$=x$, blue$=\phi$

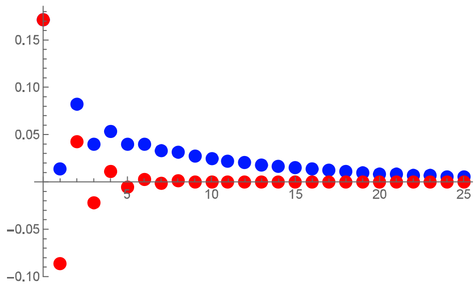

Figure 3: HUS for $\frac{-2}{h} < \lambda < \frac{-1}{h}$: red$=x$, blue$=\phi$

exists in $\mathbb{C}$*, and*

$$x(t) := \left(\lim_{t\to\infty} \phi(t)(1+\lambda h)^{\frac{-t}{h}}\right)(1+\lambda h)^{\frac{t}{h}}$$

is the unique solution such that

$$|\phi(t) - x(t)| < \frac{\varepsilon h}{|\lambda h + 2|}, \qquad t \in \{0, h, 2h, 3h, \cdots\}.$$

Example 2.1. Let $\lambda < \frac{-2}{h}$. See Figure 2.

Theorem 2.2 (Onitsuka 2017)**.** *: Let* $\frac{-2}{h} < \lambda < \frac{-1}{h}$*. Given* $\varepsilon > 0$*, suppose* $\phi : \mathbb{Z} \to \mathbb{C}$ *satisfies*

$$|\Delta_h \phi(t) - \lambda\phi(t)| \le \varepsilon, \qquad t \in \{0, h, 2h, 3h, \cdots\}.$$

If x *is a solution with*

$$|\phi(0) - x(0)| < \frac{\varepsilon h}{\lambda h + 2},$$

then

$$|\phi(t) - x(t)| < \frac{\varepsilon h}{\lambda h + 2}, \qquad t \in \{0, h, 2h, 3h, \cdots\}.$$

Example 2.2. Let $\frac{-2}{h} < \lambda < \frac{-1}{h}$, $\varepsilon = 0.1$. See Figure 3.

Result: HUS Theorem 5 (Onitsuka 2017): $\frac{-1}{h} < \lambda < 0$. Given $\varepsilon > 0$, suppose $\phi : \mathbb{Z} \to \mathbb{C}$ satisfies

$$|\Delta_h \phi(t) - \lambda\phi(t)| \le \varepsilon, \qquad t \in \{0, h, 2h, 3h, \cdots\}.$$

If x is a solution with

$$|\phi(0) - x(0)| < \frac{\varepsilon}{|\lambda|}$$

then

$$|\phi(t) - x(t)| < \frac{\varepsilon}{|\lambda|} \qquad t \in \{0, h, 2h, 3h, \cdots\}.$$

Result: HUS Theorem 6 (Onitsuka 2017): $\lambda > 0$. Given $\varepsilon > 0$, suppose $\phi : \mathbb{Z} \to \mathbb{C}$ satisfies

$$|\Delta_h \phi(t) - \lambda\phi(t)| \leq \varepsilon, \qquad t \in \{0, h, 2h, 3h, \cdots\}.$$

Then $\lim_{t\to\infty} \phi(t)(1 + \lambda h)^{\frac{-t}{h}}$ exists, and the function

$$x(t) := \left(\lim_{t\to\infty} \phi(t)(1 + \lambda h)^{\frac{-t}{h}}\right)(1 + \lambda h)^{\frac{t}{h}}$$

is the unique solution such that $|\phi(t) - x(t)| \leq \frac{\varepsilon}{\lambda}$ for all $t \in \{0, h, 2h, 3h, \cdots\}$.

Summary for discrete delta case: $\Delta_h x = \lambda x$. Equation $\Delta_h x = \lambda x$ has HUS for $\lambda \neq \frac{-2}{h}, \frac{-1}{h}, 0$, and the best HUS constant K is

$$K_1 = \frac{h}{|\lambda h + 2|}$$

if $\lambda < \frac{-2}{h}$ or $\frac{-2}{h} < \lambda < \frac{-1}{h}$, and

$$K_2 = \frac{1}{|\lambda|}$$

if $\frac{-1}{h} < \lambda < 0$ or $\lambda > 0$.

Summary for discrete nabla case: $\nabla_h x = \lambda x$. Equation $\nabla_h x = \lambda x$ has HUS for $\lambda \neq 0, \frac{1}{h}, \frac{2}{h}$, and the best HUS constant K is

$$K_1 = \frac{h}{|\lambda h - 2|}$$

if $\frac{1}{h} < \lambda < \frac{2}{h}$ or $\lambda > \frac{2}{h}$, and

$$K_2 = \frac{1}{|\lambda|}$$

if $\lambda < 0$ or $0 < \lambda < \frac{1}{h}$.

Unifying Idea: Imaginary Hilger Circle. See Figure 4.

Zero Hilger Real Part.

Definition 2.1 (1999)**.** For $\lambda \in \mathbb{C} \setminus \{\frac{-1}{h}\}$, the Hilger real part of λ is defined by

$$\text{Re}_h(\lambda) := \frac{|\lambda h + 1| - 1}{h}.$$

Theorem 2.3 (2019)**.** *If* $\lambda \in \mathbb{C}$ *with* $|\lambda h + 1| = 1$, *that is if* $\text{Re}_h(\lambda) = 0$, *then*

$$\Delta_h x(t) = \lambda x(t)$$

is not Hyers–Ulam stable.

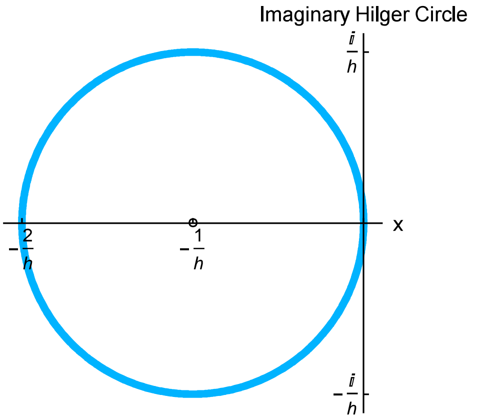

Figure 4: The (left) imaginary Hilger circle (blue)

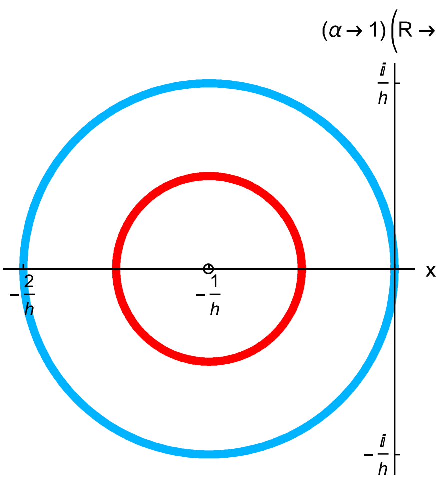

Figure 5: The (left) Hilger real part is negative (red)

Non-Zero Hilger Real Part.

Theorem 2.4 (2019)**.** *If* $\lambda \in \mathbb{C}\backslash\{-1/h\}$ *with* $|\lambda h + 1| \neq 1$*, then*

$$\Delta_h x(t) = \lambda x(t)$$

has Hyers–Ulam stability with minimum HUS constant

$$\frac{h}{|1 - |\lambda h + 1||} = \frac{1}{|\operatorname{Re}_h(\lambda)|}$$

on $h\mathbb{Z}$*.*

Delta Case: (Left) Hilger Real Part. See Figure 5.
Delta Case: (Left) Hilger Real Part. See Figure 6.
Nabla Case: (Right) Hilger Circle. See Figure 7.
Non-Zero Hilger Real Part: Nabla Case.

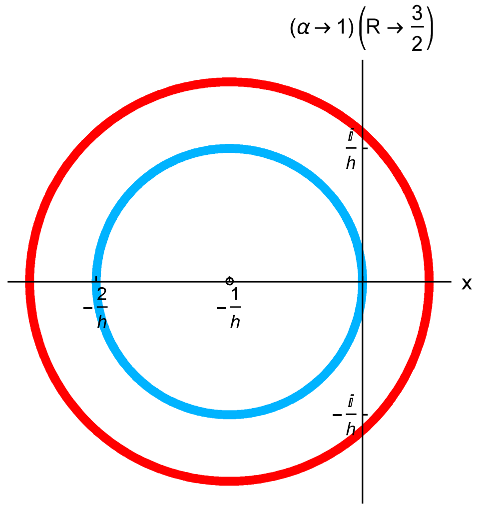

Figure 6: The (left) Hilger real part is positive (red)

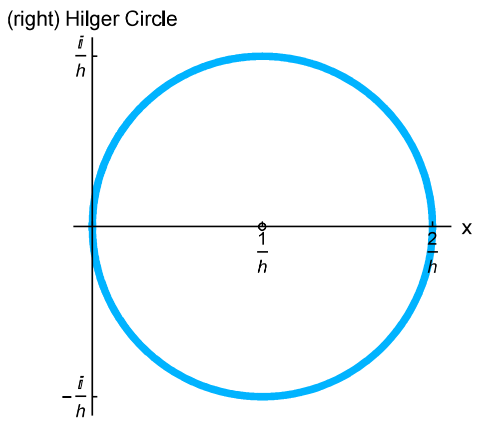

Figure 7: The (right) imaginary Hilger circle (blue)

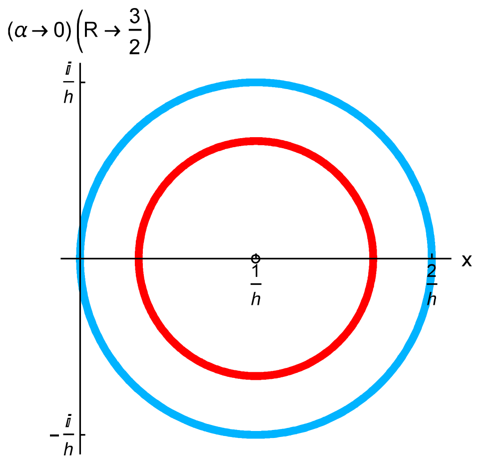

Figure 8: The (right) Hilger real part is positive (red)

Theorem 2.5 (2019)**.** *If* $\lambda \in \mathbb{C}\backslash\{1/h\}$ *with* $|\lambda h - 1| \neq 1$*, then*

$$\nabla_h x(t) = \lambda x(t)$$

has Hyers–Ulam stability with minimum HUS constant

$$\frac{h}{|1 - |\lambda h - 1||} = \frac{1}{\left|\widehat{\mathrm{Re}}_h(\lambda)\right|}$$

on $h\mathbb{Z}$*, where*

$$\widehat{\mathrm{Re}}_h(\lambda) := \frac{1 - |\lambda h - 1|}{h}.$$

Nabla Case: (Right) Hilger Real Part. See Figure 8.

Nabla Case: (Right) Hilger Real Part. See Figure 9.

Diamond$-\alpha$ Operator: Proportional Combination of Delta and Nabla Cases. Define the linear combination of discrete difference operator as the diamond-alpha operator given via

$$\Diamond_\alpha x(t) := \alpha \Delta_h x(t) + (1 - \alpha)\nabla_h x(t), \quad \alpha \in [0, 1].$$

Final Unification: Ellipses! See Figure 10.

Elliptical Coordinates Left: $\mathrm{Re}_{(h,\alpha)}(\lambda) < 0$. See Figure 11.

Elliptical Coordinates Left: $\mathrm{Re}_{(h,\alpha)}(\lambda) > 0$. See Figure 12.

Elliptical Coordinates Right: $\mathrm{Re}_{(h,\alpha)}(\lambda) < 0$. See Figure 13.

Elliptical Coordinates Right: $\mathrm{Re}_{(h,\alpha)}(\lambda) > 0$. See Figure 14.

Imaginary $\Diamond_\alpha$ Ellipses.

Definition 2.2 (2019)**.** The imaginary $\Diamond_\alpha$ ellipse $\mathcal{E}_{(h,\alpha)}$ is the set of all $\lambda \in \mathbb{C}$ such that

$$\lambda = \frac{(1 - 2\alpha)(1 - \cos\theta) + i \sin\theta}{h} \in \mathcal{E}_{(h,\alpha)} \quad \text{for any} \quad \theta \in [0, 2\pi].$$

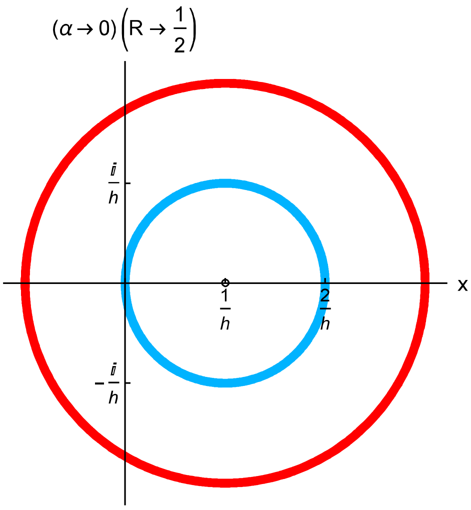

Figure 9: The (right) Hilger real part is negative (red)

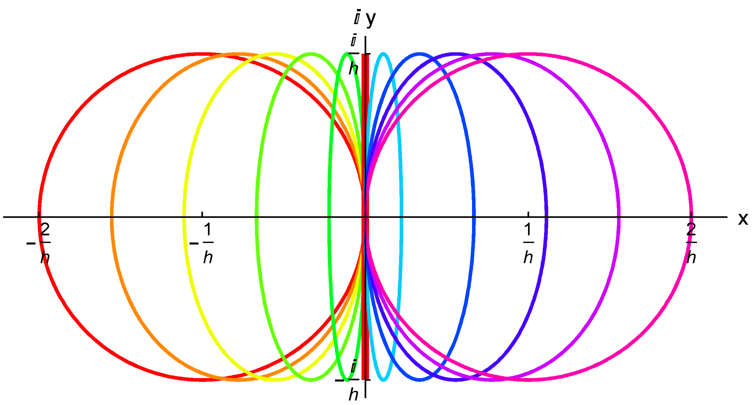

Figure 10: Progression of ellipses, from left Hilger circle to the right Hilger circle

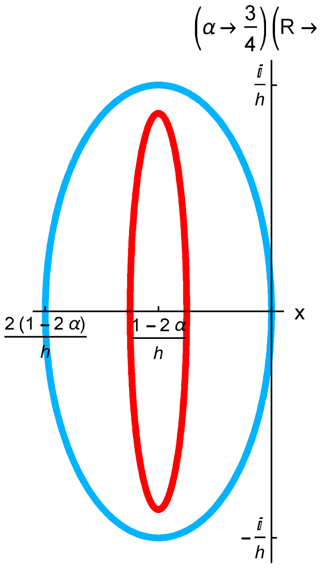

Figure 11: The imaginary ellipse in blue, elliptical coordinates for λ in red

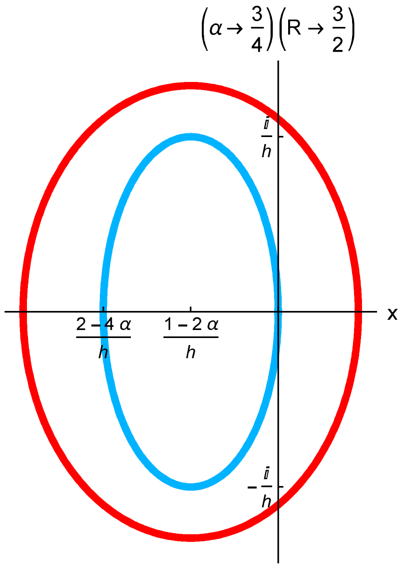

Figure 12: The imaginary ellipse in blue, elliptical coordinates for λ in red

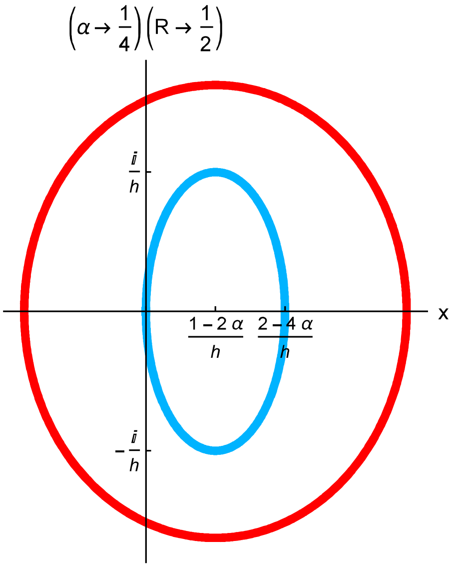

Figure 13: The imaginary ellipse in blue, elliptical coordinates for λ in red

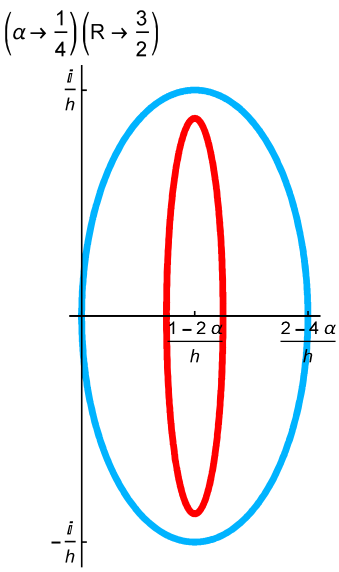

Figure 14: The imaginary ellipse in blue, elliptical coordinates for λ in red

Remark 2.1 (2019)**.** Note that $\mathcal{E}_{(h,1)} = \mathbb{Y}_h$, the (left) Hilger circle, and $\mathcal{E}_{(h,0)} = \widehat{\mathbb{Y}}_h$, the (right) Hilger circle, respectively. For $\lambda \notin \mathcal{E}_{(h,\alpha)}$, we represent λ via

$$\lambda = \frac{R(1-2\alpha) + [\alpha(R^2+1)-1]\cos\theta}{hR} + i\frac{[1+\alpha(R^2-1)]\sin\theta}{hR}$$

for $\theta \in [0, 2\pi]$.

Elliptical Real Part.

Definition 2.3 (2019)**.** Let $\alpha \in [0,1]$ and $h > 0$ be given. For any $\lambda \in \mathbb{C}$, λ can be expressed in terms of the imaginary ellipse $\mathcal{E}_{(h,\alpha)}$. Define the elliptical real part of λ to be

$$\mathrm{Re}_{(h,\alpha)}(\lambda) := \begin{cases} \dfrac{(R-1)(1-\alpha-R\alpha)}{hR} & : \alpha \in \left(0, \frac{1}{2}\right], \\ \dfrac{(R-1)(\alpha R+\alpha-1)}{hR} & : \alpha \in \left(\frac{1}{2}, 1\right). \end{cases}$$

Radial Solutions!

Remark 2.2 (2019)**.** Elliptical form of the eigenvalue λ:

$$\lambda = \frac{R(1-2\alpha) + [\alpha(R^2+1)-1]\cos\theta}{hR} + i\frac{[1+\alpha(R^2-1)]\sin\theta}{hR}$$

Radial form of the eigenfunctions:

$$x(t) = c_1 \left(Re^{i\theta}\right)^{\frac{t}{h}} + c_2 \left(\frac{\alpha-1}{\alpha Re^{i\theta}}\right)^{\frac{t}{h}},$$

for $t \in \mathbb{H}$ and arbitrary constants $c_1, c_2 \in \mathbb{C}$.

No branch cuts. Converges to expected form.

Main $\Diamond_\alpha$ Result.

Theorem 2.6 (2019)**.** *For any $\alpha \in [0,1]$ and $\theta \in [0, 2\pi]$, if $\lambda \neq \frac{(1-2\alpha)(1-\cos\theta)+i\sin\theta}{h} \in \mathbb{C}$, then*

$$\Diamond_\alpha x(t) = \lambda x(t)$$

has Hyers–Ulam stability on $\mathbb{H}$, with best HUS constant

$$\frac{hR}{|R-1||\alpha R+\alpha-1|} = \frac{1}{\left|\mathrm{Re}_{(h,\alpha)}(\lambda)\right|}$$

where $\lambda \in \mathbb{C} \setminus \mathcal{E}_{(h,\alpha)}$ for $R > 0$ with $R \neq 1$ and $R \neq \frac{1-\alpha}{\alpha}$, and $\mathrm{Re}_{(h,\alpha)}(\lambda)$ is the elliptical real part of λ given on the previous slide. □

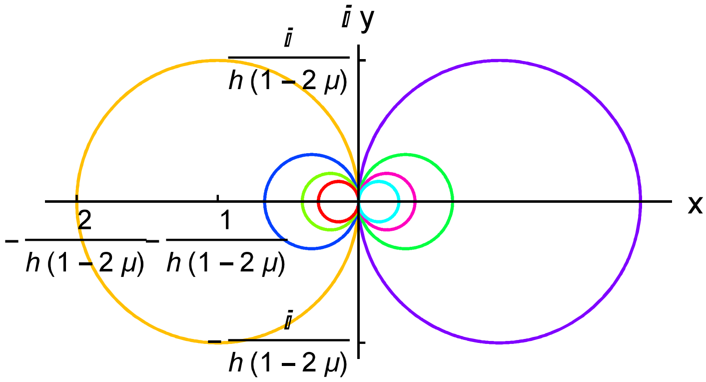

Figure 15: Various imaginary μ-circles for $\mu \in [0, 1]$, with $0 \le \mu < \frac{1}{2}$ generating the left μ-circles, and $\frac{1}{2} < \mu \le 1$ the right μ-circles. For $\mu = \frac{1}{2}$, the circle becomes infinite in diameter, namely the standard imaginary axis. These are unstable manifolds in the HUS sense.

Discrete Cayley Equation. The discrete Cayley equation (1.5) is another way to discretize $x' = \lambda x$, via

$$\Delta_h x(t) = \lambda \langle x(t) \rangle_\mu, \quad \lambda \in \mathbb{C} \backslash \left\{ \frac{1}{h\mu}, \frac{-1}{h(1-\mu)} \right\}, \tag{2.1}$$

where $\mu \in [0, 1]$ and

$$\langle x(t) \rangle_\mu := \mu x(t + h) + (1 - \mu) x(t).$$

No HUS for the Disrete Cayley Equation.

Theorem 2.7 (2019)**.** *Define the μ-real part of λ to be*

$$\mathrm{Re}_h^\mu(\lambda) := \frac{|1 + h\lambda(1-\mu)| - |1 - h\lambda\mu|}{h}, \qquad \lim_{h\to 0} \mathrm{Re}_h^\mu(\lambda) = \mathrm{Re}(\lambda). \tag{2.2}$$

If $\lambda \in \mathbb{C} \backslash \left\{ \frac{1}{h\mu}, \frac{-1}{h(1-\mu)} \right\}$ with

$$\mathrm{Re}_h^\mu(\lambda) = 0,$$

then (1.5) is unstable in the Hyers–Ulam sense.

$$\Delta_h x(t) = \lambda \left[\mu x(t+h) + (1-\mu)x(t) \right], \quad \mu \in [0,1]. \text{ See Figure 15.}$$
$$\Delta_h x(t) = \lambda \left[\mu x(t+h) + (1-\mu)x(t) \right], \quad \mu \in [0,1].$$

Theorem 2.8 (2019)**.** *Pick any $\lambda \in \mathbb{C} \backslash \left\{ \frac{1}{h\mu}, \frac{-1}{h(1-\mu)} \right\}$ such that $\mathrm{Re}_h^\mu(\lambda) \neq 0$. Then (1.5) is Hyers–Ulam stable, and the best HUS constant in the minimal sense is*

$$K = \frac{h}{||1 + h\lambda(1-\mu)| - |1 - h\lambda\mu||} = \frac{1}{|\mathrm{Re}_h^\mu(\lambda)|}.$$

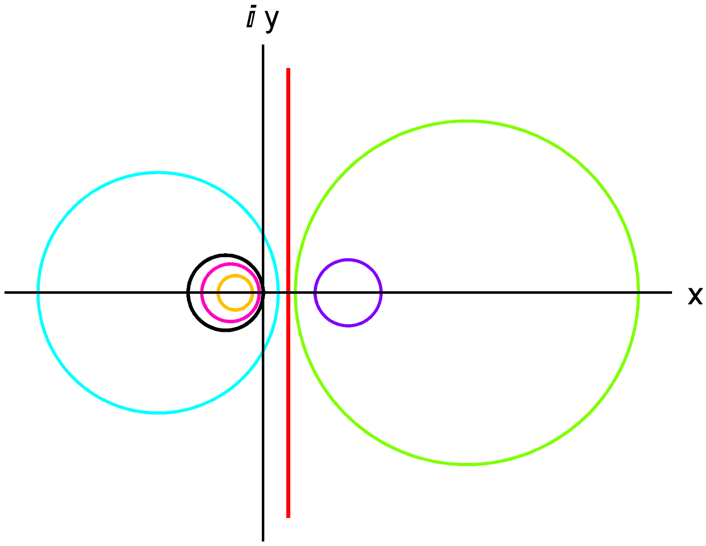

Figure 16: Let $\mu = \frac{1}{4}$, $h = 1$, $R > 0$. The circles represent parametrizations of λ. Black: imaginary $\frac{1}{4}$-circle (unstable). Orange & Maroon: $0 < R < 1$. Blue: $1 < R < 3$. Red: $R = \frac{1-\mu}{\mu} = 3$. Green & Purple: $R > 3$ (all stable).

Cayley Equation: Radial Solutions. Let $R > 0$ be a radial parameter, and use this to parametrize the complex coefficient λ via

$$\lambda = \frac{-1 + \mu + R^2\mu + R(1-2\mu)\cos(w) + iR\sin(w)}{h[(1-\mu)^2 + R\mu\,(R\mu + 2(1-\mu)\cos(w))]}. \tag{2.3}$$

Then

$$\mathrm{Re}_h^{\mu}(\lambda) = \frac{R-1}{h|1+\mu(Re^{iw}-1)|}$$

when $R \neq 1$, and the Cayley equation has radial solutions

$$x(t) = c(Re^{iw})^{\frac{t}{h}}.$$

HUS: $\Delta x(t) = \lambda\left[\frac{1}{4}x(t+1) + \frac{3}{4}x(t)\right]$. See Figure 16.

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