時空間変調下での散逸ソリトンのダイナミクス

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Abstract

The complex Ginzburg-Landau equation (CGLE) is a general model of spatially extended nonequilibrium systems. In this paper, an analytical method for solving a variable coefficient CGLE (VCCGLE) is presented to obtain exact solutions. Variable transformations for space and time variables with coefficient functions yield an imaginary time advection equation related to a complex valued characteristic curve. The VCCGLE is transformed into the nonlinear Schrödinger equation (NLSE) on the complex valued characteristic curve. This result indicates that the exact solutions of the NLSE generate that of the VCCGLE. Examples of the exact solutions of the VCGLE are presented through those of the NLSE.

1 Introduction

Spatiotemporal dynamics in dissipative systems have attracted the interest of researchers in the past few decades [1, 2]. In this literature, complex wave patters play dominant roles in various fields, such as fluid convections [3], fiber optics [4], chemical reactions [5] and biological systems [6]. In particular, localized dissipative waves, known as *dissipative solitons*, serve variety of the spatiotemporal dynamics in the dissipative systems [7]. Thus the nature of the dissipative solitons have been studied intensively by analytical, experimental and numerical methods to obtain the knowledge of the related systems.

As a fundamental model of the dissipative systems, the complex Ginzburg-Landau equation (CGLE) has been introduced by means of singular perturbation methods [8, 9]. The parameters of the CGLE is derived from the original evolution equations of the dissipative systems. Thus the emergence of the dissipative solitons can be predicted by estimating specific values of the parameters of the CGLE. Based on the idea, the control method of the dissipative solitons were developed in and applied to the optical fiber telecommunications [10].

Recently, the dissipative solitons under spatiotemporal modulations were began to be investigated in the area of the Bose-Einstein condensation [11, 12, 13, 14] and signal processing in optical fibers [15, 16, 17, 18]. In these systems, the CGLE is modified to have space and/or time dependent variable coefficients originated from external magnetic fields and/or optical lattices [19, 20]. To investigate the behavior of the dissipative solitons in these systems, analytical approximation techniques with numerical methods have been employed: a secant hyperbolic form anzats takes the dynamical system of the parameters of the dissipative solitons [21], the method of variational approximation provides the criteria of stability [22], steady state assumption derives nonlinear eigenvalue problems which determine the shape of the spatial modes of the dissipative solitons [23].

Despite the previous studies of the dissipative solitons under spatiotemporal modulations, fully analytical methods, which provide exact solutions, have yet to be developed except to the situation that

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temporal modulations only exist [24]. In this study, therefore, we develop an analytical method for solving a variable coefficient CGLE (VCCGLE) with its solvable condition. As applications of the proposed method, we investigate the spatiotemporal dynamics of the dissipative solitons of the VCCGLE in related physical systems.

2 Formal solutions of the variable coefficient complex Ginzburg-Landau equation

The CGLE is derived from nonlinear partial differential equations of the dissipative systems, based on the assumption that the spatiotemporal dynamics can be described by slow amplitude evolution on a carrier wave. In order to incorporate the effect of inhomogeneity in media and temporal modulation, space and time dependent variable coefficients are introduced into the CGLE in the form:

$$\mathrm{i}\frac{\partial \psi}{\partial t} + p(x,t)\frac{\partial^2 \psi}{\partial x^2} + q(x,t)|\psi|^2\psi = [-\omega(t) + \mathrm{i}\gamma(t)]\psi, \tag{1}$$

where $\psi(x,t)$ describes slowly evolving amplitude of the systems. The coefficient functions $p(x,t)$ and $q(x,t)$ are given by $p(x,t) = p_r(x,t) + \mathrm{i}p_i(x,t)$ and $q(x,t) = q_r(x,t) + \mathrm{i}q_i(x,t)$, where $p_r(x,t)$, $p_i(x,t)$, $q_r(x,t)$, and $q_i(x,t)$ are real valued functions, $\gamma(t)$, and $\omega(t)$ are positive definite real valued functions. It is assumed that $p(-x,t)q(-x,t) = p(x,t)q(x,t)$. In the literature of nonlinear wave theory, $p(x,t)$, $q(x,t)$, $\gamma(t)$, and $\omega(t)$ correspond to linear dispersion and dissipation, nonlinear saturation, linear gain or loss, and frequency modulation coefficients, respectively.

2.1 Variable transformations

The complex valued function $\psi(x,t)$ is transformed into

$$\psi(x,t) = \exp[\Gamma(t) + \mathrm{i}\Omega(t)]\varphi(x,t), \tag{2}$$

where $\Gamma(t)$ and $\Omega(t)$ are defined by

$$\Gamma(t) = \int^t \gamma(t')\mathrm{d}t', \tag{3}$$

$$\Omega(t) = \int^t \omega(t')\mathrm{d}t'. \tag{4}$$

Substituting Eqs. (2), (3) and (4) into Eq. (1) with a transformed variable

$$\tau = \int^t q(x,t')\mathrm{e}^{2\Gamma(t')}\mathrm{d}t' \tag{5}$$

and a transformed coefficient function

$$r(x,t)^2 = \frac{p(x,t)}{q(x,t)}\mathrm{e}^{-2\Gamma(t)}, \tag{6}$$

one obtains a variable coefficient nonlinear Schrödinger equation (NLSE) as

$$\frac{\partial \varphi}{\partial \tau} + r(x,\tau)^2\frac{\partial^2 \varphi}{\partial x^2} + |\varphi|^2\varphi = 0. \tag{7}$$

In order to obtain transformed variables which reduce Eq. (7) to the NLSE without the variable coefficients, a complex valued characteristic curve is introduced. Suppose $\xi(x,\tau)$ is a transformed variable satisfying an imaginary time advection equation

$$\mathrm{i}\frac{\partial \xi}{\partial \tau} - \left[\frac{1}{2}\frac{\partial}{\partial x}r(x,\tau)^2\right]\frac{\partial \xi}{\partial x} = 0. \tag{8}$$

On the characteristic curve of Eq. (8), the variable coefficient NLSE in Eq. (7) is transformed into

$$\frac{\partial \varphi}{\partial \tau} + \left(r(\xi, \tau) \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} \right)^2 \varphi + |\varphi|^2 \varphi = 0. \tag{9}$$

Introducing the variable transformation as

$$\eta = \int^{\xi} \frac{1}{r(\xi', \tau)} \left(\frac{\partial \xi'}{\partial x} \right)^{-1} \mathrm{d}\xi', \tag{10}$$

Eq. (9) yields the NLSE with respect to η and τ as

$$\frac{\partial \varphi}{\partial \tau} + \frac{\partial^2 \varphi}{\partial \eta^2} + |\varphi|^2 \varphi = 0. \tag{11}$$

Since the exact solutions of the NLSE can be obtained by several methods, the inverse variable transformations for the solutions of Eq. (11) provide those of the VCCGLE in Eq. (1).

2.2 Solvable condition of the imaginary time advection equation

In order to solve the imaginary time advection equation in Eq. (8), without loss of generality, the solvable conditions with respect to variable coefficients are introduced. Suppose $p(x, t)$ and $q(x, t)$ are given by

$$p(x, t) = X_p(x) T_p(t), \tag{12}$$

$$q(x, t) = X_q(x) T_q(t), \tag{13}$$

where $X_l(x)$ and $T_l(t)$ $(l = p, q)$ are positive real functions, respectively. With Eqs. (5), (12) and (13), the imaginary time advection equation in Eq. (8) is rewritten as

$$\frac{\mathrm{i}}{T_p(t)} \frac{\partial \xi}{\partial t} - \frac{1}{2} X_q(x) \frac{\partial}{\partial x} \left(\frac{X_p(x)}{X_q(x)} \right) \frac{\partial \xi}{\partial x} = 0. \tag{14}$$

By the method of characteristics [29], the corresponding characteristic equations are derived as

$$\frac{\mathrm{d}t}{\mathrm{d}s} = -\frac{\mathrm{i}}{T_p(t)}, \tag{15}$$

$$\frac{\mathrm{d}x}{\mathrm{d}s} = -\frac{1}{2} X_q(x) \frac{\partial}{\partial x} \left(\frac{X_p(x)}{X_q(x)} \right), \tag{16}$$

where s is an auxiliary variable of the characteristic curve. From Eqs. (15) and (16), the characteristic curve is derived as

$$\kappa = -\int^{x} \frac{2}{X_q(x')} \left[\frac{\partial}{\partial x} \left(\frac{X_p(x')}{X_q(x')} \right) \right]^{-1} \mathrm{d}x' + \mathrm{i} \int^{t} T_p(t') \mathrm{d}t', \tag{17}$$

and thus the solution of the imaginary time advection equation is obtained as

$$\xi(x, t) = \xi_0(\kappa), \tag{18}$$

where $\xi_0(\cdot)$ is an initial function of Eq. (8). Moreover, if $X_q(x)$ is constant, the transformed variable η is reduced to

$$\eta = \int^{x} \frac{\mathrm{d}x'}{r(x', \tau)} \tag{19}$$

by the inversion function theorem. Under these solvable conditions, the specific forms of transformed variables can be calculated.

3 Exact solutions of the nonlinear Scrödinger equation

With the use of the variable transformations in the previous section, one can obtain the exact solutions of the VCCGLE through those of the NLSE. In this section, thus, exact solutions of the NLSE are briefly reviewed.

Hirota's bilinear method was developed to solve integrable nonlinear wave equations by algebraic procedures. The D-operator for differentiable functions $f(z)$ and $g(z)$ is defined by [28]

$$D_z(f\cdot g) = \left[\frac{\partial f(z)}{\partial z}g(z') - f(z)\frac{\partial g(z')}{\partial z'}\right]_{z'=z}. \tag{20}$$

The complex valued function $\varphi(\eta,\tau)$ is assumed to be a rational function of real $F(\eta,\tau)$ and complex $G(\eta,\tau)$ functions as

$$\varphi(\eta,\tau) = \frac{G(\eta,\tau)}{F(\eta,\tau)}. \tag{21}$$

The following relations between partial derivatives and D-operators

$$\frac{\partial\varphi}{\partial\tau} = \frac{D_\tau(G\cdot F)}{F^2}, \tag{22}$$

$$\frac{\partial^2\varphi}{\partial\eta^2} = \frac{D_\eta^2(G\cdot F)}{F^2} - \frac{D_\eta^2(F\cdot F)}{GF} \tag{23}$$

for Eq. (11) yield the bilinear forms with respect to F and G as

$$D_\tau(G\cdot F) + D_\eta^2(G\cdot F) = \lambda GF, \tag{24}$$

$$D_\eta^2(F\cdot F) - |G|^2 = \lambda F^2. \tag{25}$$

The auxiliary function $\lambda(\eta,\tau)$ is introduced to incorporate boundary conditions into the bilinear forms in Eqs. (24) and (25). The exact solutions of the NLSE are obtained from algebraic procedures for the bilinear forms.

Suppose F and G are expanded with respect to an infinitesimal parameter ϵ as

$$F = 1 + \epsilon^2 F_2 + \epsilon^4 F_4 + \cdots, \tag{26}$$

$$G = \epsilon G_1 + \epsilon^3 G_3 + \cdots. \tag{27}$$

Substituting Eqs. (26) and (27) into Eqs. (24) and (25), one obtains an ϵ-hierarchy of the bilinear forms. Although the perturbation expansions in Eqs. (26) and (27) are infinite series, the ϵ-hierarchy of the bilinear forms are truncated at finite order. In particular, the one soliton solution is immediately obtained under $\lambda = 0$ as follows:

$$\varphi(\eta,\tau) = A\mathrm{sech}(K\eta - V\tau)\mathrm{e}^{\mathrm{i}\theta_0}, \tag{28}$$

where A, K, V, and θ_0 are constant parameters. Since the NLSE describes the propagating envelope of a carrier wave, the soliton solution in Eq. (28) is called an envelope soliton. Figure 1 shows spatiotemporal dynamics of magnitude of the envelope soliton in Eq. (28). As well known, a unimodal shape propagates with constant velocity in this figure.

The NLSE has a rational function solution [30]. Suppose F and G in Eq. (21) are polynomial functions of x and t, their coefficients are determined sequentially by direct substitution. Depending on the highest order of the polynomials, the rational function solutions of the NLSE exhibit different forms. The possible lowest order polynomials of F and G yield the following rational function solution

$$\varphi(\eta,\tau) = \left[1 - \frac{4(1+2\mathrm{i}\tau)}{1+4\eta^2+4\tau^2}\right]\mathrm{e}^{\mathrm{i}\tau}. \tag{29}$$

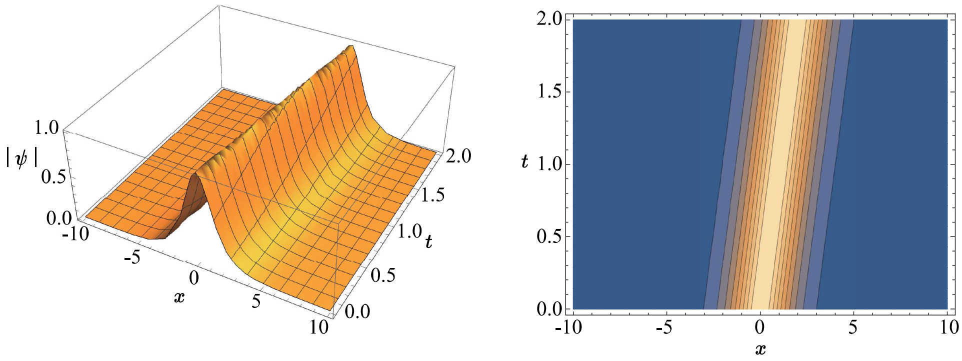

Fig. 1: Spatiotemporal dynamics of the envelope soliton solution. The parameters in Eq. (28) are fixed as $A = K = V = \theta_0 = 1.0$.

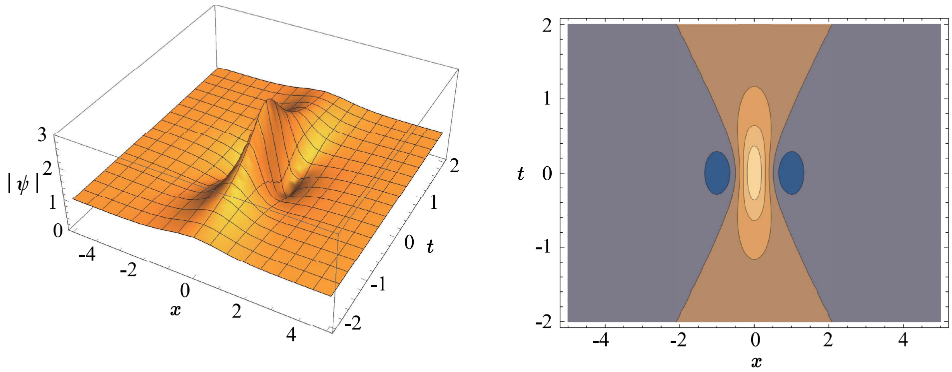

Fig. 2: Spatiotemporal dynamics of the Peregrine soliton.

The spatiotemporally localized soliton in Eq. (29) is known as the Peregrine soliton, which is used as a mathematical model of rogue waves and freak waves [31, 32]. In Fig. 2, it is seen that a localized wave emerges around the center of filed. It disappears after a short living time.

In addition, a periodic wave solution is derived from the NLSE [33, 34]. To obtain an oscillating solution the following ansatz is introduced as

$$\varphi(\eta, \tau) = [\rho(\eta, \tau) + \sigma(\tau)]\, \mathrm{e}^{\mathrm{i}\theta(\tau)}, \tag{30}$$

where $\rho(\eta, \tau)$ is a complex valued function, $\sigma(\tau)$ and $\theta(\tau)$ are real valued functions. Substituting Eq. (30) into Eq. (11) provides the set of differential equations with respect to ρ, σ and θ. Through a cumbersome calculation with integrable conditions, a periodic solution is obtained as

$$\varphi(\eta, \tau) = \frac{\cos(\sqrt{2}\eta) + \mathrm{i}\sqrt{2}\sinh\tau}{\cos(\sqrt{2}\eta) - \sqrt{2}\cosh\tau} \mathrm{e}^{\mathrm{i}\tau}. \tag{31}$$

This periodic solution shows breathing of localized wave trains, which is known as the Akhmediev breather, as is shown in Fig. 3. In this figure, periodically aligned localized waves emerge around the center of field.

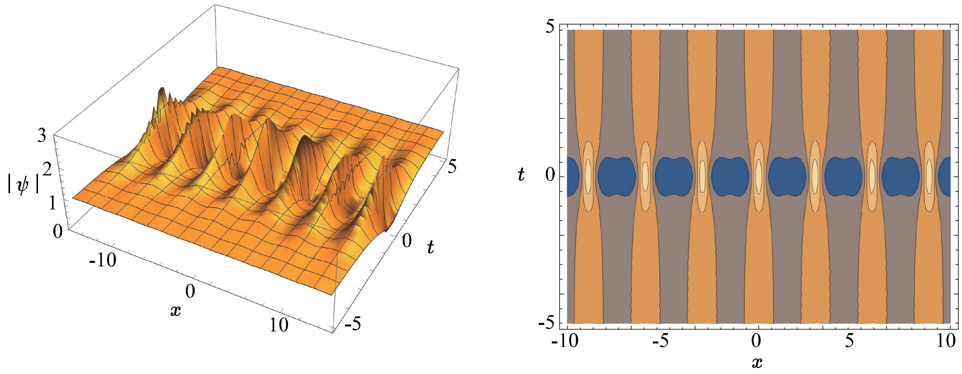

Fig. 3: Spatiotemporal dynamics of the Akhmediev breather.

4 Examples of related physical systems

Some relevant physical models of the VCCGLE appear in the field of plasma fluid, optical lattice and the Bose-Einstein condensation. In previous works, both analytical and numerical approximation methods have been introduced to investigate the spatiotemporal dynamics of the relevant models. The proposed method in this paper, on the other hand, provides the exact solution of the VCCGLE related to those of the NLSE presented in Sec. 3.

4.1 Nonlinear waves in plasma systems

In the plasma system where an electron beam is injected, an unstable NLSE is derived from electromagnetic fluid equations. As a generalized model of slowly varying amplitude modulations, the NLSE with space variable coefficients has been introduced as follows:

$$\mathrm{i}\frac{\partial\psi}{\partial x}+p(x)\frac{\partial^2\psi}{\partial t^2}+q(x)|\psi|^2\psi=0. \tag{32}$$

Interchange of variables x and t in Eq. (32), one obtains a stable variable coefficient NLSE considered in the system of soliton equation with slowly varying variables [25]:

$$\mathrm{i}\frac{\partial\psi}{\partial t}+p(t)\frac{\partial^2\psi}{\partial x^2}+q(t)|\psi|^2\psi=0. \tag{33}$$

Since this equation is a reduced form of the VCCGLE in Eq. (1), exact solutions with the corresponding variable transformations can be obtained. As an example, here, the time variable coefficients in Eq. (33) are given as

$$p(t)=1+a_p\cos(\omega_p t+\delta_p), \tag{34}$$

$$q(t)=1+a_q\sin(\omega_q t+\delta_q), \tag{35}$$

where a_p, a_q, ω_p, ω_q, δ_p and δ_q are positive real parameters. In this case, auxiliary variables are readily obtained from Eqs. (5) and (19) as

$$\tau=t+\frac{a_q}{\omega_q}[\cos(\omega_q t+\delta_q)-1], \tag{36}$$

$$\eta=\sqrt{\frac{1+a_q\sin(\omega_q t+\delta_q)}{1+a_p\cos(\omega_p t+\delta_p)}}x. \tag{37}$$

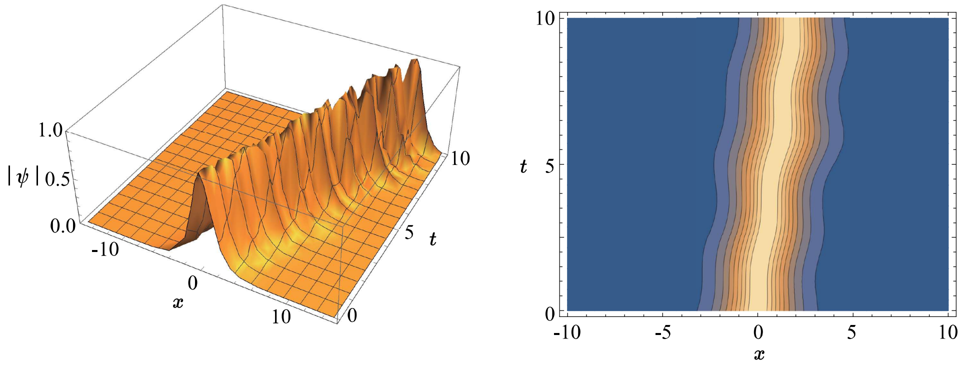

Fig. 4: Spatiotemporal dynamics of the envelope soliton under periodic temporal modulation. The coefficients of $p(t)$ and $q(t)$ are fixed as follows: $a_p = 0.1$, $\omega_p = 10$, $\delta_p = 0$, $a_q = 0.1$, $\omega_q = 5$, $\delta_q = 0$.

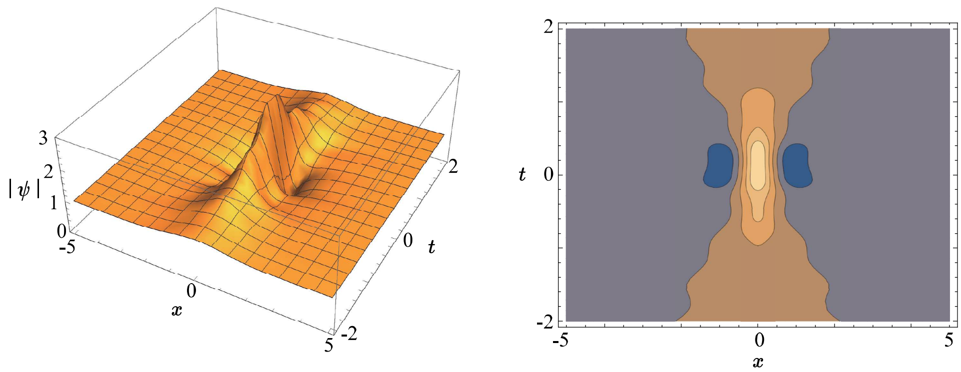

Fig. 5: Spatiotemporal dynamics of the Peregrine soliton under periodic temporal modulation. The coefficients of $p(t)$ and $q(t)$ are fixed as follows: $a_p = 0.1$, $\omega_p = 10$, $\delta_p = 0$, $a_q = 0.1$, $\omega_q = 5$, $\delta_q = 0$.

Periodic temporal modulations in x and t directions are expected to appear by influence of the variable coefficients in Eqs. (34) and (35). Figure. 4 shows a propagating envelope soliton under the periodic temporal modulation. In these pictures, the velocity of the envelope soliton varies periodically while the shape of it is invariant. In Fig. 5, the Peregrine soliton is also affected by the periodic temporal modulation. Its tails spread periodically in time direction after the wave emerging suddenly. The spatiotemporal dynamics of the Akhmediev breathers under periodic temporal modulation is exhibited in Fig. 6. The influence of temporal modulation is observed clearly far from central domain.

4.2 Nonlinear optical lattice

The system of nonlinear optical lattices is given by the NLSE or CGLE with spatially modulated variable coefficients for linear and/or nonlinear terms [26]. For the sake of brevity, we consider the case that only nonlinear terms are influenced by spatial modulation. In this case, the NLSE with a linear damping and a spatial variable coefficient is presented as

$$\mathrm{i}\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + q(x)|\psi|^2\psi = \mathrm{i}\gamma\psi. \tag{38}$$

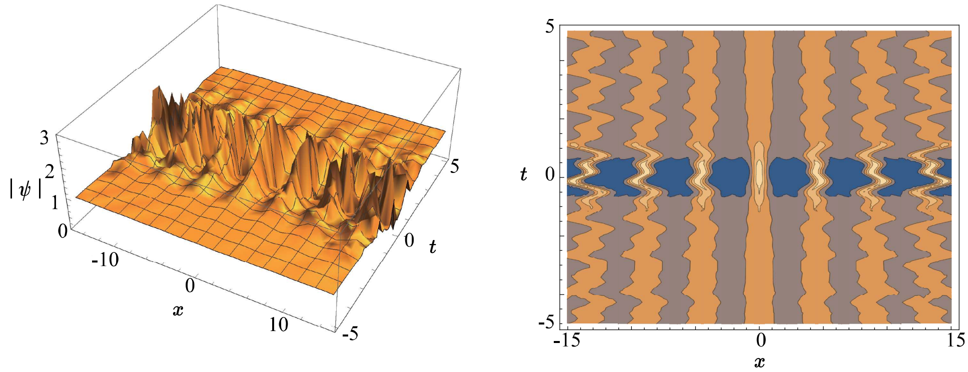

Fig. 6: Spatiotemporal dynamics of the Akhmediev breather under periodic temporal modulation. The coefficients of $p(t)$ and $q(t)$ are fixed as follows: $a_p = 0.1$, $\omega_p = 10$, $\delta_p = 0$, $a_q = 0.1$, $\omega_q = 5$, $\delta_q = 0$.

The linear damping term with the positive constant γ leads to $\Gamma = \gamma t$ in Eq. (3). In this example, the variable coefficient $q(x)$ is given as

$$q(x) = 1 - k^2 \sin^2 x, \ (0 < k < 1), \tag{39}$$

which appears in $\mathcal{PT}$-symmetric systems [27]. Transformed variables are derived from $\Gamma(t)$ and $q(x)$ as

$$\tau = \frac{1}{2\gamma}(1 - k^2 \sin^2 x)(\mathrm{e}^{2\gamma t} - 1) \tag{40}$$

$$\eta = \mathrm{e}^{2\gamma t} F(|x|, k), \tag{41}$$

where $F(\phi, k)$ is the incomplete elliptic integral of the first kind. With these transformed variables, the exact solutions presented in Sec. 3 provide the spationtemporal dynamics of Eq. (38).

Figure 7 shows an envelope soliton in the nonlinear optical lattice. It is observed that the shape of the envelope soliton changes under acceleration. The symmetry breaking of the Peregrine soliton on the time direction is observed in Fig. 8. In fact, as is confirmed in Eqs. (40) and (41), both τ and η no longer have symmetry with respect to the time variable t. With existence of the linear damping and spatial modulation, the Akhmediev breather focus on the time direction and then converges into a plane wave. To the best of my knowledge, it is first time that such an anomalous dynamics of the nonlinear wave propagation under the optical lattice are observed.

5 Conclusion

In this paper, the analytical method for solving the VCCGLE was presented. On the characteristic curve, which introduces the imaginary time advection equation, the VCCGLE can be transformed into the NLSE with respected to the transformed variables. Also, the solvable condition for the imaginary time advection equation was introduced in order to obtain the characteristic curve analytically. Examples of the VCCGLE were investigated related to the plasma systems and optical lattices, where it was demonstrated that the proposed method yields the exact solutions of the VCCGLE with closed forms of the transformed variables.

A space dependent coefficient function for the linear term was not considered in this paper. In other words, as a model of the Bose-Einstein condensation, Eq. (1) is not applicable to describe the

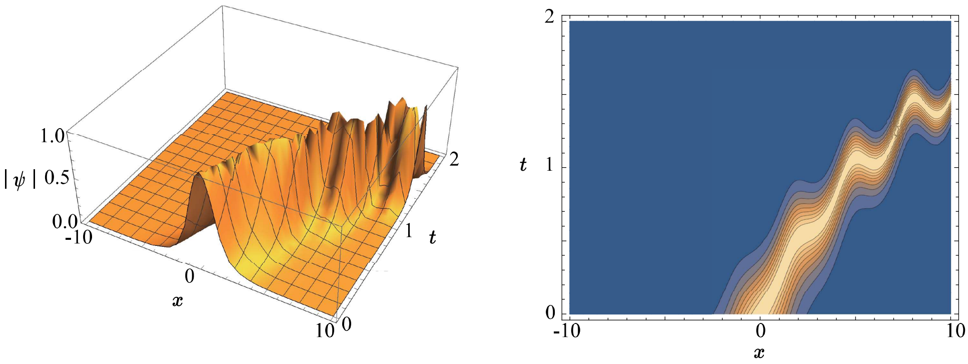

Fig. 7: Spatiotemporal dynamics of the envelope soliton under linear damping and spatial modulation. The coefficients are fixed as follows: $\gamma = 0.5$, $k = 0.5$.

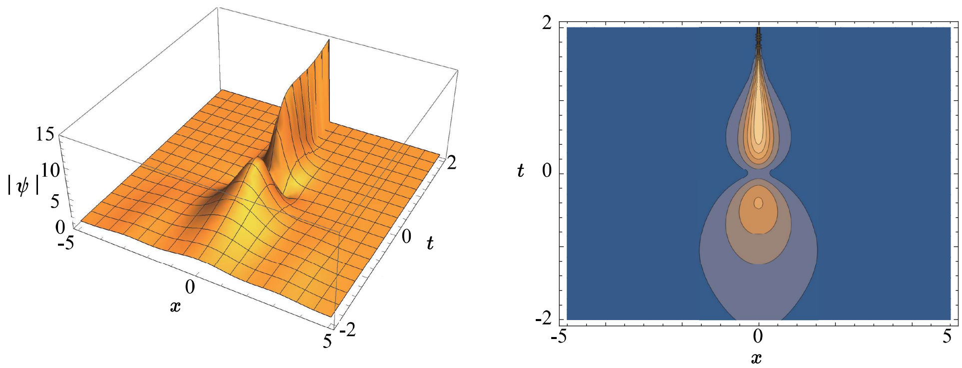

Fig. 8: Spatiotemporal dynamics of the Peregrine soliton under linear damping and spatial modulation. The coefficients are fixed as follows: $\gamma = 0.5$, $k = 0.5$.

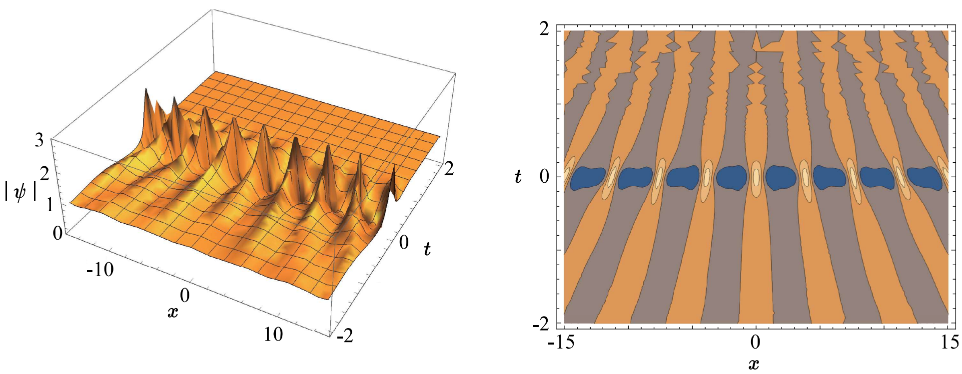

Fig. 9: Spatiotemporal dynamics of the Akhmediev breather under linear damping and spatial modulation. The coefficients are fixed as follows: $\gamma = 0.5$, $k = 0.5$.

spatiotemporal dynamics of macroscopic wave functions. Incorporating the space variable coefficient into the linear term of the VCCGLE will be considered near future.

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