Fukuda-Okasaka-Fujimoto Model System of Mixed Ionized Gas Dynamics

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Abstract

The aim of this paper is to study a one dimensional model system of equations for ionized gas dynamics at high temperature where the gas is a mixture of two kinds of monatomic gas. In addition to the mass density, pressure, temperature and particle velocity, degrees of ionization of both gases are also involved. By assuming that the local thermal equilibrium is attained, Saha's ionization equations are added. Thus the equations are supplemented by the first and second law of thermodynamics, a single equation of state and, in addition, a set of thermodynamic equations.

The equations constitute a strictly hyperbolic system, which guarantees that the initial value problem is well-posed locally in time for sufficiently smooth initial data. The geometric properties of the system are rather complicated: in particular, we prove the existence of a region where convexity (genuine nonlinearity) fails for forward and backward characteristic fields. Also we study thermodynamic properties of shock waves by a detailed analysis of Hugoniot loci, which is employed for the study of existence and uniqueness of solutions to the Riemann Problem. We prove that the Grüneisen coefficient is positive and the Liu-Smith strong condition is satisfied, which shows that the Riemann problem is well-posed.

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1 Introduction

A shock wave is a propagating discontinuity of density, pressure, temperature and etc., which is supersonic with respect to the gaseous medium ahead of it and subsonic with respect to that behind it. Behind a shock wave, not only pressure but also temperature increases abruptly and the gas is heated to high temperatures. In the gas behind the shock front, almost all molecules become dissociated and finally some of the atoms become ionized: $\mathrm{X} \rightleftarrows \mathrm{X}^+ + \mathrm{e}^-$. Numerous spectroscopic measurements of atomic parameters and thermodynamic equilibrium of plasma thus generated have been done, for example, in various Helium-Hydrogen mixtures (Fukuda-Sugiyama [10], Fukuda-Sugiyama-Okasaka [11]).

The model system of mixed ionized gas dynamics that we discuss in this paper is proposed by Fukuda-Okasaka-Fujimoto [9][1] for the purpose of providing a theoretical basis for their observations. The system consists of equations of macroscopic motion for 1-d mixed gas dynamics. Its particular nature is: degree of ionization of each gas is considered to be a thermodynamic

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[1] An English translation of [9] is available upon request to F. Asakura.

variable. The aim of this paper is to exhibit principal results obtained in Asakura [1] and to discuss the existence and uniqueness of solutions to the Riemann problem. For a single monatomic ionized gas, studies have been done in Asakura-Corli [2, 3, 4]. This model system is similar to the ideal dissociating diatomic gas model studied by Lighthill [13].

Basic thermodynamic variables are denoted in this paper by T : temperature, p : pressure, ρ : mass density, $v = 1/\rho$: specific volume, e : specific internal energy and s : specific entropy. The flow velocity is denoted by u and the specific total energy by $\mathcal{E} = \frac{1}{2}u^2 + e$. The system of equations of one-dimensional motion for gas dynamics consists of the following three conservation laws: conservation of mass, momentum and energy

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ (\rho \mathcal{E})_t + (\rho \mathcal{E} u + pu)_x = 0, \end{cases} \tag{1.1}$$

which are supplemented by the first and second law of thermodynamics

$$de = TdS - pdv, \tag{1.2}$$

a single equation of state and a set of thermodynamic equations. For brevity we will call s, e and $\mathcal{E}$, the entropy, internal and total energy, respectively.

Let us consider one mole mixture of monatomic gases A and B whose ionization reactions are $\mathrm{A} \rightleftarrows \mathrm{A}^+ + \mathrm{e}^-$, $\mathrm{B} \rightleftarrows \mathrm{B}^+ + \mathrm{e}^-$. We denote the number of atoms and ions for each gas by $N_{\mathrm{a}}^{\mathrm{A}}, N_{\mathrm{a}}^{\mathrm{B}}$ and $N_{\mathrm{i}}^{\mathrm{A}}, N_{\mathrm{i}}^{\mathrm{B}}$, respectively. The number of electrons are denoted by N_{e}. Note that

$$N_{\mathrm{a}}^{\mathrm{A}} + N_{\mathrm{i}}^{\mathrm{A}} + N_{\mathrm{a}}^{\mathrm{B}} + N_{\mathrm{i}}^{\mathrm{B}} = N_0 : \text{ Avogadro Number}, \quad N_{\mathrm{e}} = N_{\mathrm{i}}^{\mathrm{A}} + N_{\mathrm{i}}^{\mathrm{B}}.$$

The concentration of atoms, ions and electrons are defined by $n_{\mathrm{a}}^{\mathrm{A}} = \frac{N_{\mathrm{a}}^{\mathrm{A}}}{V}, n_{\mathrm{a}}^{\mathrm{B}} = \frac{N_{\mathrm{a}}^{\mathrm{B}}}{V}, n_{\mathrm{i}}^{\mathrm{A}} = \frac{N_{\mathrm{i}}^{\mathrm{A}}}{V}, n_{\mathrm{i}}^{\mathrm{B}} = \frac{N_{\mathrm{i}}^{\mathrm{B}}}{V}, n_{\mathrm{e}} = \frac{N_{\mathrm{e}}}{V}$, respectively.

By denoting $G_{\mathrm{a}}^{\mathrm{A}}, G_{\mathrm{a}}^{\mathrm{B}}$: the partition functions of the neutral state, $G_{\mathrm{i}}^{\mathrm{A}}, G_{\mathrm{i}}^{\mathrm{B}}$: same for the 1-ionized state, and $\chi^{\mathrm{A}}, \chi^{\mathrm{B}}$: first ionization potentials, the coupled Saha's laws for mixed monatomic gas are presented as the following.

$$\frac{n_{\mathrm{i}}^{\mathrm{A}} n_{\mathrm{e}}}{n_{\mathrm{a}}^{\mathrm{A}}} = \frac{2G_{\mathrm{i}}^{\mathrm{A}}}{G_{\mathrm{a}}^{\mathrm{A}}} \left(\frac{2\pi m_{\mathrm{e}} kT}{h^2} \right)^{\frac{3}{2}} e^{-\frac{\chi^{\mathrm{A}}}{kT}}, \quad \frac{n_{\mathrm{i}}^{\mathrm{B}} n_{\mathrm{e}}}{n_{\mathrm{a}}^{\mathrm{B}}} = \frac{2G_{\mathrm{i}}^{\mathrm{B}}}{G_{\mathrm{a}}^{\mathrm{B}}} \left(\frac{2\pi m_{\mathrm{e}} kT}{h^2} \right)^{\frac{3}{2}} e^{-\frac{\chi^{\mathrm{B}}}{kT}} \tag{1.3}$$

For A: hydrogen atom, we have $\chi^{\mathrm{A}} = 13.59844$ eV and for B: helium atom, $\chi^{\mathrm{B}} = 24.58741$ eV. First ionization temperatures are

$$T_{\mathrm{A}} = \frac{\chi^{\mathrm{A}}}{k} = 1.5780 \times 10^5, T_{\mathrm{B}} = \frac{\chi^{\mathrm{B}}}{k} = 2.8532 \times 10^5.$$

Note that $T_{\mathrm{A}} < T_{\mathrm{B}} < 2T_{\mathrm{A}}$. We have also $\frac{2G_{\mathrm{i}}^{\mathrm{A}}}{G_{\mathrm{a}}^{\mathrm{A}}} = 1$, $\frac{2G_{\mathrm{i}}^{\mathrm{B}}}{G_{\mathrm{a}}^{\mathrm{B}}} = 4$. We will assume that a local thermodynamic equilibrium is everywhere attained: that is, the coupled Saha's laws (1.3) hold everywhere even in presence of shock waves.

The *degree of ionization* and *fraction* for each gas are defined by $\alpha_{\mathrm{A}} = \frac{n_{\mathrm{i}}^{\mathrm{A}}}{n_{\mathrm{a}}^{\mathrm{A}} + n_{\mathrm{i}}^{\mathrm{A}}}, \alpha_{\mathrm{B}} = \frac{n_{\mathrm{i}}^{\mathrm{B}}}{n_{\mathrm{a}}^{\mathrm{B}} + n_{\mathrm{i}}^{\mathrm{B}}}$, and $\beta = \frac{N_{\mathrm{a}}^{\mathrm{A}} + N_{\mathrm{i}}^{\mathrm{A}}}{N_0} = \frac{n_{\mathrm{a}}^{\mathrm{A}} + n_{\mathrm{i}}^{\mathrm{A}}}{n_0}, 1 - \beta = \frac{N_{\mathrm{a}}^{\mathrm{B}} + N_{\mathrm{i}}^{\mathrm{B}}}{N_0} = \frac{n_{\mathrm{a}}^{\mathrm{B}} + n_{\mathrm{i}}^{\mathrm{B}}}{n_0}$, respectively. The density and molar mass of each gas are denoted by ρ_{A}, ρ_{B} and M_{A}, M_{B}, respectively. The pressure is a sum of partial pressures with respect to atoms, ions and electrons:

$$p = p_{\mathrm{a}} + p_{\mathrm{i}} + p_{\mathrm{e}} = p_{\mathrm{a}} + 2p_{\mathrm{i}}, \quad p_{\mathrm{j}} = kn_{\mathrm{j}}T \ (\mathrm{j} = \mathrm{a}, \mathrm{i}, \mathrm{e}).$$

Then by setting $\alpha = \beta\alpha_{\mathrm{A}} + (1-\beta)\alpha_{\mathrm{B}}$

$$\begin{aligned} p &= p_{\mathrm{a}}^{\mathrm{A}} + 2p_{\mathrm{i}}^{\mathrm{A}} + p_{\mathrm{a}}^{\mathrm{B}} + 2p_{\mathrm{i}}^{\mathrm{B}} = k\left(n_{\mathrm{a}}^{\mathrm{A}} + 2n_{\mathrm{i}}^{\mathrm{A}} + n_{\mathrm{a}}^{\mathrm{B}} + 2n_{\mathrm{i}}^{\mathrm{B}}\right)T \\ &= k\left[\left(n_{\mathrm{a}}^{\mathrm{A}} + n_{\mathrm{i}}^{\mathrm{A}}\right)(1+\alpha_{\mathrm{A}}) + \left(n_{\mathrm{a}}^{\mathrm{B}} + n_{\mathrm{i}}^{\mathrm{B}}\right)(1+\alpha_{\mathrm{B}})\right]T = kn_0\,(1+\alpha)\,T. \end{aligned}$$

By noticing $1 + \frac{n_{\mathrm{a}}^{\mathrm{A}}}{n_{\mathrm{i}}^{\mathrm{A}}} = \frac{1}{\alpha^{\mathrm{A}}}, 1 + \frac{n_{\mathrm{a}}^{\mathrm{B}}}{n_{\mathrm{i}}^{\mathrm{B}}} = \frac{1}{\alpha^{\mathrm{B}}}$, Saha's laws have the forms

$$\begin{aligned} \frac{n_{\mathrm{i}}^{\mathrm{A}} n_{\mathrm{e}}}{n_{\mathrm{a}}^{\mathrm{A}}} &= \left(\frac{\alpha_{\mathrm{A}}}{1-\alpha_{\mathrm{A}}}\right)\left(n_{\mathrm{i}}^{\mathrm{A}} + n_{\mathrm{i}}^{\mathrm{B}}\right) = \frac{n_0\alpha_{\mathrm{A}}\alpha}{1-\alpha_{\mathrm{A}}} = \frac{2G_{\mathrm{i}}^{\mathrm{A}}}{G_{\mathrm{a}}^{\mathrm{A}}}\left(\frac{2\pi m_{\mathrm{e}}kT}{h^2}\right)^{\frac{3}{2}} e^{-\frac{T_{\mathrm{A}}}{T}}, \\ \frac{n_{\mathrm{i}}^{\mathrm{B}} n_{\mathrm{e}}}{n_{\mathrm{a}}^{\mathrm{B}}} &= \left(\frac{\alpha_{\mathrm{B}}}{1-\alpha_{\mathrm{B}}}\right)\left(n_{\mathrm{i}}^{\mathrm{A}} + n_{\mathrm{i}}^{\mathrm{B}}\right) = \frac{n_0\alpha_{\mathrm{B}}\alpha}{1-\alpha_{\mathrm{B}}} = \frac{2G_{\mathrm{i}}^{\mathrm{B}}}{G_{\mathrm{a}}^{\mathrm{B}}}\left(\frac{2\pi m_{\mathrm{e}}kT}{h^2}\right)^{\frac{3}{2}} e^{-\frac{T_{\mathrm{B}}}{T}}. \end{aligned}$$

Thus we conclude that thermodynamic state equation has the form

$$\begin{aligned} p &= \frac{(1-\alpha_{\mathrm{A}})(1+\alpha)}{\alpha_{\mathrm{A}}\alpha}\frac{2G_{\mathrm{i}}^{\mathrm{A}}}{G_{\mathrm{a}}^{\mathrm{A}}}\frac{(2\pi m_{\mathrm{e}})^{\frac{3}{2}}(kT)^{\frac{5}{2}}}{h^3}e^{-\frac{T_{\mathrm{A}}}{T}} \\ &= \frac{(1-\alpha_{\mathrm{B}})(1+\alpha)}{\alpha_{\mathrm{B}}\alpha}\frac{2G_{\mathrm{i}}^{\mathrm{B}}}{G_{\mathrm{a}}^{\mathrm{B}}}\frac{(2\pi m_{\mathrm{e}})^{\frac{3}{2}}(kT)^{\frac{5}{2}}}{h^3}e^{-\frac{T_{\mathrm{B}}}{T}}. \end{aligned} \tag{1.4}$$

Also we have a compatibility condition

$$\frac{2G_{\mathrm{i}}^{\mathrm{A}}}{G_{\mathrm{a}}^{\mathrm{A}}}\frac{1-\alpha_{\mathrm{A}}}{\alpha_{\mathrm{A}}}e^{-\frac{T_{\mathrm{A}}}{T}} = \frac{2G_{\mathrm{i}}^{\mathrm{B}}}{G_{\mathrm{a}}^{\mathrm{B}}}\frac{1-\alpha_{\mathrm{B}}}{\alpha_{\mathrm{B}}}e^{-\frac{T_{\mathrm{B}}}{T}}. \tag{1.5}$$

We assume $T_{\mathrm{A}} < T_{\mathrm{B}} \leq 2T_{\mathrm{A}}$ and moreover

- Gases are well mixed so that: $\rho = \beta\rho_{\mathrm{A}} + (1-\beta)\rho_{\mathrm{B}}$
- Pressure of each gas has the form

$$p_{\mathrm{A}} = \frac{R\rho_{\mathrm{A}}}{M_{\mathrm{A}}}T(1+\alpha_{\mathrm{A}}), \quad p_{\mathrm{B}} = \frac{R\rho_{\mathrm{B}}}{M_{\mathrm{B}}}T(1+\alpha_{\mathrm{B}})$$

- Specific enthalpies are defined by

$$h_{\mathrm{A}} = \frac{5R}{2M_{\mathrm{A}}}T(1+\alpha_{\mathrm{A}}) + \frac{RT_{\mathrm{A}}}{M_{\mathrm{A}}}\alpha_{\mathrm{A}}, \quad h_{\mathrm{B}} = \frac{5R}{2M_{\mathrm{B}}}T(1+\alpha_{\mathrm{B}}) + \frac{RT_{\mathrm{B}}}{M_{\mathrm{B}}}\alpha_{\mathrm{B}}$$

- Local thermodynamic equilibrium is everywhere attained
- Macroscopic motion of the gas flow is one-dimensional.

We deduce from the above assumptions that the total pressure is

$$p = \beta p_{\mathrm{A}} + (1-\beta)p_{\mathrm{B}} = \beta\frac{R\rho_{\mathrm{A}}}{M_{\mathrm{A}}}T(1+\alpha_{\mathrm{A}}) + (1-\beta)\frac{R\rho_{\mathrm{B}}}{M_{\mathrm{B}}}T(1+\alpha_{\mathrm{B}}).$$

Thus

$$\frac{p}{\rho} = \frac{\beta\frac{R}{V}T(1+\alpha_{\mathrm{A}}) + (1-\beta)\frac{R}{V}T(1+\alpha_{\mathrm{B}})}{\beta\frac{M_{\mathrm{A}}}{V} + (1-\beta)\frac{M_{\mathrm{B}}}{V}} = \frac{RT\left[1+\beta\alpha_{\mathrm{A}} + (1-\beta)\alpha_{\mathrm{B}}\right]}{\beta M_{\mathrm{A}} + (1-\beta)M_{\mathrm{B}}}.$$

Denoting $\alpha = \beta\alpha_{\mathrm{A}} + (1-\beta)\alpha_{\mathrm{B}}$ and $M = \beta M_{\mathrm{A}} + (1-\beta)M_{\mathrm{B}}$, we obtain

$$p = \frac{R}{M}\rho T(1+\alpha) \tag{1.6}$$

which is the equation of state. The total specific enthalpy is

$$h = \frac{\beta M_{\mathrm{A}} h_{\mathrm{A}} + (1-\beta)M_{\mathrm{B}} h_{\mathrm{B}}}{\beta M_{\mathrm{A}} + (1-\beta)M_{\mathrm{B}}} = \frac{5RT}{2M}(1+\alpha) + \frac{R}{M}\left[\beta T_{\mathrm{A}}\alpha_{\mathrm{A}} + (1-\beta)T_{\mathrm{B}}\alpha_{\mathrm{B}}\right] \tag{1.7}$$

After showing some basic calculus lemmas, we construct physical entropy functions in Section 2. The system (1.1) is shown to be strictly hyperbolic and characteristic fields are computed in Section 3. However, unlike the ideal polytropic case, the forward and backward characteristic fields of the system are not genuinely nonlinear and we study the set where this happens in Section 4. We refer to [7], [18] for more information on systems of conservation laws. We study in Section 5 the relation between α_{A} and α_{B}. A detailed study of Hugoniot loci of the system is carried out in Section 6. Though Hugoniot loci are monotone in (T, α)-plane in a single monatomic case, they are not always monotone in the present mixed monatomic case: If β is sufficiently small, then they lose monotonicity at some base state. Thus the degree of ionization does not always increase across the shock front, even if the temperature increases. However we prove that the pressure actually increases as the temperature increases. In section 7, we study the existence and uniqueness of solutions to the Riemann Problem and prove that the Grüneisen coefficient is positive and the Liu-Smith strong condition is satisfied, which shows that the Riemann problem is well-posed.

2 Construction of Entropy Function

Introducing the *specific enthalpy* $h = e + pv$, we have

Proposition 2.1 (Maxwell's relations)**.**

$$\left(\frac{\partial s}{\partial p}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_p, \quad \left(\frac{\partial s}{\partial T}\right)_p = \frac{1}{T}\left(\frac{\partial h}{\partial T}\right)_p$$

As usual, the subscript T or p above means that the derivative is computed by holding the subscripted variable fixed. The specific volume v is expressed by (1.6) as $v = \frac{RT}{Mp}(1+\alpha)$ and the enthalpy is (1.7). The dimensionless entropy η is defined by $\eta = \frac{M}{R}S$. Consequently we have by Proposition 2.1

Lemma 2.1.

$$\left(\frac{\partial \eta}{\partial p}\right)_T = -\frac{1}{p}\left[1 + \alpha + T\left(\frac{\partial \alpha}{\partial T}\right)_p\right] \tag{2.1}$$

$$\left(\frac{\partial \eta}{\partial T}\right)_p = \frac{5}{2T}(1+\alpha) + \beta\left(\frac{5}{2} + \frac{T_{\mathrm{A}}}{T}\right)\left(\frac{\partial \alpha_{\mathrm{A}}}{\partial T}\right)_p + (1-\beta)\left(\frac{5}{2} + \frac{T_{\mathrm{A}}}{T}\right)\left(\frac{\partial \alpha_{\mathrm{B}}}{\partial T}\right)_p \tag{2.2}$$

Saha Equations: Setting

$$T_{\mathrm{A}} = \frac{\chi^{\mathrm{A}}}{k}, \quad T_{\mathrm{B}} = \frac{\chi^{\mathrm{B}}}{k}, \quad \mu_{\mathrm{A}}^{-1} = \frac{2G_{\mathrm{i}}^{\mathrm{A}}}{G_{\mathrm{a}}^{\mathrm{A}}}\frac{(2\pi m_{\mathrm{e}})^{\frac{3}{2}}k^{\frac{5}{2}}}{h^3}, \quad \mu_{\mathrm{B}}^{-1} = \frac{2G_{\mathrm{i}}^{\mathrm{B}}}{G_{\mathrm{a}}^{\mathrm{B}}}\frac{(2\pi m_{\mathrm{e}})^{\frac{3}{2}}k^{\frac{5}{2}}}{h^3},$$

we have from (1.4) and (1.5)

Lemma 2.2. *Saha's equations have the forms*

$$\left(\frac{1}{\alpha_{\mathrm{A}}}-1\right)\left(\frac{1}{\alpha}-1\right)=\frac{\mu_{\mathrm{A}}pe^{\frac{T_{\mathrm{A}}}{T}}}{T^{\frac{5}{2}}},\quad \left(\frac{1}{\alpha_{\mathrm{B}}}-1\right)\left(\frac{1}{\alpha}-1\right)=\frac{\mu_{\mathrm{B}}pe^{\frac{T_{\mathrm{B}}}{T}}}{T^{\frac{5}{2}}} \tag{2.3}$$

and the compatibility condition

$$\left(\frac{1}{\alpha_{\mathrm{A}}}-1\right)\frac{e^{-\frac{T_{\mathrm{A}}}{T}}}{\mu_{\mathrm{A}}}=\left(\frac{1}{\alpha_{\mathrm{B}}}-1\right)\frac{e^{-\frac{T_{\mathrm{B}}}{T}}}{\mu_{\mathrm{B}}}. \tag{2.4}$$

Computation of $\left(\frac{\partial\alpha_{\mathrm{A}}}{\partial p}\right)_T, \left(\frac{\partial\alpha_{\mathrm{A}}}{\partial T}\right)_p, \left(\frac{\partial\alpha_{\mathrm{B}}}{\partial p}\right)_T, \left(\frac{\partial\alpha_{\mathrm{B}}}{\partial T}\right)_p$**:** For the sake of brevity, we set

$$q_{\mathrm{A}}=\alpha_{\mathrm{A}}(1-\alpha_{\mathrm{A}}),\ q_{\mathrm{B}}=\alpha_{\mathrm{B}}(1-\alpha_{\mathrm{B}}),\ q=\beta q_{\mathrm{A}}+(1-\beta)q_{\mathrm{B}}.$$

Differentiating Saha's equations, we have two Pfaff equations

$$\frac{\alpha(1+\alpha)+\beta q_{\mathrm{A}}}{\alpha_{\mathrm{A}}^2\alpha^2}d\alpha_{\mathrm{A}}+\frac{(1-\alpha_{\mathrm{A}})(1-\beta)}{\alpha_{\mathrm{A}}\alpha^2}d\alpha_{\mathrm{B}}=-\frac{\mu_{\mathrm{A}}pe^{\frac{T_{\mathrm{A}}}{T}}}{T^{\frac{5}{2}}}\left[\frac{dp}{p}-\left(\frac{5}{2}+\frac{T_{\mathrm{A}}}{T}\right)\frac{dT}{T}\right], \tag{2.5}$$

$$\frac{(1-\alpha_{\mathrm{B}})\beta}{\alpha_{\mathrm{B}}\alpha^2}d\alpha_{\mathrm{A}}+\frac{\alpha(1+\alpha)+(1-\beta)q_{\mathrm{B}}}{\alpha_{\mathrm{B}}^2\alpha^2}d\alpha_{\mathrm{B}}=-\frac{\mu_{\mathrm{B}}pe^{\frac{T_{\mathrm{B}}}{T}}}{T^{\frac{5}{2}}}\left[\frac{dp}{p}-\left(\frac{5}{2}+\frac{T_{\mathrm{B}}}{T}\right)\frac{dT}{T}\right] \tag{2.6}$$

which constitute a system of linear equation of $d\alpha_{\mathrm{A}}$ and $d\alpha_{\mathrm{B}}$. By the inverse function theorem, we obtain

Lemma 2.3.

$$\left(\frac{\partial\alpha_{\mathrm{A}}}{\partial p}\right)_T=-\frac{\alpha(1+\alpha)q_{\mathrm{A}}}{p\left[\alpha(1+\alpha)+q\right]},\ \left(\frac{\partial\alpha_{\mathrm{B}}}{\partial p}\right)_T=-\frac{\alpha(1+\alpha)q_{\mathrm{B}}}{p\left[\alpha(1+\alpha)+q\right]} \tag{2.7}$$

$$\left(\frac{\partial\alpha_{\mathrm{A}}}{\partial T}\right)_p=\frac{\alpha(1+\alpha)q_{\mathrm{A}}}{T\left[\alpha(1+\alpha)+q\right]}\left(\frac{5}{2}+\frac{T_{\mathrm{A}}}{T}\right)+\frac{(1-\beta)q_{\mathrm{A}}q_{\mathrm{B}}(T_{\mathrm{A}}-T_{\mathrm{B}})}{T^2\left[\alpha(1+\alpha)+q\right]} \tag{2.8}$$

$$\left(\frac{\partial\alpha_{\mathrm{B}}}{\partial T}\right)_p=\frac{\alpha(1+\alpha)q_{\mathrm{B}}}{T\left[\alpha(1+\alpha)+q\right]}\left(\frac{5}{2}+\frac{T_{\mathrm{B}}}{T}\right)+\frac{\beta q_{\mathrm{A}}q_{\mathrm{B}}(T_{\mathrm{B}}-T_{\mathrm{A}})}{T^2\left[\alpha(1+\alpha)+q\right]} \tag{2.9}$$

We deduce from this lemma

$$\left(\frac{\partial\alpha_{\mathrm{A}}}{\partial T}\right)_p=-\frac{p}{T}\left(\frac{5}{2}+\frac{T_{\mathrm{A}}}{T}\right)\left(\frac{\partial\alpha_{\mathrm{A}}}{\partial p}\right)_T+\frac{(1-\beta)q_{\mathrm{A}}q_{\mathrm{B}}(T_{\mathrm{A}}-T_{\mathrm{B}})}{T^2\left[\alpha(1+\alpha)+q\right]}$$

$$\left(\frac{\partial\alpha_{\mathrm{B}}}{\partial T}\right)_p=-\frac{p}{T}\left(\frac{5}{2}+\frac{T_{\mathrm{B}}}{T}\right)\left(\frac{\partial\alpha_{\mathrm{B}}}{\partial p}\right)_T+\frac{\beta q_{\mathrm{A}}q_{\mathrm{B}}(T_{\mathrm{B}}-T_{\mathrm{A}})}{T^2\left[\alpha(1+\alpha)+q\right]}.$$

Thus we obtain useful lemmas:

Lemma 2.4.

$$-\frac{T}{p}\left(\frac{\partial\alpha}{\partial T}\right)_p=\frac{5}{2}\left(\frac{\partial\alpha}{\partial p}\right)_T+\frac{\beta T_{\mathrm{A}}}{T}\left(\frac{\partial\alpha_{\mathrm{A}}}{\partial p}\right)_T+\frac{(1-\beta)T_{\mathrm{B}}}{T}\left(\frac{\partial\alpha_{\mathrm{B}}}{\partial p}\right)_T$$

Lemma 2.5.

$$\left(\frac{\partial\eta}{\partial p}\right)_T=-\frac{1+\alpha}{p}+\beta\left(\frac{5}{2}+\frac{T_{\mathrm{A}}}{T}\right)\left(\frac{\partial\alpha_{\mathrm{A}}}{\partial p}\right)_T+(1-\beta)\left(\frac{5}{2}+\frac{T_{\mathrm{B}}}{T}\right)\left(\frac{\partial\alpha_{\mathrm{B}}}{\partial p}\right)_T \tag{2.10}$$

We will construct the physical entropy function for the present model system. Integrating (2.10) with respect to p, we have

Lemma 2.6. *The dimensionless entropy $\eta = \frac{M}{R}s$ has the form*

$$\begin{aligned}\eta(p,T) &= \log\alpha + \beta\log\alpha_{\rm A} + (1-\beta)\log\alpha_{\rm B} - 2\beta\log(1-\alpha_{\rm A}) - 2(1-\beta)\log(1-\alpha_{\rm B}) \\ &\quad + \beta\left(\frac{5}{2}+\frac{T_{\rm A}}{T}\right)\alpha_{\rm A} + (1-\beta)\left(\frac{5}{2}+\frac{T_{\rm B}}{T}\right)\alpha_{\rm B} + \mathcal{H}(T). \end{aligned} \tag{2.11}$$

where $\mathcal{H}$ is an arbitrary function of T.

By differentiating the above expression and employing (2.1) and (2.2), the form of $\mathcal{H}(T)$ is determined up to constant.

Theorem 2.1 ([1])**.** *The dimensionless entropy function has the form*

$$\begin{aligned}\eta &= \log\left[\beta\alpha_{\rm A} + (1-\beta)\alpha_{\rm B}\right] \\ &\quad + \beta\left[\log\alpha_{\rm A} - 2\log(1-\alpha_{\rm A}) + \frac{T_{\rm A}}{T}\right] + (1-\beta)\left[\log\alpha_{\rm B} - 2\log(1-\alpha_{\rm B}) + \frac{T_{\rm B}}{T}\right] \\ &\quad + \beta\left(\frac{5}{2}+\frac{T_{\rm A}}{T}\right)\alpha_{\rm A} + (1-\beta)\left(\frac{5}{2}+\frac{T_{\rm B}}{T}\right)\alpha_{\rm B} + \text{const.}\end{aligned}$$

3 Equations of Ionized Gas Dynamics

For studying thermodynamic properties of the system (1.1), the Lagrangian equations [18] are convenient

$$\begin{cases} v_t - u_\xi = 0, \\ u_t + p_\xi = 0, \\ \left(e + \frac{1}{2}u^2\right)_t + (pu)_\xi = 0 \end{cases} \tag{3.1}$$

where p : pressure, v : specific volume, e : specific internal energy and u : flow velocity. For C^1 solutions, equation $(3.1)_3$ can be written as [2] $s_t = 0$.

Characteristic speeds and vector fields: The associated quasi-linear equations are

$$p_t - \frac{u_x}{v_p} = 0, \quad u_t + p_x = 0, \quad s_t = 0$$

in the (p,u,s) state space and we find by direct computation that the characteristic speeds and the corresponding vector fields are

$$\lambda_\pm(U) = \pm\frac{1}{\sqrt{-v_p}}, \quad \lambda_0(U) = 0, \quad \boldsymbol{r}_\pm(U) = \begin{bmatrix} \pm 1 \\ \sqrt{-v_p} \\ 0 \end{bmatrix}, \quad \boldsymbol{r}_0(U) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We note that characteristic speeds and characteristic vectors are all thermodynamic quantities.

[2] The equation $s_t = 0$ is transformed to $(\rho S)_t + (\rho u S)_x = 0$ in Eulerian coordinates.

For further computation, we adopt (p, u, T) as a set of state variables. Since $v_t - u_\xi = v_p p_t + v_T T_t - u_\xi = 0$ and $\eta_t = \eta_p p_t + \eta_T T_t = 0$, we can write system (3.1) in the form

$$\begin{cases} p_t - \frac{\eta_T}{v_p\eta_T - v_T\eta_p} u_\xi = 0, \\ u_t + p_\xi = 0, \\ T_t + \frac{\eta_p}{v_p\eta_T - v_T\eta_p} u_\xi = 0. \end{cases} \tag{3.2}$$

Characteristic speeds and vector fields are computed as the following.

Lemma 3.1. *The characteristic speeds and the corresponding characteristic vector fields of the system* (3.2) *are*

$$\lambda_\pm = \pm\sqrt{-\frac{\eta_T}{v_p\eta_T - v_T\eta_p}}, \quad \lambda_0 = 0, \quad \boldsymbol{r}_\pm = \begin{bmatrix} \pm 1 \\ \frac{1}{\sqrt{-\frac{\eta_T}{v_p\eta_T - v_T\eta_p}}} \\ \mp\frac{\eta_p}{\eta_T} \end{bmatrix}, \quad \boldsymbol{r}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The 0*-characteristic field is linearly degenerate; a pair of Riemann invariants for* λ_0 *is* $\{u, p\}$*. A Riemann invariant for both* $\lambda_\pm$ *is* η*. The characteristic speeds of system* (1.1) *are then* $u + \frac{1}{\rho}\lambda_\pm$ *and* u*.*

Computation of $\lambda_\pm$: Fore the sake of brevity, let us introduce the quantities:

$$\begin{aligned} q &= \beta q_{\mathrm{A}} + (1-\beta) q_{\mathrm{B}}, \\ \Sigma &= \alpha(1+\alpha) + \beta q_{\mathrm{A}} + (1-\beta) q_{\mathrm{B}} = \alpha(1+\alpha) + q \\ \Phi &= \beta q_{\mathrm{A}} \left(\frac{5}{2} + \frac{T_{\mathrm{A}}}{T}\right)^2 + (1-\beta) q_{\mathrm{B}} \left(\frac{5}{2} + \frac{T_{\mathrm{B}}}{T}\right)^2, \\ \Psi &= \beta q_{\mathrm{A}} \left(\frac{15}{4} + \frac{3T_{\mathrm{A}}}{T} + \frac{T_{\mathrm{A}}^2}{T^2}\right) + (1-\beta) q_{\mathrm{B}} \left(\frac{15}{4} + \frac{3T_{\mathrm{B}}}{T} + \frac{T_{\mathrm{B}}^2}{T^2}\right), \\ \Omega &= \frac{\beta(1-\beta) q_{\mathrm{A}} q_{\mathrm{B}} \left(T_{\mathrm{A}} - T_{\mathrm{B}}\right)^2}{T^2}. \end{aligned}$$

Substituting (2.7), (2.8), (2.9) into (2.2) and (2.10), we obtain

$$\left(\frac{\partial \eta}{\partial p}\right)_T = -\frac{1+\alpha}{p} - \frac{\alpha(1+\alpha)\left[\beta\left(\frac{5}{2} + \frac{T_{\mathrm{A}}}{T}\right) q_{\mathrm{A}} + (1-\beta)\left(\frac{5}{2} + \frac{T_{\mathrm{B}}}{T}\right) q_{\mathrm{B}}\right]}{p\Sigma} \tag{3.3}$$

$$\left(\frac{\partial \eta}{\partial T}\right)_p = \frac{5}{2T}(1+\alpha) + \frac{\alpha(1+\alpha)\Phi + \Omega}{T\Sigma}. \tag{3.4}$$

Since $v = \frac{a^2}{p} T(1+\alpha)$ $(a^2 = \frac{R}{M})$, we have by applying Lemma 2.3

$$a^{-2}\left(\frac{\partial v}{\partial p}\right)_T = -\frac{T(1+\alpha)}{p^2} - \frac{T\alpha(1+\alpha)q}{p^2\Sigma}, \quad a^{-2}\left(\frac{\partial v}{\partial T}\right)_p = -\left(\frac{\partial \eta}{\partial p}\right)_T.$$

Theorem 3.1 ([1])**.** *The characteristic speeds* $\lambda_\pm$ *have the forms* $\lambda_\pm = \pm\lambda$ *where*

$$\lambda = \frac{p}{a\sqrt{T}(1+\alpha)} \sqrt{\frac{\frac{5}{2}(1+\alpha)\Sigma + \alpha(1+\alpha)\Phi + \Omega}{\frac{3}{2}\Sigma + \alpha\Psi + \Omega}}. \tag{3.5}$$

Remark 3.1 (Isentropes)**.** *In the (p,u,T) state space, an integral curve of a characteristic vector field $\boldsymbol{r}$ is a solution to the system of equations $\dfrac{d}{d\tau}\begin{bmatrix} p \\ u \\ T \end{bmatrix} = \boldsymbol{r}$ where $\boldsymbol{r}$ stands for $\boldsymbol{r}_\pm$ or $\boldsymbol{r}_0$. We have*

$$\frac{d\eta}{d\tau} = \frac{\partial\eta}{\partial p}\frac{dp}{d\tau} + \frac{\partial\eta}{\partial T}\frac{dT}{d\tau} = \pm\left(\frac{\partial\eta}{\partial p} - \frac{\partial\eta}{\partial T}\frac{\eta_p}{\eta_T}\right) = 0$$

for $\boldsymbol{r}_\pm$, and $p = \text{const.}, u = \text{const.}$ for $\boldsymbol{r}_0$. Thus the thermodynamic part of an integral curve is $\eta = const.$ for $1,2$-characteristic directions and $p = const.$ for 0-characteristic field. A curve $\eta = const.$ is called an isentrope*. Since $\left(\frac{\partial\eta}{\partial\alpha_\mathrm{A}}\right)_T > 0$ (see the proof of Lemma 7.3), an isentrope is represented by the graph of a differentiable function $\alpha_\mathrm{A} = \alpha_\mathrm{A}(T)$ defined on $T \in (0,\infty)$.*

4 Genuine Nonlinearity (convexity) and Inflection Loci

Now, we study the convexity of the forward and backward fields; each characteristic direction having the eigenvalue $\lambda_\pm$ is called *genuinely nonlinear* if $\boldsymbol{r}_\pm\nabla\lambda_\pm \neq 0$. We have chosen characteristic vectors $\boldsymbol{r}_\pm$ so that

$$\boldsymbol{r}_\pm\nabla\lambda_\pm = \frac{v_{pp}}{2(-v_p)^{\frac{3}{2}}} = \frac{\partial\lambda}{\partial p} - \frac{\eta_p}{\eta_T}\frac{\partial\lambda}{\partial T}. \tag{4.1}$$

Hence, genuine nonlinearity implies strict convexity (or concavity) of v as a function of p for fixed s. We refer to [15] for more insight about the failure of this condition and we will see in Theorem 6.5 that the entropy increases across the shock front if $\boldsymbol{r}_\pm\nabla\lambda_\pm > 0$.

It is convenient to consider a differential operator

$$\mathcal{R} = \Sigma\left[\eta_T\left(\frac{\partial}{\partial p}\right)_T - \eta_p\left(\frac{\partial}{\partial T}\right)_p\right]$$

which is proportional to $\boldsymbol{r}_\pm\nabla$. Computation of $\mathcal{R}\lambda$ is simple but tedious. Outline of computation is shown in [1] and the expression obtained is very heavy.

The *inflection locus* is defined to be the point set

$$\mathcal{I} = \left\{(T,\alpha_\mathrm{A});\ \boldsymbol{r}_\pm\nabla\lambda_\pm = 0,\ T > 0,\ 0 < \alpha_\mathrm{A} < 1\right\}.$$

Since $\boldsymbol{r}_+\nabla\lambda_+ = \boldsymbol{r}_-\nabla\lambda_-$, both cases lead to the same result. Since $\boldsymbol{r}_\pm\nabla\lambda_\pm > 0$ for sufficiently large T, we observe that $\mathcal{I}$ is located in a finite region. However it is difficult to get a sketch of $\mathcal{I}$ by purely mathematical reasoning and Fig. 1 shows results of numerical computations.

On the other hand, it is possible to extract from the heavy expressions asymptotics of the inflection locus for $T \to 0$. Since α_B is negligible compared with α_A, we observe that there are two branches such that $\frac{\alpha_\mathrm{A}}{T^2} \to 0$ or $\frac{\alpha_\mathrm{A}}{T^2} \to \infty$. Following theorem is a generalization of [2] Proposition 4.2.

Theorem 4.1 ([1])**.** *For $T \to 0$, the inflection locus has two branches*

$$\text{(1)}\quad \alpha_\mathrm{A} \sim \frac{60}{\beta}\left(\frac{T}{T_\mathrm{A}}\right)^3, \quad \alpha_\mathrm{B} \sim \frac{60\mu_\mathrm{A}}{\beta\mu_\mathrm{B}}\left(\frac{T}{T_\mathrm{A}}\right)^3 e^{-\frac{T_\mathrm{B}-T_\mathrm{A}}{T}}$$

$$\text{(2)}\quad \alpha_\mathrm{A} \sim \frac{1}{\beta}\left(\frac{T}{T_\mathrm{A}}\right)^{\frac{3}{2}}, \quad \alpha_\mathrm{B} \sim \frac{\mu_\mathrm{A}}{\beta\mu_\mathrm{B}}\left(\frac{T}{T_\mathrm{A}}\right)^{\frac{3}{2}} e^{-\frac{T_\mathrm{B}-T_\mathrm{A}}{T}}$$

and we conclude that the characteristic directions of $\lambda_\pm$ are not genuinely nonlinear in a neighbourhood of $(T,\alpha_\mathrm{A}) = (0,0)$.

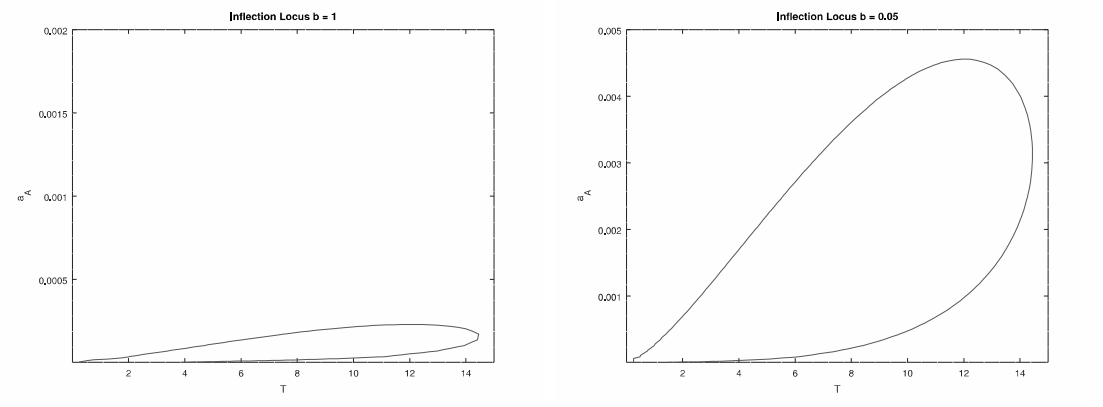

Figure 1: $T_{\mathrm{A}} = 1576.0, T_{\mathrm{B}} = 2853.2$ left: $\beta = 1$(monatomic), right: $\beta = 0.05$.

5 Compatibility Condition

The compatibility condition (2.4) constitutes a thermodynamic state space.

Lemma 5.1. *The compatibility condition has the form*

$$\alpha_{\mathrm{B}} = \frac{\mu_{\mathrm{A}}\alpha_{\mathrm{A}} e^{-\frac{T_{\mathrm{B}}-T_{\mathrm{A}}}{T}}}{\mu_{\mathrm{A}}\alpha_{\mathrm{A}} e^{-\frac{T_{\mathrm{B}}-T_{\mathrm{A}}}{T}} + \mu_{\mathrm{B}}(1-\alpha_{\mathrm{A}})}. \tag{5.1}$$

If $\alpha_{\mathrm{A}} \to 0$, *then* $\alpha_{\mathrm{B}} \to 0$ *and we have*

$$\alpha_{\mathrm{B}} = \frac{\mu_{\mathrm{A}}}{\mu_{\mathrm{B}}}\alpha_{\mathrm{A}} e^{-\frac{T_{\mathrm{B}}-T_{\mathrm{A}}}{T}} \left[1 + O(1)\alpha_{\mathrm{A}}\right]. \tag{5.2}$$

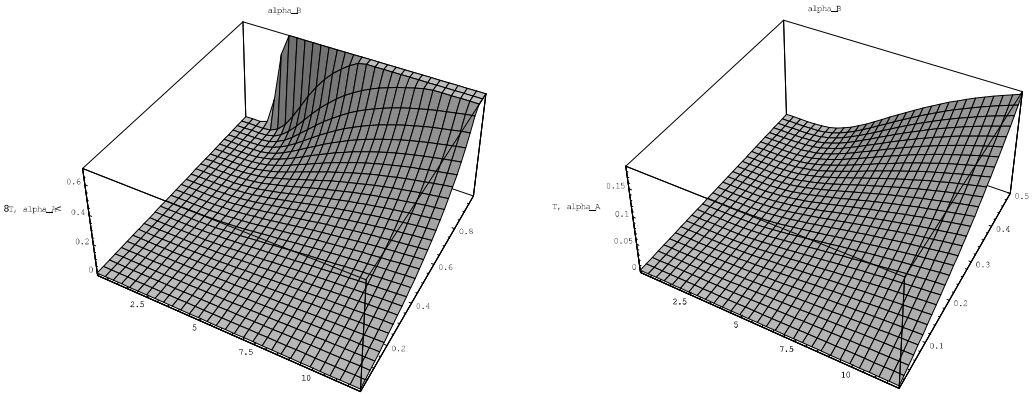

Figure 2: State space $T_{\mathrm{A}} = 15, T_{\mathrm{B}} = 28$, left: $0 < \alpha_{\mathrm{A}} < 1, 0 < T < 12$, right: $0 < \alpha_{\mathrm{A}} < 0.5, 0 < T < 12$

For A: hydrogen atom and B: helium atom, $\frac{\mu_{\mathrm{A}}}{\mu_{\mathrm{B}}} = 4$.

Incidentally, we find

$$\alpha_{\mathrm{B}}(1-\alpha_{\mathrm{B}}) = \frac{\mu_{\mathrm{A}}\mu_{\mathrm{B}}\alpha_{\mathrm{A}}(1-\alpha_{\mathrm{A}}) e^{-\frac{T_{\mathrm{B}}-T_{\mathrm{A}}}{T}}}{\left[\mu_{\mathrm{A}}\alpha_{\mathrm{A}} e^{-\frac{T_{\mathrm{B}}-T_{\mathrm{A}}}{T}} + \mu_{\mathrm{B}}(1-\alpha_{\mathrm{A}})\right]^2}$$

and we have derivatives of $\alpha_{\rm B}$ in the forms

$$\left(\frac{\partial\alpha_{\rm B}}{\partial T}\right)_{\alpha_{\rm A}} = \frac{(T_{\rm B}-T_{\rm A})\alpha_{\rm B}(1-\alpha_{\rm B})}{T^2}, \quad \left(\frac{\partial\alpha_{\rm B}}{\partial\alpha_{\rm A}}\right)_{T} = \frac{\alpha_{\rm B}(1-\alpha_{\rm B})}{\alpha_{\rm A}(1-\alpha_{\rm A})}$$

showing that $\left(\frac{\partial\alpha_{\rm B}}{\partial T}\right)_{\alpha_{\rm A}}, \left(\frac{\partial\alpha_{\rm B}}{\partial\alpha_{\rm A}}\right)_{T} > 0$. By setting for brevity $q = \beta q_{\rm A} + (1-\beta)q_{\rm B}$, $Q_{\rm BA} = \frac{(1-\beta)(T_{\rm B}-T_{\rm A})q_{\rm B}}{T}$, derivatives of α take the forms

Lemma 5.2.

$$\left(\frac{\partial\alpha}{\partial T}\right)_{\alpha_{\rm A}} = \frac{Q_{\rm BA}}{T}, \quad \left(\frac{\partial\alpha}{\partial\alpha_{\rm A}}\right)_{T} = \frac{q}{q_{\rm A}}. \tag{5.3}$$

In the following sections, we shall adopt T and $\alpha_{\rm A}$ as a set of independent thermodynamic state variables.

6 Thermodynamic Hugoniot Loci

In the one-dimensional gas dynamics, two constant states separated by a shock front $x = \sigma t$ constitute a *weak solution,* if and only if the *Rankine-Hugoniot conditions*

$$\begin{cases} \sigma[\rho] = [\rho u], \\ \sigma[\rho u] = [\rho u^2 + p], \\ \sigma[\rho E] = [\rho u E + p u]. \end{cases} \tag{6.1}$$

are satisfied. Here we denote $[\rho] = \rho_+ - \rho_-$, where $\rho_\pm$ denote the right and left limits, respectively, of ρ with respect to x at $x = \sigma t$; the same notation is used for the other variables.

If $[\rho] = 0$ then $[u] = 0$ by $(6.1)_1$ and $[p] = 0$ by $(6.1)_2$; in this case, $\sigma = u_\pm$: the speed is equal to the flow velocity and the discontinuity is called a *contact discontinuity*. From now on we discuss the discontinuity corresponding to the forward and backward characteristic fields having charcteristic speeds $\lambda_\pm$ and assume $[\rho] \neq 0$. In this case σ is eliminated from the first equation and by substituting it into the other two equations, the conditions (6.1) are reduced to

$$\begin{cases} (u_+ - u_-)^2 + (p_+ - p_-)(v_+ - v_-) &= 0: & \text{kinetic condition,} \\ e_+ - e_- + \frac{1}{2}(p_+ + p_-)(v_+ - v_-) &= 0: & \text{thermodynamic condition.} \end{cases} \tag{6.2}$$

In the following, we consider a single forward shock front; we fix a constant state (p_+, u_+, T_+) and consider $(p, u, T) = (p_-, u_-, T_-)$ as a set of state variables. Under this notation, (6.2) is a set of equations for *Hugoniot locus* of (p_+, u_+, T_+). For brevity, we call solutions to $(6.2)_1$ and $(6.2)_2$, respectively, the *kinetic* and *thermodynamic Hugoniot loci.*

In this section we will give a precise description of thermodynamic Hugoniot loci for the present model system and evaluate, in particular, change of the thermodynamic variables along them; this analysis is fundamental for the study of shock waves (see [3]).

The right thermodynamic state is denoted by (p_+, T_+) and the left state (p, T). The thermodynamic Rankine-Hugoniot condition is written as[3]

$$\begin{aligned} &T\,(1+\alpha)\left(4+\frac{p_+}{p}\right) + 2\left[\beta T_{\rm A}\alpha_{\rm A} + (1-\beta)T_{\rm B}\alpha_{\rm B}\right] \\ &= T_+\left(1+\alpha^+\right)\left(4+\frac{p}{p_+}\right) + 2\left[\beta T_{\rm A}\alpha_{\rm A}^+ + (1-\beta)T_{\rm B}\alpha_{\rm B}^+\right] \end{aligned}$$

[3]For the sake of convenience, we adopt the notation $\alpha_{\rm A}^\pm$ instead of $\alpha_{{\rm A}\pm}$.

By (1.4) the pressure is expressed as

$$p=\frac{(1-\alpha_{\rm A})(1+\alpha)}{\mu_{\rm A}\alpha_{\rm A}\alpha}T^{\frac{5}{2}}e^{-\frac{T_{\rm A}}{T}}=\frac{(1-\alpha_{\rm B})(1+\alpha)}{\mu_{\rm B}\alpha_{\rm B}\alpha}T^{\frac{5}{2}}e^{-\frac{T_{\rm B}}{T}}. \tag{6.3}$$

and thus

$$\frac{p_-}{p_+}=\frac{(1-\alpha_{\rm A}^-)(1+\alpha^-)\alpha_{\rm A}^+\alpha^+}{(1-\alpha_{\rm A}^+)(1+\alpha^+)\alpha_{\rm A}^-\alpha^-}\left(\frac{T_-}{T_+}\right)^{\frac{5}{2}}e^{-\frac{T_{\rm A}}{T_-}+\frac{T_{\rm A}}{T_+}},\quad \frac{v_-}{v_+}=\frac{p_+T_-(1+\alpha^-)}{p_-T_+(1+\alpha^+)}. \tag{6.4}$$

Consequently by setting $T=T_-,\alpha=\alpha^-,\alpha_{\rm A}=\alpha_{\rm A}^-$ and $\alpha_{\rm B}=\alpha_{\rm B}^-$, and defining

$$\begin{aligned}H(T,\alpha_{\rm A})=&\,T\,(1+\alpha)\left[4+\frac{(1-\alpha_{\rm A}^+)(1+\alpha^+)\alpha_{\rm A}\alpha}{(1-\alpha_{\rm A})(1+\alpha)\alpha_{\rm A}^+\alpha^+}\left(\frac{T_+}{T}\right)^{\frac{5}{2}}e^{-\frac{T_{\rm A}}{T_+}+\frac{T_{\rm A}}{T}}\right]\\&+2\left[\beta\alpha_{\rm A}T_{\rm A}+(1-\beta)\alpha_{\rm B}T_{\rm B}\right]\\&-T_+\left(1+\alpha^+\right)\left[4+\frac{(1-\alpha_{\rm A})(1+\alpha)\alpha_{\rm A}^+\alpha^+}{(1-\alpha_{\rm A}^+)(1+\alpha^+)\alpha_{\rm A}\alpha}\left(\frac{T}{T_+}\right)^{\frac{5}{2}}e^{-\frac{T_{\rm A}}{T}+\frac{T_{\rm A}}{T_+}}\right]\\&-2\left[\beta\alpha_{\rm A}^+T_{\rm A}+(1-\beta)\alpha_{\rm B}^+T_{\rm B}\right],\end{aligned} \tag{6.5}$$

the thermodynamic Rankine-Hugoniot condition is represented by $H(T,\alpha_{\rm A})=0$.

Asymptotics: We have the following asymptotic formulas.

Theorem 6.1 ([1])**.** *On the thermodynamic Hugoniot locus* (6.5)*, if* $T\to 0$*, then* $\alpha_{\rm A},\alpha_{\rm B}\to 0$ *and by setting*

$$A=\sqrt{\frac{\alpha_{\rm A}^+\alpha^+\left\{4\left(1+\alpha^+\right)+2\left[\beta\alpha_{\rm A}^+\frac{T_{\rm A}}{T_+}+(1-\beta)\alpha_{\rm B}^+\frac{T_{\rm B}}{T_+}\right]\right\}e^{\frac{T_{\rm A}}{T_+}}}{\left[\beta+\frac{\mu_{\rm A}}{\mu_{\rm B}}(1-\beta)\right](1-\alpha_{\rm A}^+)(1+\alpha^+)}},$$

we have

$$\alpha_{\rm A}\sim A\left(\frac{T}{T_+}\right)^{\frac{3}{4}}e^{-\frac{T_{\rm A}}{2T}},\quad \alpha_{\rm B}\sim\frac{A\mu_{\rm A}}{\mu_{\rm B}}\left(\frac{T}{T_+}\right)^{\frac{3}{4}}e^{-\frac{2T_{\rm B}-T_{\rm A}}{2T}}. \tag{6.6}$$

On the other hand, if $T\to\infty$*, then* $\alpha_{\rm A},\alpha_{\rm B}\to 1$ *and*

$$1-\alpha_{\rm A}\sim\frac{4(1-\alpha_{\rm A}^+)}{\alpha_{\rm A}^+\alpha^+}\left(\frac{T}{T_+}\right)^{-\frac{3}{2}}e^{-\frac{T_{\rm A}}{T_+}},\ 1-\alpha_{\rm B}\sim\frac{4\mu_{\rm A}(1-\alpha_{\rm A}^+)}{\mu_{\rm B}\alpha_{\rm A}^+\alpha^+}\left(\frac{T}{T_+}\right)^{-\frac{3}{2}}e^{-\frac{T_{\rm A}}{T_+}} \tag{6.7}$$

Loss of Monotonicity: For a single monatomic gas, Hugoniot loci are graphs of strictly increasing functions in (T,α) plane ([2], [3]). We will show in this subsection that it is not always the case for mixed monatomic gases.

By direct computation, we have

$$\begin{aligned}\left(\frac{\partial H}{\partial\alpha_{\rm A}}\right)_T=&\frac{T\,(1+\alpha)}{q_{\rm A}}\left[1+\frac{q}{\alpha(1+\alpha)}\right]\frac{p_+}{p}+\frac{T_+\left(1+\alpha^+\right)}{q_{\rm A}}\left[1+\frac{q}{\alpha(1+\alpha)}\right]\frac{p}{p_+}\\&+\frac{Tq}{q_{\rm A}}\left(4+\frac{p_+}{p}\right)+\frac{2}{q_{\rm A}}\left[\beta q_{\rm A}T_{\rm A}+(1-\beta)q_{\rm B}T_{\rm B}\right],\end{aligned} \tag{6.8}$$

showing that $\left(\frac{\partial H}{\partial\alpha_{\rm A}}\right)_T>0$. We have

Theorem 6.2 ([1]). *For every $T > 0$, there is a unique $0 < \alpha_{\rm A} < 1$ such that $H(T, \alpha_{\rm A}) = 0$ and the function $\alpha_{\rm A} = \alpha_{\rm A}(T)$ is differentiable.*

In the same way

$$\left(\frac{\partial H}{\partial T}\right)_{\alpha_{\rm A}} = 4(1+\alpha)\left[1 + \frac{Q_{\rm BA}}{1+\alpha}\left(1 + \frac{T_{\rm B}}{2T}\right)\right]$$
$$- (1+\alpha)\left(\frac{3}{2} + \frac{T_{\rm A}}{T} - \frac{Q_{\rm BA}}{\alpha}\right)\frac{p_+}{p} - (1+\alpha^+)\frac{T_+}{T}\left[\frac{5}{2} + \frac{T_{\rm A}}{T} - \frac{Q_{\rm BA}}{\alpha(1+\alpha)}\right]\frac{p}{p_+}. \tag{6.9}$$

Theorem 6.3 ([1]). *If β is sufficiently close to 0, then*

$$\frac{d\alpha_{\rm A}}{dT}(T_+) = -\frac{\left(\frac{\partial H}{\partial T}\right)_T (T_+, \alpha_{\rm A}^+)}{\left(\frac{\partial H}{\partial \alpha_{\rm A}}\right)_{\alpha_{\rm A}} (T_+, \alpha_{\rm A}^+)} < 0,$$

showing that $\alpha_{\rm A}$ is a decreasing function of T in a neighbourhood of $T = T_+$,

Outline of Proof: Let us study the sign of $\left(\frac{\partial H}{\partial T}\right)_{\alpha_{\rm A}}$ at $(T_+, \alpha_{\rm A}^+)$. We find by the above expression that $\left(\frac{\partial H}{\partial T}\right)_{\alpha_{\rm A}} > 0$ if and only if

$$F(T, \alpha_{\rm A}) = \frac{1}{4}\left(\frac{3}{2} + \frac{T_{\rm A}}{T} - \frac{Q_{\rm BA}}{\alpha}\right)\frac{p_+}{p}$$
$$+ \frac{T_+(1+\alpha^+)}{4T(1+\alpha)}\left[\frac{5}{2} + \frac{T_{\rm A}}{T} - \frac{Q_{\rm BA}}{\alpha(1+\alpha)}\right]\frac{p}{p_+} - \left[1 + \frac{Q_{\rm BA}}{1+\alpha}\left(1 + \frac{T_{\rm B}}{2T}\right)\right]$$

is negative. If $\beta = 0$, then

$$F(T_+, \alpha_{\rm A}^+) = \frac{T_{\rm A}}{2T}\left[1 - \frac{1-\alpha_{\rm B}}{1+\alpha_{\rm B}}\left(\frac{2+5\alpha_{\rm B}}{2} + \frac{T_{\rm B}\alpha_{\rm B}}{T}\right)\frac{T_{\rm B} - T_{\rm A}}{T_{\rm A}}\right].$$

Obviously for any $\alpha_{\rm B} > 0$, there is some $T > 0$ so that the above expression is negative.

Pressure Change: Let us study the behaviour of the pressure. By direct computation, we have

$$\frac{1}{p}\frac{dp}{dT} = \frac{\left[1 + \frac{q}{\alpha(1+\alpha)}\right]\left(\frac{\partial H}{\partial T}\right)_{\alpha_{\rm A}} + \frac{q_{\rm A}}{T}\left[\frac{5}{2} + \frac{T_{\rm A}}{T} - \frac{Q_{\rm BA}}{\alpha(1+\alpha)}\right]\left(\frac{\partial H}{\partial \alpha_{\rm A}}\right)_T}{q_{\rm A}\left(\frac{\partial H}{\partial \alpha_{\rm A}}\right)_T}.$$

Substituting (6.8) and (6.9) into the above expression, we obtain

Theorem 6.4 ([1]). *The pressure p strictly increases along the Hugoniot locus as the temperature T increases.*

It is well known that (Bethe [5], [12, §86]): if (p_0, u_0, s_0) and (p_1, u_1, s_1) are connected by a shock front, then

$$s_1 - s_0 = \frac{1}{12T_0}\left(\frac{\partial^2 v}{\partial p_0^2}\right)_s (p_1 - p_0)^3 + O(1)(p_1 - p_0)^4. \tag{6.10}$$

By virtue of this theorem, the physical entropy inceases as the temperature increases on condition that $v_{pp}\big|_{p=p_0} > 0$. For discontinuities with arbitrary amplitude, the following theorem is known.

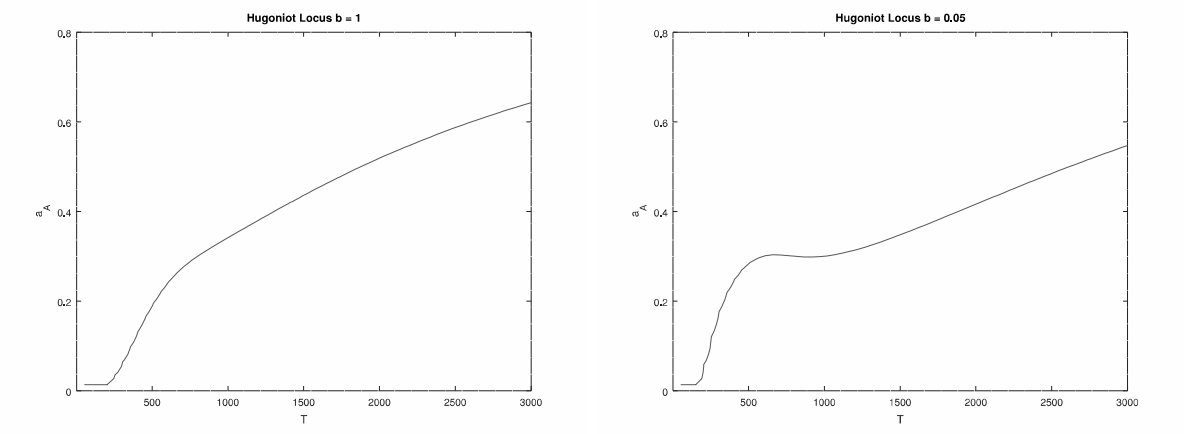

Figure 3: $T_{\mathrm{A}} = 1576.0, T_{\mathrm{B}} = 2853.2, T = 800, \alpha_{\mathrm{A}}^{+} = 0.3$ left: $\beta = 1$(single monatomic), right: $\beta = 0.05$

Theorem 6.5 (Bethe-Weyl)**.** *The thermodynamic Hugoniot locus of the state* (v_0, s_0) *intersects each isentrope at least once. Moreover, if* $p_{vv} > 0$ (*equivalently* $v_{pp} > 0$) *along an isentrope, then* $|u - \sigma| < c$*, if* $v_1 < v_0$*. while the opposite inequalities hold if* $v_1 > v_0$*.*

The kinetic condition (6.2) shows that $v_1 < v_0$ if and only if $p_1 > p_0$. Hence the Lax condition (see [7], [18]) is satisfied even for large $|p_1 - p_0|$ as long as $p_1 > p_0$. Proof is found in [5], [15, (3.44)] and Weyl [20]. We may also call this "shock wave", however the physical entropy does not necessarily increase.

Next theorem in [5] guarantees increase of the physical entropy. Let Γ denote the *Grüneisen coeficient* Γ whose definition is found and $\Gamma > 0$ is shown in the next section

Theorem 6.6 (Bethe)**.** *Suppose that* $p_{vv} > 0$ (*equivalently* $v_{pp} > 0$) *and* $\Gamma \geq -2$*. Then the thermodynamic Hugoniot locus of the state* (v_0, s_0) *intersects each isentrope exactly once and* $s_1 > s_0$ *if* $v_1 < v_0$*, while* $s_1 < s_0$ *if* $v_1 > v_0$*.*

7 Riemann Problem

In this section we consider the Lagrangian system of the ionized gas dynamics mainly in the (p, u, s) state space. For the sake of brevity, we denote $U = {}^{T}(p, u, s)$. The *Riemann problem* is the initial value problem with special initial data

$$U(\xi, 0) = \begin{cases} U_L & \text{for} \quad \xi < 0, \\ U_R & \text{for} \quad \xi > 0. \end{cases}$$

where $U_L = {}^{T}(p_L, u_L, s_L), U_R = {}^{T}(p_R, u_R, s_R)$ are constant states. We will construct a self-similar (weak) solution to the initial value problem.

Rarefaction Curves The integral curves of $\boldsymbol{r}_j(U)$ $(j = 0, 1, 2)$ are represented by

$$\begin{aligned} \widehat{\mathcal{R}}_1(U): &\quad u + \int \sqrt{-v_p}\, dp = \text{const}, \quad s = \text{const}, \\ \widehat{\mathcal{R}}_0(U): &\quad p = \text{const}, \quad u = \text{const}, \\ \widehat{\mathcal{R}}_2(U): &\quad u - \int \sqrt{-v_p}\, dp = \text{const}, \quad s = \text{const}. \end{aligned}$$

As we have mentioned in Remark 3.1, rarefaction curves are isentropes in the thermodynamic state space. For the Riemann data U_L, U_R, we define the *forward 1-rarefaction curve* $\widehat{\mathcal{R}}_1^F(U_L)$ and the *backward 2-rarefaction curve* $\widehat{\mathcal{R}}_2^B(U_R)$ as the following.

$$\begin{array}{lrcll} \widehat{\mathcal{R}}_1^F(U_L): & u-u_L & = & -\displaystyle\int_{p_L}^{p}\sqrt{-v_p}\,dp, & p\le p_L, \\ & s-s_L & = & 0 & \\ \widehat{\mathcal{R}}_2^B(U_R): & u-u_R & = & \displaystyle\int_{p_R}^{p}\sqrt{-v_p}\,dp, & p\le p_R. \\ & s-s_R & = & 0 & \end{array} \tag{7.1}$$

If $U\in\widehat{\mathcal{R}}_1^F(U_L)$, the state U_L is connected to U from left to right by a 1-centred rarefaction wave $U=U\left(\frac{x}{t}\right)$; if $U\in\widehat{\mathcal{R}}_2^B(U_R)$, U is connected to U_R from left to right by a 2-centred rarefaction wave (see Dafermos [7], Smoller [18]).

Hugoniot Loci Two states separated by a shock front $\xi=\Sigma t$ constitute a Lagrangian weak solution if and only if the kinetic and thermodynamic conditions (6.2) are satisfied. The Lagrangian shock speed [4] satisfies $\Sigma^2=-\frac{p_+-p_-}{v_+-v_-}$. In section 6 we have proved that the thermodynamic Hugoniot loci is represented by a smooth curve in the $(T,\alpha_{\rm A})$ plane. In order to study the Riemann problem we have to consider them in the (p,s) plane, and somtimes (v,s) and (p,v) planes. Following argument and notations are due to Menikoff-Plohr [15].

From now on, we denote by Σ, in particular, the positive Lagrangian shock speed $\sqrt{-\frac{p_+-p_-}{v_+-v_-}}$. We have two branches of Hugoniot loci such that

$$u-u_0=\pm\frac{p-p_0}{\Sigma}\quad\text{for}\quad p\ge p_0\quad\text{and}\quad \left.\frac{du}{dp}\right|_{p=p_0}=\frac{\pm1}{\Sigma|_{p=p_0}}=\pm\sqrt{v_p(p_0,s_0)}.$$

We will show these two branches are strictly increasing and decreaing ones for all $p\ge p_0$.

The *adiabatic exponent* γ and *Grüneisen coeficient* are defined respectively by

$$\gamma=-\frac{v}{p}\left(\frac{\partial p}{\partial v}\right)_s,\quad \Gamma=-\frac{v}{T}\frac{\partial^2 e}{\partial s\partial v}=\frac{v}{T}\left(\frac{\partial p}{\partial s}\right)_v=v\left(\frac{\partial p}{\partial e}\right)_v.$$

Obviouly we have

$$dp=\frac{1}{v}\left(T\Gamma ds-p\gamma dv\right). \tag{7.2}$$

Denoting $H=e-e_0+\frac{1}{2}(p+p_0)(v-v_0)$, we have by the thermodynamic Hugoniot condition together with the first and second law of thermodynamics

$$Tds=dH-\frac{1}{2}(v-v_0)dp+\frac{1}{2}(p-p_0)dv. \tag{7.3}$$

Lemma 7.1. *If $2\gamma\ge\Gamma$, then the theromodynamic Hugoniot locus is represented by the graph of a smooth function $v=v(p)$ for $p\ge p_0$ in the (p,v) plane and the specific entropy s is a smooth function of p defined for $p\ge p_0$. Derivatives have the forms*

$$-\frac{p}{v}\frac{dv}{dp}=\frac{1+\frac{\Gamma(v-v_0)}{2v}}{\gamma-\frac{\Gamma(p-p_0)}{2p}},\quad \frac{ds}{dp}=-\frac{(v-v_0)\left[\gamma+\frac{v(p-p_0)}{p(v-v_0)}\right]}{2T\left[\gamma-\frac{\Gamma(p-p_0)}{2p}\right]}. \tag{7.4}$$

[4] The Eulerian shock speed is $\sigma=u_\pm+\frac{\Sigma}{\rho_\pm}$.

Proof. For the Hugoniot locus, we have by (7.2) and (7.3)

$$\begin{aligned} dp &= \frac{\Gamma}{2v}\left[dH + (p-p_0)dv - (v-v_0)dp\right] - \frac{p\gamma}{v}dv \\ &= \frac{\Gamma}{2v}dH - \frac{\Gamma(v-v_0)}{2v}dp - \frac{p}{v}\left[\gamma - \frac{\Gamma(p-p_0)}{2p}\right] \end{aligned}$$

which gives

$$\frac{\Gamma}{2v}dH = \left[1 + \frac{\Gamma(v-v_0)}{2v}\right]dp + \frac{p}{v}\left[\gamma - \frac{\Gamma(p-p_0)}{2p}\right]dv.$$

Suppose that $\gamma \geq \frac{1}{2}\Gamma$. Then for $p \geq p_0$

$$\gamma - \frac{\Gamma(p-p_0)}{2p} = \gamma - \frac{1}{2}\Gamma + \frac{\Gamma p_0}{2p} \geq \frac{\Gamma p_0}{2p} > 0.$$

Thus we find that there is a smooth function $v = v(p)$ defined for $p \geq p_0$ such that $H(p, v(p)) = 0$ and hence the first claim follows.

We consider s as a function of (p, v), and then a function of p by substituting $v = v(p)$. Since

$$Tds = -\frac{1}{2}\left[(v-v_0) - (p-p_0)\frac{dv}{dp}\right]dp$$

Substituting the first expression (7.4) into the above expression, we obtain the second. □

Let us next consider the (full) Hugoniot locus and, in particular, its projection onto the (p, u) plane. By the above lemma, we may suppose that v is a smooth function of p. Hence by the kinetic Hugoniot condition, u is a smooth function of p, which is the projection onto the (p, u) plane. Notice that $v < v_0$ if $p > p_0$. The following theorem is due to Liu [14] and Smith [17], and formulation and proof are to [15].

Theorem 7.1 (Liu-Smith)**.** *Suppose that $\gamma \geq \Gamma$. Then the projection onto the (p, u) plane of each Hugoniot locus of U_0 is represented by the graph of a smooth function $u = u(p)$ for $p \geq p_0$ in the (p, u) plane and*

$$\frac{du}{dp} = -\frac{(v-v_0)\left[\gamma - \frac{\Gamma(p-p_0)}{p} + \frac{v\Sigma^2}{p}\right]}{2(u-u_0)\left[\gamma - \frac{\Gamma(p-p_0)}{2p}\right]} \neq 0 \quad \textit{for} \quad p > p_0.$$

Proof. By (7.4) and the kinetic Hugoniot condition, we find that

$$\begin{aligned} 2(u-u_0)du &= -(p-p_0)dv - (v-v_0)dp \\ &= -\frac{(v-v_0)\left[\gamma - \frac{\Gamma(p-p_0)}{p} - \frac{v(p-p_0)}{p(v-v_0)}\right]}{\gamma - \frac{\Gamma(p-p_0)}{2p}}dp. \end{aligned}$$

Proof of Lemma 7.1 shows that $\gamma - \frac{\Gamma(p-p_0)}{2p} > 0$ for $p \geq p_0$. We have

$$\gamma - \frac{\Gamma(p-p_0)}{p} - \frac{v(p-p_0)}{p(v-v_0)} = \gamma - \Gamma + \frac{\Gamma p_0}{p} + \frac{v\Sigma^2}{p} > \gamma - \Gamma > 0$$

for $p > p_0$ which shows that $\frac{du}{dp} \neq 0$ for $p > p_0$, Also we have observed that, in a neighbourhood of (p_0, u_0), there are two branches of solutions each of which is represented by a smooth function $u(p)$ and $\left.\frac{du}{dp}\right|_{p=p_0} \neq 0$. Thus we obtain the theorem. □

Let us consider the positive Lagrangian shock speed as a function of U and U_0 denoted by $\Sigma(U, U_0)$. For the Riemann data U_L, U_R, we define the *forward 1-shock curve* $\widehat{\mathcal{S}}_1^F(U_L)$ and the *backward 2-shock curve* $\widehat{\mathcal{S}}_2^B(U_R)$ as the following.

$$\begin{array}{lrcll} & u - u_L &=& -\dfrac{p - p_L}{\Sigma(U, U_L)} & \\ \widehat{\mathcal{S}}_1^F(U_L): & & & & p \geq p_L, \\ & s - s_L &=& \displaystyle\int_{P_L}^{p} \frac{(v - v_0)\left[\gamma - \frac{v\Sigma^2}{p}\right]}{2T\left[\gamma - \frac{\Gamma(p-p_0)}{2p}\right]}\, dp & \\ & u - u_R &=& \dfrac{p_R - p}{\Sigma(U_R, U)} & \\ \widehat{\mathcal{S}}_2^B(U_R): & & & & p \geq p_R. \\ & s - s_R &=& \displaystyle\int_{P_R}^{p} \frac{(v - v_0)\left[\gamma - \frac{v\Sigma^2}{p}\right]}{2T\left[\gamma - \frac{\Gamma(p-p_0)}{2p}\right]}\, dp & \end{array} \tag{7.5}$$

Since we have proved in Theorem 6.4 that the pressure p is an increasing function of the temperature T, we may consider the above curves are admissible branches. If a states is located on $\widehat{\mathcal{S}}_1^F(U_L)$ or $\widehat{\mathcal{S}}_2^B(U_R)$, the jump discontinuity connecting U and U_L or U_R, respectively, is simply called a *shock wave*. The line of discontinuity $\xi = \Sigma t$ is actually a shock front. Note that the above branches complement the rarefaction curves (7.1).

If $U \in \widehat{\mathcal{S}}_1^F(U_L)$, the state U_L is connected to U from left to right by a 1-shock wave; if $U \in \widehat{\mathcal{S}}_2^B(U_R)$, U is connected to U_R from left to right by a 2-shock wave (see [7], [18]).

If $p_+ = p_-$, we have a discontinuity

$$u_+ = u_-, \quad p_+ = p_-, \quad s_+ \neq s_-, \quad \Sigma = 0 \tag{7.6}$$

which coincides with the integral curve $R_0(U_-)$. This type of discontinuity is called a *contact discontinuity*.

Solution to the Riemann Problem Suppose that $\gamma \geq \Gamma$: the Liu-Smith strong condition is satisfied. In order to solve the Riemann problem, we define the *forward* 1*-wave curve* $\widehat{\mathcal{W}}_1^F(U_L)$ and the *backward* 2*-wave curve* $\widehat{\mathcal{W}}_2^B(U_R)$ to be

$$\widehat{\mathcal{W}}_1^F(U_L) = \begin{cases} \widehat{\mathcal{R}}_1^F(U_L) & (p \leq p_L) \\ \widehat{\mathcal{S}}_1^F(U_L) & (p > p_L) \end{cases}, \quad \widehat{\mathcal{W}}_1^F(U_R) = \begin{cases} \widehat{\mathcal{R}}_2^B(U_R) & (p \leq p_R) \\ \widehat{\mathcal{S}}_2^B(U_R) & (p > p_R) \end{cases}. \tag{7.7}$$

Each wave curve is a C^2-curve with Lipschitz continuous second derivative and represents all realisable rarefaction waves and shock waves. If $(p, u, s) \in \widehat{\mathcal{W}}_1^F(U_L)$, then there is a 1-rarefaction wave or shock wave connecting (p_L, u_L, s_L) and (p, u, s). If, on the other hand, $(p, u, s) \in \widehat{\mathcal{W}}_2^B(U_R)$, then there is a 2-rarefaction waves or shock wave connecting (p, u, s) and (p_R, u_R, s_R).

Let $\mathcal{W}_1^F(U_L)$ and $\mathcal{W}_2^B(U_R)$ be the *projection* of $\widehat{\mathcal{W}}_1^F(U_L)$ and $\widehat{\mathcal{W}}_2^B(U_R)$, respectively, onto the (p, u)-plane. We find by Theorem 7.1

Lemma 7.2. *If $\gamma \geq \Gamma$, $\mathcal{W}_1^F(U_L)$ and $\mathcal{W}_1^B(U_R)$, respectively, are represented by the graphs of strictly decreasing and increasing, respectively, functions of p defined for $0 < p < \infty$.*

Let $U_L = {}^T(p_L, u_L, s_L)$ and $U_R = {}^T(p_R, u_R, s_R)$ be given Riemann data. The Riemann problem is solved in the following way: If two curves $\mathcal{W}_1^F(U_L)$ and $\mathcal{W}_2^B(U_R)$, have an intersection

point (p_m, u_m), then the state $(p_m, u_m, s_m^-) \in \widehat{\mathcal{W}}_1^F(U_L)$ and $(p_m, u_m, s_m^+) \in \widehat{\mathcal{W}}_2^B(U_L)$ are connected by a contact discontinuity (see Smoller [18] Chap. 18 for the details). In most cases (for example: except for the isothermal gas $\gamma = 1$), we may assume

$$\int_0^{p_0} \sqrt{-v_p}\, dp < \infty \quad \text{for any} \quad p_0 > 0.$$

If this is the case, both $\mathcal{W}_1^F(U_L)$ and $\mathcal{W}_2^B(U_R)$ have limit points at $p = 0$, namely

$$u_0^F = u_L + \int_0^{p_L} \sqrt{-v_p}\, dp, \quad u_0^B = u_R - \int_0^{p_R} \sqrt{-v_p}\, dp.$$

Since $\mathcal{W}_1^F(U_L)$ and $\mathcal{W}_1^B(U_R)$, respectively, are represented by the graphs of strictly decreasing and increasing, respectively, functions of p, there is a unique intersection point if and only if $u_0^B < u_0^F$. Thus we obtain

Theorem 7.2. *Suppose that $\gamma \geq \Gamma$. If*

$$u_R - u_L < \int_0^{p_R} \sqrt{-v_p}\, dp + \int_0^{p_L} \sqrt{-v_p}\, dp$$

then there is a unique solution to the Riemann problem.

Computation of Dimensionless Quantities Finally we will verify the Liu-Smith strong condition for our model system. We have adopt α_{A}, T as state variables so that

$$p = p(\alpha_{\mathrm{A}}, T), \quad v = v(\alpha_{\mathrm{A}}, T), \quad s = s(\alpha_{\mathrm{A}}, T) = \frac{R}{M}\eta(\alpha_{\mathrm{A}}, T).$$

Let us first represent γ and Γ in terms of derivatives with respect to α_{A} and T.

Lemma 7.3.

$$\gamma = \frac{\begin{vmatrix} \frac{q_{\mathrm{A}}}{p}\frac{\partial p}{\partial \alpha_{\mathrm{A}}} & \frac{T}{p}\frac{\partial p}{\partial T} \\ q_{\mathrm{A}}\frac{\partial \eta}{\partial \alpha_{\mathrm{A}}} & T\frac{\partial \eta}{\partial T} \end{vmatrix}}{\begin{vmatrix} q_{\mathrm{A}}\frac{\partial \eta}{\partial \alpha_{\mathrm{A}}} & T\frac{\partial \eta}{\partial T} \\ \frac{q_{\mathrm{A}}}{v}\frac{\partial v}{\partial \alpha_{\mathrm{A}}} & \frac{T}{v}\frac{\partial v}{\partial T} \end{vmatrix}}, \quad \Gamma = (1+\alpha)\frac{\begin{vmatrix} \frac{q_{\mathrm{A}}}{p}\frac{\partial p}{\partial \alpha_{\mathrm{A}}} & \frac{T}{p}\frac{\partial p}{\partial T} \\ \frac{q_{\mathrm{A}}}{v}\frac{\partial v}{\partial \alpha_{\mathrm{A}}} & \frac{T}{v}\frac{\partial v}{\partial T} \end{vmatrix}}{\begin{vmatrix} q_{\mathrm{A}}\frac{\partial \eta}{\partial \alpha_{\mathrm{A}}} & T\frac{\partial \eta}{\partial T} \\ \frac{q_{\mathrm{A}}}{v}\frac{\partial v}{\partial \alpha_{\mathrm{A}}} & \frac{T}{v}\frac{\partial v}{\partial T} \end{vmatrix}}$$

Proof. We have

$$ds = \frac{\partial s}{\partial \alpha_{\mathrm{A}}} d\alpha_{\mathrm{A}} + \frac{\partial s}{\partial T} dT, \quad dv = \frac{\partial v}{\partial \alpha_{\mathrm{A}}} d\alpha_{\mathrm{A}} + \frac{\partial v}{\partial T} dT, \quad dp = \frac{\partial p}{\partial \alpha_{\mathrm{A}}} d\alpha_{\mathrm{A}} + \frac{\partial p}{\partial T} dT.$$

Solving the first two equations for $d\alpha_{\mathrm{A}}$ and dT and substituting them into the third, we obtain expressions of $\left(\frac{\partial p}{\partial v}\right)_s$ and $\left(\frac{\partial p}{\partial s}\right)_v$ in terms of derivatives with respect to α_{A} and T. We note that, by arranging quantities, γ and Γ are represented in terms of dimensionless quantities as above. □

The form of η is shown in Theorem 2.1 and we have by (6.3)

$$v = \frac{1}{\rho} = \frac{R\mu_{\mathrm{A}}\alpha_{\mathrm{A}}\alpha}{M(1-\alpha_{\mathrm{A}})} T^{-\frac{3}{2}} e^{\frac{T_{\mathrm{A}}}{T}} = \frac{R\mu_{\mathrm{B}}\alpha_{\mathrm{B}}\alpha}{M(1-\alpha_{\mathrm{B}})} T^{-\frac{3}{2}} e^{\frac{T_{\mathrm{B}}}{T}}.$$

We have by direct computations

$$
\begin{aligned}
q_{\mathrm{A}}\frac{\partial \eta}{\partial \alpha_{\mathrm{A}}} &= 1+\alpha+\beta\left(\frac{1}{\alpha}+\frac{5}{2}+\frac{T_{\mathrm{A}}}{T}\right)q_{\mathrm{A}}+(1-\beta)\left(\frac{1}{\alpha}+\frac{5}{2}+\frac{T_{\mathrm{B}}}{T}\right)q_{\mathrm{B}},\\
T\frac{\partial \eta}{\partial T} &= \frac{(1-\beta)(T_{\mathrm{B}}-T_{\mathrm{A}})q_{\mathrm{B}}}{T}\left(\frac{1}{\alpha}+\frac{5}{2}+\frac{T_{\mathrm{B}}}{T}\right)-\frac{T_{\mathrm{A}}(1+\alpha)}{T},\\
\frac{q_{\mathrm{A}}}{v}\frac{\partial v}{\partial \alpha_{\mathrm{A}}} &= 1+\frac{\beta q_{\mathrm{A}}+(1-\beta)q_{\mathrm{B}}}{\alpha}, \qquad \frac{T}{v}\frac{\partial v}{\partial T} = \frac{(1-\beta)(T_{\mathrm{B}}-T_{\mathrm{A}})q_{\mathrm{B}}}{\alpha T}-\left(\frac{3}{2}+\frac{T_{\mathrm{A}}}{T}\right),\\
\frac{q_{\mathrm{A}}}{p}\frac{\partial p}{\partial \alpha_{\mathrm{A}}} &= -1-\frac{\beta q_{\mathrm{A}}+(1-\beta)q_{\mathrm{B}}}{\alpha(1+\alpha)}, \quad \frac{T}{p}\frac{\partial p}{\partial T},= -\frac{(1-\beta)(T_{\mathrm{B}}-T_{\mathrm{A}})q_{\mathrm{B}}}{\alpha(1+\alpha)T}+\left(\frac{5}{2}+\frac{T_{\mathrm{A}}}{T}\right).
\end{aligned}
$$

We first compute Γ : For the sake of brevity, we set $q=\beta q_{\mathrm{A}}+(1-\beta)q_{\mathrm{B}}$ and $Q_{\mathrm{BA}}=\frac{(1-\beta)(T_{\mathrm{B}}-T_{\mathrm{A}})q_{\mathrm{B}}}{T}$. The denominator is computed as

$$
\begin{aligned}
\begin{vmatrix} q_{\mathrm{A}}\dfrac{\partial \eta}{\partial \alpha_{\mathrm{A}}} & T\dfrac{\partial \eta}{\partial T} \\ \dfrac{q_{\mathrm{A}}}{v}\dfrac{\partial v}{\partial \alpha_{\mathrm{A}}} & \dfrac{T}{v}\dfrac{\partial v}{\partial T} \end{vmatrix} &= -Q_{\mathrm{BA}}\left(\frac{3}{2}+\frac{T_{\mathrm{B}}}{T}\right)-\frac{3}{2}(1+\alpha)-\frac{\beta Q_{\mathrm{BA}}(T_{\mathrm{B}}-T_{\mathrm{A}})q_{\mathrm{A}}}{\alpha T}\\
&\quad -\left[\beta\frac{T_{\mathrm{A}}q_{\mathrm{A}}}{T}+(1-\beta)\frac{T_{\mathrm{B}}q_{\mathrm{B}}}{T}\right]\left(\frac{3}{2}+\frac{T_{\mathrm{A}}}{T}\right)-\left[\frac{3q}{2\alpha}\left(1+\frac{5}{2}\alpha+\frac{T_{\mathrm{A}}\alpha}{T}\right)\right]<0,
\end{aligned}
$$

and the numerator is

$$
\begin{vmatrix} \dfrac{q_{\mathrm{A}}}{p}\dfrac{\partial p}{\partial \alpha_{\mathrm{A}}} & \dfrac{T}{p}\dfrac{\partial p}{\partial T} \\ \dfrac{q_{\mathrm{A}}}{v}\dfrac{\partial v}{\partial \alpha_{\mathrm{A}}} & \dfrac{T}{v}\dfrac{\partial v}{\partial T} \end{vmatrix} = -\left(1+\frac{Q_{\mathrm{BA}}}{1+\alpha}\right)-q\left[\frac{1}{\alpha}+\frac{1}{1+\alpha}\left(\frac{3}{2}+\frac{T_{\mathrm{A}}}{T}\right)\right]<0.
$$

Consequently, we conclude that $\Gamma>0$.

Next we compute γ : Note that γ has the same denominator as Γ. By denoting $\Sigma=\alpha(1+\alpha)+q$, $Q_T=\frac{\beta q_{\mathrm{A}}T_{\mathrm{A}}+(1-\beta)q_{\mathrm{B}}T_{\mathrm{B}}}{T}$, the numerator is computed as

$$
\begin{vmatrix} \dfrac{q_{\mathrm{A}}}{p}\dfrac{\partial p}{\partial \alpha_{\mathrm{A}}} & \dfrac{T}{p}\dfrac{\partial p}{\partial T} \\ q_{\mathrm{A}}\dfrac{\partial \eta}{\partial \alpha_{\mathrm{A}}} & T\dfrac{\partial \eta}{\partial T} \end{vmatrix} = -\frac{5\Sigma}{2\alpha}-\left(\frac{5}{2}q+Q_T\right)\left(\frac{5}{2}+\frac{T_{\mathrm{A}}}{T}\right)-Q_{\mathrm{BA}}\left[\left(\frac{5}{2}+\frac{T_{\mathrm{B}}}{T}\right)+\frac{\beta q_{\mathrm{A}}(T_{\mathrm{B}}-T_{\mathrm{A}})}{\alpha(1+\alpha)T}\right]<0.
$$

Finally we verify the Liu-Smith strong condition.

Theorem 7.3.

$$\gamma>\Gamma>0$$

Proof. Since the denominator is negative, we compute

$$
\begin{aligned}
&-\begin{vmatrix} \dfrac{q_{\mathrm{A}}}{p}\dfrac{\partial p}{\partial \alpha_{\mathrm{A}}} & \dfrac{T}{p}\dfrac{\partial p}{\partial T} \\ q_{\mathrm{A}}\dfrac{\partial \eta}{\partial \alpha_{\mathrm{A}}} & T\dfrac{\partial \eta}{\partial T} \end{vmatrix}+(1+\alpha)\begin{vmatrix} \dfrac{q_{\mathrm{A}}}{p}\dfrac{\partial p}{\partial \alpha_{\mathrm{A}}} & \dfrac{T}{p}\dfrac{\partial p}{\partial T} \\ \dfrac{q_{\mathrm{A}}}{v}\dfrac{\partial v}{\partial \alpha_{\mathrm{A}}} & \dfrac{T}{v}\dfrac{\partial v}{\partial T} \end{vmatrix}\\
&=\frac{3\Sigma}{2\alpha}+\left(\frac{3}{2}q+Q_T\right)\left(\frac{5}{2}+\frac{T_{\mathrm{A}}}{T}\right)+Q_{\mathrm{BA}}\left[\left(\frac{3}{2}+\frac{T_{\mathrm{B}}}{T}\right)+\frac{\beta q_{\mathrm{A}}(T_{\mathrm{B}}-T_{\mathrm{A}})}{\alpha(1+\alpha)T}\right]>0.
\end{aligned}
$$

Thus the theorem follows. □

8 Conclusions and Discussions

In this paper, we have studied a model system for macroscopic motion of an ionized gas which is a mixture of two monatomic gas A and B; the mixture ratio is $\beta : 1 - \beta$. This model system is proposed by [9] and consists of three conservation laws in one space dimension together with the first and second law of thermodynamics which are supplemented by an equation of state and two more thermodynamic equations called Saha's laws. We have assumed that T_{B} : the first ionization temperature of the gas B is higher than T_{A} : that of the gas A and $2T_{\mathrm{A}} \geq T_{\mathrm{B}}$. Note that A: hydrogen and B: helium satisfy these assumptions.

The physical entropy functions are constructed and it is remarkable that they are expressed in terms of elementary functions. Saha's two equations bring about a compatibility condition involving $\alpha_{\mathrm{A}}, \alpha_{\mathrm{B}}$ and T. It is shown that α_{B} is a differentiable function of α_{A} and T whose graph constitutes the thermodynamic state space. We propose that (T, α_{A}) is a suitable pair of independent thermodynamic state variables.

The system of conservation laws is shown to constitute a strictly hyperbolic system, which implies that the initial-value problem is well-posed locally in time for sufficiently smooth initial data. Characteristic fields are computed and geometric properties are studied: unlike the polytropic (non-ionized) case, the convexity (genuine nonlinearity) of the forward and backward characteristic fields of the system is lost and the set where this happens is determined in a neighbourhood of $T = \alpha_{\mathrm{A}} = 0$. Whole set is located in a finite region in (T, α_{A}) plane but it is difficult to get its full picture by purely mathematical reasoning; only pictures by numerical computation are presented.

A detailed study of the thermodynamic Hugoniot locus is performed. The Hugoniot locus represented by a smooth graph in the (T, α_{A}) plane. While the thermodynamic Hugoniot locus is monotone in (T, α) plane in a single monatomic case, for the mixed monatomic case it is shown that: if β is sufficiently small, then it loses monotonicity at some base state. Thus the degree of ionization does not always increase across the shock front, even if the temperature increases. However the pressure is actually proved to increase as the temperature increases which ensures that $T > T_+$ is the admissible branch.

By adopting the branch of increasing temperature (hence increasing pressure) of the Hugoniot locus as a shock curve, the existence and uniqueness of solutions to the Riemann Problem is studied. It is shown that the Grüneisen coefficient is positive and the Liu-Smith strong condition is satisfied, which shows that: (1) the physical entropy increases across the shock wave in the genuinely nonlinear region, (2) the Riemann problem has a unique solution as long as the vacuum state is not involved.

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