Numerical Analysis on the Long-Time Behavior of Moving Body in a Rarefied Gas

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Abstract

We consider the unsteady motion of an infinite plate under an external elastic force acting on the plate in its perpendicular direction. The plate is immersed in a rarefied gas, and thus the plate motion is coupled with the surrounding gas behavior. We perturb the plate from its equilibrium position and release it at an initial time. Subsequently, the plate starts an oscillating motion around the equilibrium position, but the displacement from the equilibrium decays as time goes on because of the drag force exerted by the surrounding gas. We numerically analyze the long-time behavior of the plate using the Bhatnagar-Gross-Krook (BGK) model of the Boltzmann equation as the basic equation for the rarefied gas. It is shown that the displacement from the equilibrium decays as time goes on with a rate $t^{-3/2}$, where t is a time variable.

1 Introduction

Let us consider the translational rectilinear motion of a moving body along the x_1-axis immersed in an infinite expanse of a rarefied gas. The x_1-components of position and velocity of the body are denoted by $X(t)$ and $V(t)$, respectively, with a time variable t. The gas behavior is described by the molecular velocity distribution function $f(\boldsymbol{x}, \boldsymbol{\xi}, t)$, where $\boldsymbol{x} = (x_1, x_2, x_3)$ is the Cartesian coordinate system and $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ is the molecular velocity. The unsteady motion of the body is initiated by the action of an external elastic force $F = F_{\rm el} \equiv -K_{\rm el}X(t)$ in the x-direction with a positive parameter $K_{\rm el}$. The equation of motion is then written as

$$\frac{\mathrm{d}X}{\mathrm{d}t} = V(t), \quad M\frac{\mathrm{d}V}{\mathrm{d}t} = F - G[f], \quad (X(0) = X_0,\ V(0) = V_0), \tag{1}$$

where M is the mass of the body, X_0 and V_0 are initial displacement and velocity, respectively, and $G[f]$ is the drag force exerted by the surrounding gas; note that $G[f]$ is dependent on

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the gas behavior. The velocity distribution function is determined by a kinetic equation and its initial and boundary conditions:

$$\frac{\partial f}{\partial t} + \boldsymbol{\xi} \cdot \frac{\partial f}{\partial \boldsymbol{x}} = \frac{1}{k} Q[f], \tag{2a}$$

$$f(\boldsymbol{x}, \boldsymbol{\xi}, t) = f_0 \quad \text{for} \ \ t = 0, \tag{2b}$$

$$f(\boldsymbol{x}, \boldsymbol{\xi}, t) = f_{\mathrm{w}} \quad \text{for} \ \ \boldsymbol{x} \in \mathsf{S}(X(t)), \quad \boldsymbol{\xi} \in \mathsf{Z}(V(t)). \tag{2c}$$

In Eq. (2), k is the Knudsen number that indicates the degree of gas rarefaction. $Q[f]$ is the collision term between gas molecules, f_0 is an initial distribution, and f_{w} is the boundary condition; $\mathsf{S}(X(t))$ is the surface of the body and $\mathsf{Z}(V(t))$ is the molecular velocity for reflected molecules. This is basically a coupled problem between the moving body $(X(t), V(t))$ and the gas behavior $f(\boldsymbol{x}, \boldsymbol{\xi}, t)$, that is, we have to deal with the moving boundary problem for a rarefied gas [19]. The motion decays as time goes on due to the drag force, and we investigate the rate of decay for a sufficiently large time. In this paper, we review the results of Refs. [19, 20], where the body is an infinite plate on which the external elastic force acts in its perpendicular direction; the collision term $Q[f]$ is described by the Bhatnagar-Gross-Krook (BGK) model of the Boltzmann equation [3, 22]. Before going in detail of the analysis, the overview of moving boundary problems for the rarefied gas dynamics is given in the introduction as follows.

First, let us consider the case with a simplified setting. If the drag is modelled as $G = \alpha_0 V(t)$ with a constant $\alpha_0 (> 0)$, such as the case of Stokes drag for a steady motion of a sphere, the decay rate is exponential with time, that is, $|X(t)| \approx C_1 \exp(-C_2 t)$ holds for large t. In the present text, C_i $(i = 1, 2, \cdots)$ are some positive constants. On the contrary, if we include in G the Basset term that accounts for the memory of the motion in the past, the decay rate becomes algebraic, that is, $|X(t)| \approx C_3 t^{-\gamma}$ for a spherical body in a Stokes fluid [8], where $\gamma = 3/2$ for $V_0 = 0$ and $\gamma = 1/2$ for $V_0 \neq 0$. Therefore, the memory effect is necessary to obtain a correct time-asymptotic behavior. Now, let us go back to our problem (1) and (2). A highly rarefied gas characterized by $k \to \infty$ is called a free-molecular gas. The following power law decay is obtained for a large time in Refs. [6, 18] if the body is immersed in the free-molecular gas:

$$|X(t)| \approx C_4 t^{-n}, \quad \text{for} \quad t \gg 1, \tag{3}$$

where n is an integer. Denoting $d = 1$, 2, and 3 by the spatial dimension of the problem, it was mathematically proven in Ref. [6] that $n = 2 + d$ for the specular reflection boundary condition under some smallness assumption on X_0 and V_0. On the contrary, it was numerically demonstrated that $n = 1 + d$ for the diffuse reflection boundary condition [17, 18]. The decay rate in the case of the free-molecular gas depends on the boundary condition and the spatial dimension, and is different from the case of Stokes fluids [8]. This is attributed to the memory effect particular to the free-molecular gas: *recollisions* [5]. Let us focus on a gas molecule that hits the moving body at time $t = t_1$. The molecule is reflected by the body and travels in the gas domain for $t > t_1$ with a constant velocity, which is determined by a boundary condition that depends on the velocity of the body $V(t_1)$. However, the velocity of the body

at subsequent times, $V(t)$, is time-dependent, and thus the body may catch up the reflected molecule at a later time $t = t_2 (> t_1)$. Such a molecule makes recollision and exerts the memory effect, since it transfers the information in the past time t_1 to the future time t_2 via the collision.

The power law decay (3) is caused by recollision, thereby the manner of decay is expected to change when the effect of recollision is eliminated by inter-molecular collisions, i.e., $k < \infty$. The case of $k = O(1)$ is numerically investigated in Ref. [20] based on the BGK model of the Boltzmann equation. Due to the computational difficulty, the problem was restricted to the case of $d = 1$, i.e., the body is an infinitely wide plate perpendicular to the x_1-direction. The decay rate for a large time in this case is

$$|X(t)| \approx C_5 t^{-3/2}, \quad \text{for} \quad t \gg 1, \tag{4}$$

which is qualitatively same as the case of Stokes fluid [8]. The mathematical proof for Eq. (4) has not been obtained yet. Both the mathematical and numerical results are absent for the case of the full Boltzmann equation.

Finally, we close this section by introducing some related researches. For more detail, readers are referred to a text book [4]. The case of a constant external force, i.e., $F = \text{const.}$ was first studied in Ref. [5] using the specular reflection boundary condition. The geometry of the body was a d-dimensional disk perpendicular to the external force. In this case, the velocity $V(t)$ approaches a terminal velocity V_∞ with a rate $|V(t) - V_\infty| \approx C_6 t^{-m}$, with $m = 2 + d$. The result was later improved to a general convex body in Ref. [7] and was extended to the case of diffuse reflection boundary condition in Ref. [1], where the power was $m = 1 + d$. The numerical study [2] complemented Ref. [1] by eliminating a smallness assumption on $|V_\infty - V_0|$. More recently, the case with a more generalized boundary condition is investigated in Ref. [10] that obtained $m = p + 3$ with $p \in (0, 2]$ for $d = 3$. The case of a specific class of concave body was investigated mathematically and numerically in Refs. [16] and [21], respectively: the former used the specular reflection boundary condition and the latter the diffuse reflection boundary condition. The rate was independent from d and $m = 3$ [16] and $m = 2$ [21] for the concave body; the slower rate was attributed to the increase of recolliding molecules trapped at the concave part. Note that the case of an elastic body was also treated in Ref. [9] for $F = 0$. It was shown in Ref. [14] that the presence of a static infinite wall placed behind the moving body changed the rate to $m = d - 1$ ($d \geq 2$). When the terminal velocity satisfies $0 \leq V_\infty < V_0$, the sign of $V(t) - V_\infty$ changes during the time development [6]. The condition for this velocity reversal to happen was investigated in Ref. [12]. Above references stated in this paragraph treat the free-molecular gas, where the molecular path is a straight line. The effect of the loss of the memory on the long-time behavior was included by introducing background obstacles that interact with the gas molecules [18, 15]. This was the first step to understand the role of inter-molecular collision in the memory effect. The case where the gas molecules are subject to an external force is investigated in Ref. [11].

2 Formulation

We formulate the problem using non-dimensional variables. For more detailed description, the readers are referred to Ref. [20].

Let us consider an infinite plate, whose position and velocity are denoted as $x_1 = x_w(t) \in \mathbb{R}$ and $v_w(t) \in \mathbb{R}$, respectively, where $t \geq 0$ is a time variable. The plate is immersed in an infinite expanse of a rarefied gas. An external force $-x_w(t)$ acts on the plate in its perpendicular direction, which we denote by $x_1 \in \mathbb{R}$. The equation of motion of the plate is thus written as

$$\frac{dx_w}{dt} = v_w, \quad \frac{dv_w}{dt} = -x_w - \frac{G}{\mathcal{M}}, \quad (x_w(0) = x_{w0},\ v_w(0) = v_{w0}), \tag{5}$$

where G is the drag force exerted by the surrounding gas, $\mathcal{M}$ the non-dimensional mass of the plate, and x_{w0} and v_{w0} are the initial position and velocity, respectively. The drag force G acting on the plate is given as

$$G = G_+ + G_-, \quad G_\pm = \pm \int_{\mathbb{R}^3} [\zeta_1 - v_w(t)]^2 f(x_w(t) \pm 0, \boldsymbol{\zeta}, t) d\boldsymbol{\zeta}, \tag{6}$$

where the velocity distribution function of gas molecules is denoted by $f(x_1, \boldsymbol{\zeta}, t) \geq 0$ with $\boldsymbol{\zeta} \in \mathbb{R}^3$ the molecular velocity. Note that ζ_1 is the velocity component in the x_1 direction.

The basic equation for the rarefied gas is the BGK equation. For the present spatially one-dimensional setting, we can eliminate the velocity component ζ_2 and ζ_3 from the BGK equation by taking the marginal of f [13]:

$$g(x_1, \zeta_1, t) = \int_{\mathbb{R}^2} f(x_1, \boldsymbol{\zeta}, t) d\zeta_2 d\zeta_3, \quad h(x_1, \zeta_1, t) = \int_{\mathbb{R}^2} (\zeta_2^2 + \zeta_3^2) f(x_1, \boldsymbol{\zeta}, t) d\zeta_2 d\zeta_3. \tag{7}$$

Then, the BGK equation is written as

$$\frac{\partial g}{\partial t} + \zeta_1 \frac{\partial g}{\partial x_1} = \frac{1}{\mathsf{K}} \rho (M - g), \quad \frac{\partial h}{\partial t} + \zeta_1 \frac{\partial h}{\partial x_1} = \frac{1}{\mathsf{K}} \rho (TM - g), \tag{8a}$$

$$M = \frac{\rho}{(\pi T)^{1/2}} \exp\left(-\frac{(\zeta_1 - u_1)^2}{T} \right), \tag{8b}$$

$$\rho = \int_{\mathbb{R}} g d\zeta_1, \quad \rho u_1 = \int_{\mathbb{R}} \zeta_1 g d\zeta_1, \quad \frac{3}{2} \rho T = \int_{\mathbb{R}} [(\zeta_1 - u_1)^2 g + h] d\zeta_1, \tag{8c}$$

where M is the local Maxwellian distribution, ρ the density, u_1 the flow velocity in the x_1 direction, T the temperature, and $\mathsf{K} = (\sqrt{\pi}/2)\mathrm{Kn}$ the parameter of the order of Knudsen number Kn. Kn is the degree of gas rarefaction, i.e., the gas is rarefied when Kn is large. To be more precise, the intermolecular collision may be negligible for large Kn. In this paper, we are concerned with the case of $\mathsf{K} \approx 1$, and the readers are referred to Refs. [17, 18] for the case of $\mathsf{K} \to \infty$.

An initial condition to the system (8) is the global Maxwellian:

$$g(x_1, \zeta_1, 0) = h(x_1, \zeta_1, 0) = E(\zeta_1), \quad (E(\zeta_1) = \pi^{-1/2} \exp(-\zeta_1^2)). \tag{9}$$

The boundary conditions on the plate are the diffuse reflection boundary condition. Since the plate motion is determined by Eq. (5), the boundary condition depends on $x_w(t)$ and $v_w(t)$:

$$g(x_1, \zeta_1, 0) = \sigma_{w\pm} E(\zeta_1 - v_w(t)), \quad \text{for} \quad \zeta_1 - v_w(t) \gtrless 0,\ x_1 = x_w(t) \pm 0, \tag{10a}$$

$$h(x_1, \zeta_1, 0) = \sigma_{w\pm} E(\zeta_1 - v_w(t)), \quad \text{for} \quad \zeta_1 - v_w(t) \gtrless 0,\ x_1 = x_w(t) \pm 0, \tag{10b}$$

$$\sigma_{w\pm} = \mp 2\sqrt{\pi} \int_{\zeta_1 - v_w(t) \lessgtr 0} [\zeta_1 - v_w(t)] g(x_w(t) \pm 0, \zeta_1, t) d\zeta_1. \tag{10c}$$

Diffuse reflection (10) indicates that the gas molecules impinging on the plate is reflected according to the Maxwellian distribution with the velocity $v_w(t)$ and the temperature unity. Note that the temperature is normalized by the temperature of the boundary. The density of this Maxwellian, i.e., $\sigma_{w\pm}$, is determined in such a way that there is no net mass flux across the plate.

In this paper, we will present the numerical results for the coupled system (5)–(10). The detail of the numerical scheme is omitted here, and the readers are referred to Refs. [19, 20]. We note that the moving boundary problems in rarefied gas dynamics have an inherent difficulty: the propagation of discontinuity in the velocity distribution function. To resolve this problem, we developed the scheme based on the method of characteristics in Ref. [19] that captured accurately the essential features of the velocity distribution function. This method is called full Lagrangian scheme in this paper. The full Lagrangian scheme was more accurate than the other scheme such as a finite-difference scheme, but was computationally expensive. Therefore, we used the finite-difference scheme for the long-time computation up to $t = 10^4$ in Ref. [20]. It should be noted that the full Lagrangian scheme has validated the results of finite-difference scheme up to $t = O(10^2)$, where the discontinuities were confirmed to be small enough.

3 Results and discussion

3.1 Velocity distribution function

Before showing the result of decay rate, an important feature of the moving boundary problem is introduced. In this section, we present the result for the case with $\mathcal{M} \to \infty$, that is, the mass of the plate is so large that the oscillation never decays. In other words, the plate is forced to oscillate with the amplitude x_{w0}, where $v_{w0} = 0$ throughout the paper.

The velocity distribution function g is shown in Fig. 1 at $x_1 = x_w(t) + 0.2243$ (close to the plate) and at $x_1 = 22.3111$ (far from the plate), and at $t = 16\pi$ [19]. The parameters are set to (a) $\mathsf{K} = \infty$, (b) $\mathsf{K} = 10$, and (c) $\mathsf{K} = 1$. The solid curves and symbols are the results obtained by the full Lagrangian method and the finite-difference method, respectively.

For the results obtained by the full Lagrangian scheme, some discontinuities are observed in Fig. 1, e.g., at $\zeta_1 \approx 0.65$ in the upper figures. The origin of the discontinuity comes from the diffuse reflection boundary condition (10). Let us focus on the positive side $x_1 = x_w(t)+0$. The value of velocity distribution function g (or h) at $\zeta_1 = v_w(t) + 0$ is determined by the

Maxwellian whereas that at $\zeta_1 = v_w(t) - 0$ is computed from the BGK equation; these values are not same in general and the discontinuity is produced. Since the boundary is moving in time-dependent manner, the created discontinuity may be left in the gas domain. Moreover, there is a localized structure near $\zeta_1 \approx 0$ in the upper figures of panels (a) and (b). These are the multiple discontinuities created in the past times that are left near the boundary. Formation of discontinuities and localization are inherent in the moving boundary problem of a rarefied gas, as closely discussed in Ref. [19], especially when K is not small. It is clear from the figure that the finite-difference scheme fails to capture the correct shape of the velocity distribution function near the discontinuities.

It is seen that the magnitude of the discontinuities, g_d, decreases as K decreases. In Ref. [19], it is shown formally for the BGK equation that g_d decreases as

$$g_\mathrm{d} = A_0 \exp(-A_1 \mathsf{K}^{-1}(t-s)), \tag{11}$$

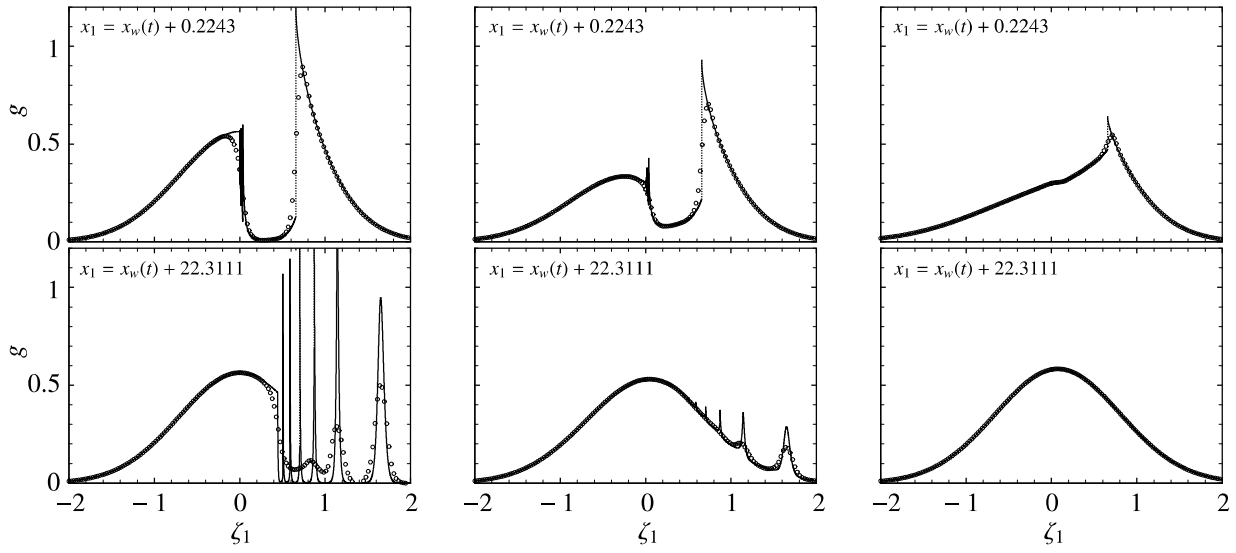
where A_0 is the value of g_d at the birth time $s(< t)$ of the discontinuity and $A_1 > 0$ is a variable related with the density of gas. In the present setting, we can assume that $A_1 = O(1)$. Therefore, g_d decays exponentially as time goes on for finite K. Since the displacement $x_w(t)$ shrinks for large t in the case of finite $\mathcal{M}$, which we treat in Sec. 3.2, the value of A_0 is expected to decrease as t increases. This fact together with the exponential decay of g_d suggests that the finite-difference scheme may work well with the case of finite $\mathcal{M}$ for large t, where the magnitude of discontinuities are expected to be small. With this consideration in mind, the decay of the plate motion will be discussed in the next section.

3.2 Decay of the oscillating plate

Figure 2 shows the displacement $|x_w(t)|$ versus time t in the double logarithmic scale for $\mathsf{K} = 1$ and $\mathcal{M} = 2$; (a) $x_{w0} = 1$, (b) $x_{w0} = 0.1$, and (c) $x_{w0} = 0.01$ [19]. The results obtained by both full Lagrangian and finite-difference schemes are plotted in the figure. For all cases, the displacement $|x_w(t)|$ decreases as time goes on, undergoing a couple of oscillations around the equilibrium position. Remarkably, the behavior of the displacement $|x_w(t)|$ agrees very well between both full Lagrangian and finite-difference schemes. Although the finite-difference scheme fails to capture the correct shape of velocity distribution function as discussed in Fig. 1, the error on the drag force G seems to be negligible for the present parameter set. It should be noted that for the case of $\mathsf{K} \to \infty$, the finite-difference scheme resulted in the wrong decay rate, as demonstrated in Ref. [19], due to the presence of discontinuity, which never be dampened.

The computation is carried out up to $t = t_\mathrm{max} = 200$ in Fig. 2. The limitation of $t_\mathrm{max} = 200$ was due to the expensive computational cost of the full Lagrangian scheme. Nonetheless, it is seen from Figs. 2(b) and (c) that the rate of decay is proportional to $t^{-3/2}$. To obtain more convincing evidence for the decay rate, the computation with much longer time is necessary.

Figure 3 shows the results of longer time computation up to $t = t_\mathrm{max} = 10^4$ obtained by the finite-difference method [20], which is computationally less expensive compared with the full Lagrangian scheme. The parameters are set to $x_{w0} = 0.1$ and $\mathsf{K} = 0.4$, 1, 2, 5; (a) $\mathcal{M} = 2$, (b) $\mathcal{M} = 1$, and (c) $\mathcal{M} = 0.5$. Panels (a)–(c) show the displacement $|x_w(t)|$ and



(a) K = ∞ (b) K = 10 (c) K = 1

Figure 1: The velocity distribution function g at $t = 16\pi$ for $x_{w0} = 1$ and $\mathcal{M} = \infty$. (a) $\mathsf{K} = \infty$, (b) $\mathsf{K} = 10$, and (c) $\mathsf{K} = 1$. The upper figures show the results at the position $x_1 = x_w(t) + 0.2243$, while the lower figures those at $x_1 = x_w(t) + 22.3111$. The results obtained by the finite-difference (circle) are shown together with those by the full Lagrangian method (solid line). This figure is reprented from Ref. [19] under the permission of Elsevier.

(d)–(f) the corresponding slopes $\alpha(x_w) \equiv d\log_{10} x_w(t)/d\log_{10} t$, namely, $\alpha = -n$ if the rate is described as $x_w(t) \approx Ct^{-n}$. The long-time computation indicates that the rate of decay is described by Eq. (4), i.e.,

$$|x_w(t)| \approx C_6 t^{-3/2} \quad \text{for} \quad t \gg 1. \tag{12}$$

4 Conclusion

In this paper, we reviewed the numerical analysis on the long-time behavior of a moving plate in a rarefied gas under the action of an external elastic force [19, 20]. The Knudsen number was set to the order unity, that is, the collisional gas was considered. Note that the earlier works were mainly concerned with the case of collisionless gas characterized by an infinite Knudsen number. The decay rate of the motion of the plate due to the drag force was shown to be proportional to $t^{-3/2}$ at large time $t = 10^4$. Using two types of numerical methods, i.e., the full Lagrangian method and the finite-difference method, the special care was taken to discuss the effect of discontinuities in the velocity distribution function, which are inherent in the moving boundary problems in rarefied gas dynamics.

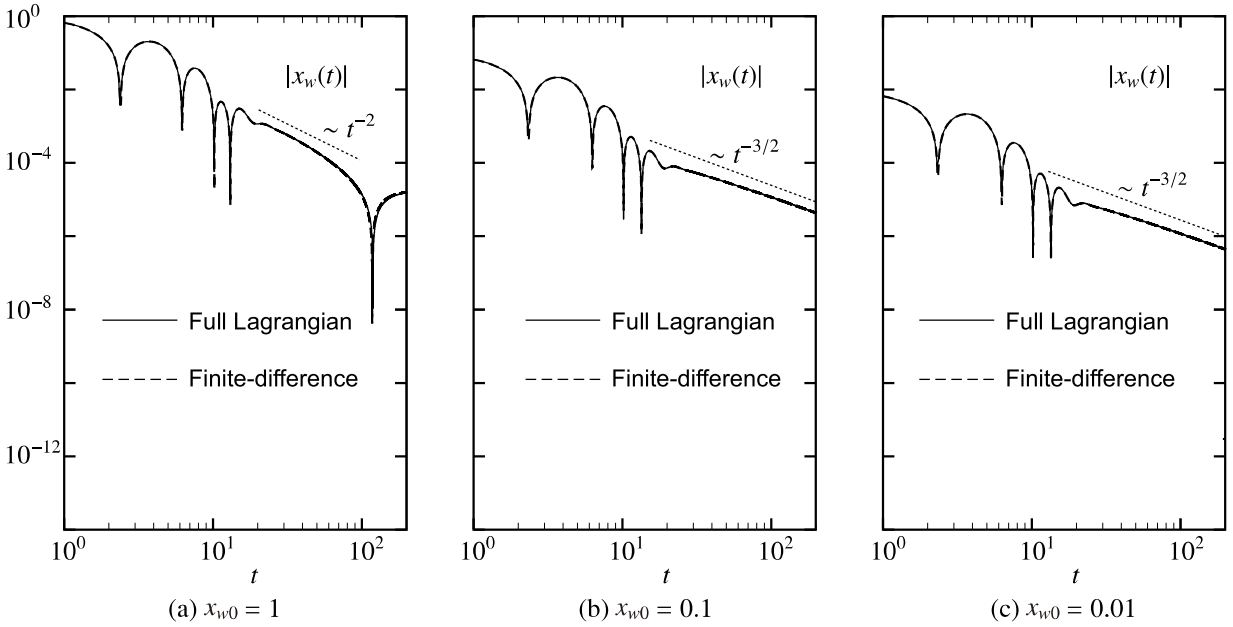


Figure 2: The displacement $|x_w(t)|$ versus t in double logarithmic scale for $\mathsf{K} = 1$. (a) $x_{w0} = 1$, (b) $x_{w0} = 0.1$, and (c) $x_{w0} = 0.01$. The solid line indicates the results obtained by the full Lagrangian scheme and the bold dashed line indicates those by the finite-difference scheme. This figure is reprent from Ref. [19] under the permission of Elsevier, where a modification is made for the consistent notation.

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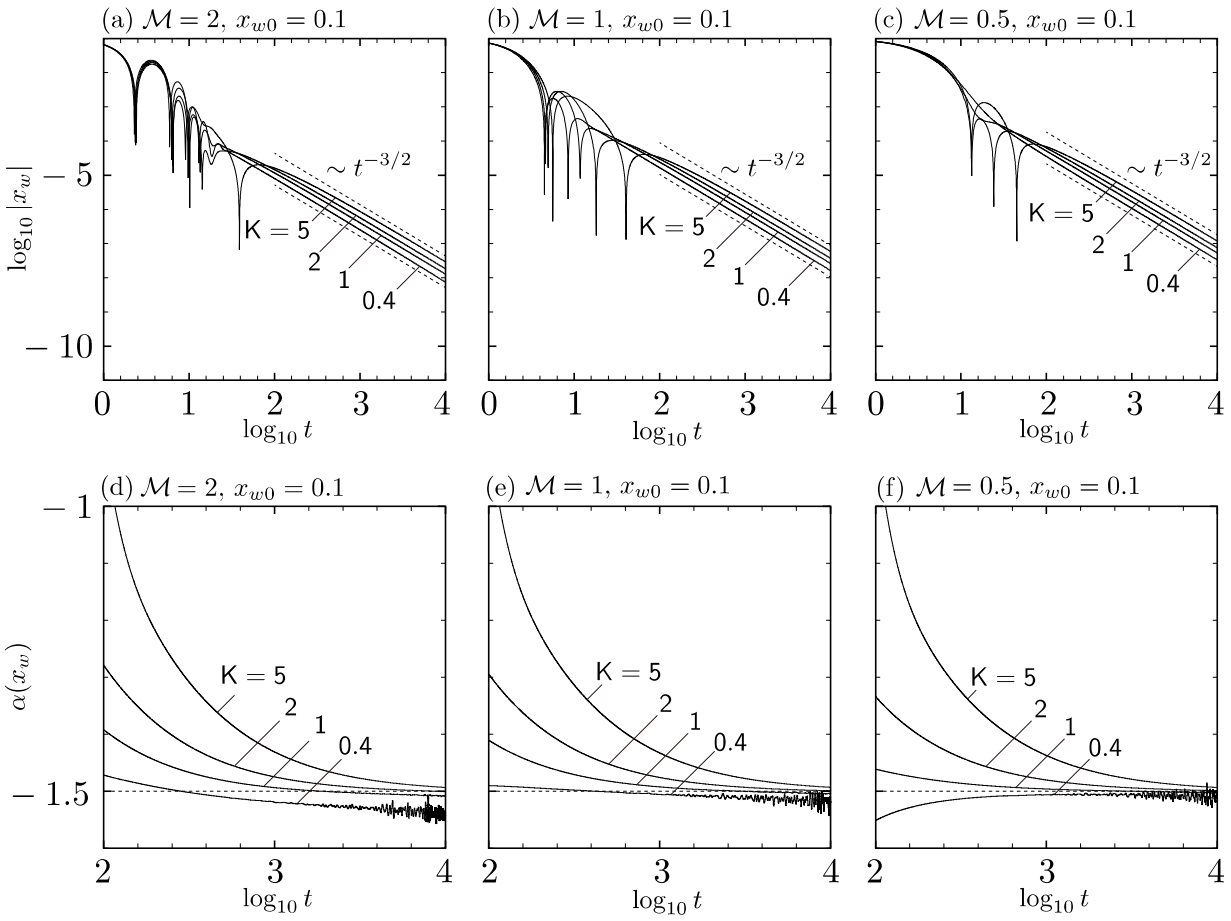

Figure 3: $\log_{10}|x_w|$ versus $\log_{10} t$ for long times at several K for $x_{w0} = 0.1$, $v_{w0} = 0$. (a) $\mathcal{M} = 2$, (b) $\mathcal{M} = 1$, (c) $\mathcal{M} = 0.5$. Panels (d), (e), and (f) show, respectively, the gradient of the curves in panels (a), (b), and (c). This figure is reprent from Ref. [20] under the permission of American Physical Society.

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