Local energy weak solution for the Navier-Stokes equations and applications

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1 Introduction

This is based on a joint work with Kyungkeun Kang and Tai-Peng Tsai [19]. The Navier-Stokes equations describe the evolution of a viscous incompressible fluid's velocity field v and its associated scalar pressure π. They are required to satisfy

$$\partial_t v - \Delta v + v \cdot \nabla v + \nabla \pi = 0, \quad \operatorname{div} v = 0 \tag{NS}$$

in the sense of distributions. For our purposes, (NS) is applied on $\mathbb{R}^3 \times (0, \infty)$ and v evolves from a prescribed, divergence free initial data $v_0 : \mathbb{R}^3 \to \mathbb{R}^3$. Solutions to (NS) has a natural scale invariance: If v satisfies (NS), then for any $\lambda > 0$ the pair (v^λ, p^λ) defined by

$$v^\lambda(x,t) = \lambda v(\lambda x, \lambda^2 t), \qquad \pi^\lambda(x,t) = \lambda^2 \pi(\lambda x, \lambda^2 t)$$

is also a solution with initial data

$$v_0^\lambda(x) = \lambda v_0(\lambda x). \tag{1.1}$$

A solution is called self-similar (SS) if $v^\lambda(x,t) = v(x,t)$ for all $\lambda > 0$ and is discretely self-similar with factor λ (i.e. v is λ-DSS) if this scaling invariance holds for a given $\lambda > 1$. Similarly, v_0 is self-similar (a.k.a. (-1)-homogeneous) if $v_0(x) = \lambda v_0(\lambda x)$ for all $\lambda > 0$ or λ-DSS if this holds for a given $\lambda > 1$. These solutions can be either forward or backward if they are defined on $\mathbb{R}^3 \times (0, \infty)$ or $\mathbb{R}^3 \times (-\infty, 0)$ respectively. In this paper we work exclusively with forward solutions and omit the qualifier "forward".

Self-similar solutions are interesting in a variety of contexts as candidates for ill-posedness or finite time blow-up of solutions to the 3D Navier-Stokes equations (see [12, 16, 17, 24, 29, 30] and the discussion in [2]. Forward self-similar solutions are compelling candidates for non-uniqueness [17, 12]. Until recently, the existence of forward self-similar solutions was only known for small data (see the references in [2]). Such solutions are necessarily unique. In [16], Jia and Šverák constructed forward self-similar solutions for large data where the data is assumed to be Hölder continuous away from the origin. This result has been generalized in a number of directions by a variety of authors [2, 3, 4, 5, 8, 21, 23, 31]; see also the survey [18].

The motivating problem is the following: It is shown in Tsai [31] that, if a λ-DSS initial data $v_0 \in C^\alpha_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$, $0 < \alpha < 1$, with $M = \|v_0\|_{C^\alpha(B_2 \setminus B_1)} < \infty$, and if $\lambda - 1 \le c_1(M)$ for some sufficiently small positive constant c_1 depending on M, then there is a λ-DSS solution v with initial data v_0 such that v is regular, that is, $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3 \times (0, \infty))$. The question is: What if we weaken the assumption of v_0 so that v_0 belongs to L^p or $L^{p,\infty}(\mathbb{R}^3)$ (i.e. weak L^p space)? Note that for $v_0 \in L^{3,\infty}(\mathbb{R}^3)$ that is λ-DSS and divergence free, Bradshaw and Tsai [2] constructed at least one λ-DSS local Leray solution. However the proof does not imply regularity of the solutions, since it is based on a weak solution approach and used compactness argument.

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Motivated by this problem, we need to study solutions whose initial data is locally in L^3, as it is also shown in [2] that, when v_0 is λ-DSS, then $v_0 \in L^{3,\infty}(\mathbb{R}^3)$ if and only if $v_0 \in L^3(B_\lambda \setminus B_1)$.

In order to state our results, we first recall the notion of the *suitable weak solution*. For any domain $\Omega \subset \mathbb{R}^3$ and open interval $I \subset (0,\infty)$, we say (v,π) is a suitable weak solution in $\Omega \times I$ if it satisfies (NS) in the sense of distributions in $\Omega \times I$,

$$v \in L^\infty(I; L^2(\Omega)) \cap L^2(I; \dot{H}^1(\Omega)), \quad \pi \in L^{3/2}(\Omega \times I),$$

and the local energy inequality:

$$\begin{aligned} &\int_\Omega |v(t)|^2 \phi(t)\, dx + 2\int_0^t \int_\Omega |\nabla v|^2 \phi\, dx\, dt \\ &\le \int_0^t \int_\Omega |v|^2 (\partial_t \phi + \Delta \phi)\, dx\, dt + \int_0^t \int_\Omega (|v|^2 + 2\pi)(v \cdot \nabla \phi)\, dx\, dt \end{aligned} \tag{1.2}$$

for all non-negative $\phi \in C_c^\infty(\Omega \times I)$. Note that no boundary condition is assumed.

The following theorem is our first main result.

Theorem 1.1. *There exist positive constants ϵ_0 and C_1 such that the following holds. Let (v,π) is a suitable weak solution of the Navier-Stokes equations (NS) in $B_1 \times (0,T_0)$, $T_0 > 0$, with divergence free initial data v_0 in the sense $\lim_{t\to 0^+} \|v(t) - v_0\|_{L^2(B_1)} = 0$. For any $M > 0$, there exists $T_1 = T_1(M) \in (0,T_0]$ such that if (v,π) satisfies*

$$\|v_0\|_{L^3(B_1)} \le \epsilon_0 \tag{1.3}$$

and

$$\|v\|^2_{L^\infty_t L^2_x \cap L^2_t H^1_x(B_1\times(0,T_1))} + \|\pi\|_{L^2_t L^{3/2}_x(B_1\times(0,T_1))} \le M, \tag{1.4}$$

then v is regular in $B_{1/4} \times (0,T_1)$ and satisfies

$$|v(x,t)| \le \frac{C_1}{\sqrt{t}} \quad \text{in} \quad B_{1/4} \times (0,T_1), \tag{1.5}$$

$$\sup_{z_0 \in B_{\frac14}\times(0,T_1)} \sup_{0<r<\infty} \frac{1}{r^2} \int_{Q_r(z_0)\cap[B_1\times(0,T_1)]} |v|^3\, dz \le 1. \tag{1.6}$$

Moreover we can choose $T_1(M) = \min\{c_1(1+M)^{-6}, T_0\}$ with some universal constant c_1.

Above, we use the notation $L^p_t L^q_x(A\times I) := L^p(I; L^q(A))$ for $A \subset \mathbb{R}^3$ and $I \subset \mathbb{R}$, and $Q_r(z) := B_r(x) \times (t-r^2, t)$ for $z = (x,t)$.

Comments for Theorem 1.1:

1. It should be noted that the constant C_1 is independent of M. Intuitively, the nonlinear term has no effect before $T_1 = T_1(M)$, and hence the solution behaves like a linear solution, and its size is given by the initial data.

2. The boundedness of π in $L^2_t L^{3/2}_x$ is natural for the Leray-Hopf weak solutions defined in $\mathbb{R}^3$, as π is given by $\pi = R_i R_j (v_i v_j)$, where $R_j = (-\Delta)^{-1/2}\partial_j$ is the Riesz transform, and

$$\|\pi\|_{L^2_t L^{3/2}_x(\mathbb{R}^3\times(0,T))} \le C\|v\|^2_{L^4_t L^3_x(\mathbb{R}^3\times(0,T))} \le C\|v\|^2_{L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(\mathbb{R}^3\times(0,T))}.$$

3. The assumption $\|\pi\|_{L^2_t L^{3/2}_x(B_1\times(0,T_1))} \le M$ can be replaced by, e.g., $\|\pi\|_{L^q(B_1\times(0,T_1))} \le M$ for some $q \in (3/2, 5/3]$. It ensures that $\int_0^T \int_{B_1} |v|^3 + |p|^{3/2} dx dt$ is small for sufficiently small $T = T(M)$ (thus $q = 3/2$ is not allowed), which is one of the key in the proof. Our choice of exponents is to maximize the time exponent, so that $T_1(M) = c(1+M)^{-m}$ has the smallest $m = 6$.

4. Theorem 1.1 is an extension of Jia-Šverák [16, Theorem 3.1], in which the initial data is assumed in $L^m(B_1)$, $m > 3$. This is similar to the extension of the mild solution theory for the scale subcritical data $v_0 \in L^m(\mathbb{R}^3)$, $m > 3$, of Fabes-Jones-Rivière [9] to the critical data $v_0 \in L^3(\mathbb{R}^3)$ of Weissler [33], Giga-Miyakawa [11], Kato [20] and Giga [10].

Our first set of applications of Theorem 1.1 is concerned with *local Leray solutions*, which are suitable weak solutions of (NS) defined in $\mathbb{R}^3 \times (0, \infty)$ that satisfy a mild decay condition at spatial infinity; see Definition 1.2. In order to state the results, we introduce the uniformly local L^q spaces. For $q \in [1, \infty)$, we say $f \in L^q_{\text{uloc}}$ if $f \in L^q_{\text{loc}}(\mathbb{R}^3)$ and

$$\|f\|_{L^q_{\text{uloc}}} = \sup_{x \in \mathbb{R}^3} \|f\|_{L^q(B_1(x))} < \infty. \tag{1.7}$$

We also denote for $\rho > 0$

$$\|f\|_{L^q_{\text{uloc},\rho}} = \sup_{x \in \mathbb{R}^3} \|f\|_{L^q(B_\rho(x))}.$$

Let E^q be the closure of $C_c^\infty(\mathbb{R}^3)$ in L^q_{uloc}-norm. Equivalently, E^q consists of those $f \in L^q_{\text{uloc}}$ with $\lim_{|x| \to \infty} \|f\|_{L^q(B_1(x))} = 0$, see [22].

Definition 1.2 (Local Leray solutions [15, 16]). *A vector field $v \in L^2_{\text{loc}}(\mathbb{R}^3 \times [0, \infty))$ is a local Leray solution to (NS) with divergence free initial data $v_0 \in E^2$ if*

1. *for some $\pi \in L^{3/2}_{\text{loc}}(\mathbb{R}^3 \times [0, \infty))$, the pair (v, π) is a distributional solution to (NS),*
2. *for any $R > 0$,*

$$\operatorname*{ess\,sup}_{0 \le t < R^2} \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} |v|^2 \, dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^{R^2} \int_{B_R(x_0)} |\nabla v|^2 \, dx \, dt < \infty, \tag{1.8}$$

3. *for all compact subsets K of $\mathbb{R}^3$ we have $v(t) \to v_0$ in $L^2(K)$ as $t \to 0^+$,*
4. *(v, π) satisfies the local energy inequality (1.2) for all non-negative $\phi \in C_c^\infty(Q)$ with all cylinder Q compactly supported in $\mathbb{R}^3 \times (0, \infty)$,*
5. *for any $R > 0$,*

$$\lim_{|x_0| \to \infty} \int_0^{R^2} \int_{B_R(x_0)} |v|^2 \, dx \, dt = 0. \tag{1.9}$$

In the following corollary we assume that the initial data belongs to $L^3(B_\delta) \cap E^2$.

Corollary 1.3. *Let ϵ_0 and C_1 be the constants from Theorem 1.1. Suppose v is a local Leray solution of the Navier-Stokes equations (NS) with divergence free initial data $v_0 \in E^2$ and there exists $\delta \in (0, \infty)$ such that*

$$\|v_0\|_{L^3(B_\delta)} \le \epsilon_0. \tag{1.10}$$

Then there exists $T_2 = T_2(\delta, N_\delta) > 0$ with $N_\delta := \frac{1}{\delta} \sup_{x_0 \in \mathbb{R}^3} \int_{B_\delta(x_0)} |v_0|^2 dx$ such that v is regular in $B_{\delta/4} \times (0, T_2)$ and satisfies

$$|v(x,t)| \le \frac{C_1}{\sqrt{t}} \quad \text{in} \quad B_{\delta/4} \times (0, T_2),$$

$$\sup_{z_0 \in B_{\delta/4} \times (0, T_2)} \sup_{0 < r < \infty} \frac{1}{r^2} \int_{Q_r(z_0) \cap [B_{\delta/4} \times (0, T_2)]} |v|^3 \, dz \le 1.$$

Furthermore, we can take $T_2 = c_2(1 + N_\delta)^{-6} \delta^2$ with some universal constant c_2.

Comments for Corollary 1.3:

1. Compared to Theorem 1.1, the local Leray solution in Corollary 1.3 is defined globally in $\mathbb{R}^3$ and the assumption (1.4) for the solution is not necessary. We also have flexibility of the radius of the ball in (1.10). Note that the time T_2 depends on the radius, which is important for our applications.

2. A result similar to Corollary 1.3 was independently obtained by Barker and Prange [1, Theorem 1]. In [1], it was proved that any local Leray solution is bounded under similar assumptions as those of Corollary 1.3, and the smallness assumption of local L^3 norm (1.10) is further relaxed to $L^{3,\infty}$ or critical Besov norms. Their approach is different to ours and relies upon the iteration method by Caffarelli, Kohn and Nirenberg [7], while ours is based on the blow-up and the compactness argument by Lin [25].

3. Consider general initial data $v_0 \in E^2$. Define

$$\rho(x) = \rho(x; v_0) = \sup \left\{ r > 0 : v_0 \in L^3(B_r(x)), \int_{B_r(x)} |v_0|^3 \leq \epsilon_0^3 \right\}.$$

   Let $\rho(x) = 0$ if such r does not exist, and let $\rho(x) = \infty$ if $\int_{\mathbb{R}^3} |v_0|^3 \leq \epsilon_0^3$. We also define

$$T(x) = c_2(1 + N_{\rho(x)})^{-6} \rho(x)^2 \in [0, \infty].$$

   For each $x \in \mathbb{R}^3$ applying Corollary 1.3 with $\delta = \rho(x)$, we see any local Leray solution v is regular in the region

$$\Omega = \{(x,t) : \ x \in \mathbb{R}^3, \ 0 < t < T(x)\},$$

$$|v(x,t)| \leq \frac{C_1}{\sqrt{t}} \quad \text{in } \Omega.$$

   Of course this is interesting only near those x with $\rho(x; v_0) > 0$.

In the next corollary we assume the initial data $v_0 \in L^3_{\mathrm{uloc}}(\mathbb{R}^3) \cap E^2$.

Corollary 1.4. *Let ϵ_0 and C_1 be the constants from Theorem 1.1. Suppose v is a local Leray solution of the Navier-Stokes equations (NS) with divergence free initial data $v_0 \in E^2$ and there exists $\delta \in (0, \infty)$ such that*

$$\|v_0\|_{L^3_{\mathrm{uloc},\delta}} \leq \epsilon_0. \tag{1.11}$$

Then there exists $T_3 = T_3(\delta) > 0$ such that v is regular in $\mathbb{R}^3 \times (0, T_3)$ and satisfies

$$|v(x,t)| \leq \frac{C_1}{\sqrt{t}}, \quad (0 < t < T_3), \tag{1.12}$$

where T_3 can be taken as $T_3 = c_3\delta^2$ with some universal constant c_3.

This result is similar to the one by Maekawa-Terasawa [27, Theorem 1.1 (iii)]. Indeed under the assumption (1.11) the authors in [27] constructed mild solutions in $L^\infty(0, T; L^3_{\mathrm{uloc}})$ and showed that such solutions satisfy (1.12) with $T = C\delta^2 \|v_0\|^{-4}_{L^3_{\mathrm{uloc},\delta}}$. We emphasize that, compared to the existence theorem of [27], Corollary 1.4 is a regularity theorem for any local Leray solution, but assuming further $v_0 \in E^2$.

In the second set of applications, we consider solutions with initial data in the *Herz spaces*. These spaces contain self-similar and DSS solutions, and are of particular interest to the study of DSS solutions since they are weighted spaces with a particular choice of centre. We now recall the definitions and basic properties of Herz spaces [14, 28, 32]. Let $A_k = \{x \in \mathbb{R}^n : 2^{k-1} \leq |x| < 2^k\}$.

For $n \in \mathbb{N}$, $s \in \mathbb{R}$ and $p, q \in (0, \infty]$, the *homogeneous Herz space* $\dot{K}^s_{p,q}(\mathbb{R}^n)$ is the space of functions $f \in L^p_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ with finite norm

$$\|f\|_{\dot{K}^s_{p,q}} = \begin{cases} \left(\sum_{k\in\mathbb{Z}} 2^{ksq} \|f\|^q_{L^p(A_k)} \right)^{1/q} & \text{if } q < \infty, \\ \sup_{k\in\mathbb{Z}} 2^{ks} \|f\|_{L^p(A_k)} & \text{if } q = \infty. \end{cases}$$

The *weak Herz space* $W\dot{K}^s_{p,q}(\mathbb{R}^n)$ are defined similarly, with $L^p(A_k)$-norm in the definition replaced by its weak version, $L^{p,\infty}(A_k)$-norm.

In what follows we take $q = \infty$, which is most suitable for our purpose. In this case, $\dot{K}^s_{p,\infty}$-norm is equivalent to

$$\|f\|_{s,p} = \sup_{x_0 \neq 0} \left\{ |x_0|^s \cdot \|f\|_{L^p(B_{\frac{|x_0|}{2}}(x))} \right\}.$$

We are interested in the Herz spaces because they seem to be natural spaces for DSS solutions of (NS). The existence problem of mild solutions of (NS) in the Herz spaces has been studied extensively by Tsutsui [32]. He proved local in time existence of mild solutions for large data in subcritical weak Herz spaces $W\dot{K}^s_{p,\infty}(\mathbb{R}^3)$, $0 \le s < 1 - 3/p$, and global existence for small data in the critical weak Herz space $W\dot{K}^0_{3,\infty}(\mathbb{R}^3)$. The following results concern the regularity of the solution for the initial data in the critical case K_p with $p \ge 3$.

Theorem 1.5. *Let ϵ_0 and C_1 be the constants from Theorem 1.1. Let v be a local Leray solution of the Navier-Stokes equations (NS) with divergence free initial data $v_0 \in E^2$. Assume further that there exists $\mu \in (0, 1)$ such that*

$$\sup_{x\neq 0} \|v_0\|_{L^3(B_{\mu|x|}(x))} \le \epsilon_0. \tag{1.13}$$

Then there exist $\sigma_1 = \sigma_1(\|v_0\|_{K_3}) > 0$, $C_2 = C_2(\|v_0\|_{K_3})$, and $\sigma_2 = \sigma_2(\mu, \|v_0\|_{K_3}) \in (0, \sigma_1]$ such that

$$\sup_{0<t<\sigma_1 r^2} \sup_{x_0\in\mathbb{R}^3} \frac{1}{r} \int_{B_r(x_0)} |v(t)|^2 \, dx + \sup_{x_0\in\mathbb{R}^3} \frac{1}{r} \int_0^{\sigma_1 r^2} \int_{B_r(x_0)} |\nabla v|^2 \, dxdt \le C_2 \tag{1.14}$$

for any $r > 0$, and v is regular in the region

$$\Sigma = \{(x, t) \ : \ 0 < t < \sigma_2 |x|^2\}$$

and satisfies

$$|v(x, t)| \le \frac{C_1}{\sqrt{t}} \quad \text{in } \Sigma. \tag{1.15}$$

Comments for Theorem 1.5:

1. We easily see

$$f \in K_3 \qquad \text{if and only if} \qquad \sup_{x_0\neq 0} \int_{B_{\mu|x_0|}(x_0)} |f|^3 \, dx < \infty \quad \text{for } any\ \mu \in (0, 1).$$

In particular, the assumption (1.13) implies $\|v_0\|_{K_3} = \sup_{x_0\neq 0} \|v_0\|_{L^3(B_{\frac{|x_0|}{2}}(x))}$ is finite (but not small in general).

2. For $v_0 \in K_p$, $p > 3$, the same conclusion of Theorem 1.5 is true, with the constants depending only on $\|v_0\|_{K_p}$. This is obtained from Theorem 1.5, since (1.13) is valid for $\mu = \min(1/2, C^{-1}(\epsilon_0/\|v_0\|_{K^p})^{p/(p-3)})$ from the following estimate:

$$\|v_0\|_{L^3(B_{\mu|x|}(x))} \le (C\mu|x|)^{1-\frac{3}{p}} \|v_0\|_{L^p(B_{\mu|x|}(x))} \le (C\mu|x|)^{1-\frac{3}{p}} \|v_0\|_{L^p(B_{|x|/2}(x))} \le C\mu^{1-\frac{3}{p}} \|v_0\|_{K_p}.$$

The following corollary answers our motivating problem:

Corollary 1.6. *(i) Let $\lambda > 1$ and v be a λ-DSS local Leray solution of the Navier-Stokes equations* (NS) *with λ-DSS divergence free data $v_0 \in L^{3,\infty}(\mathbb{R}^3)$. Then $v_0 \in K_3$,* (1.13) *holds for some $\mu \in (0,1)$, and the same conclusion of Theorem 1.5 is true.*

(ii) For any $\mu \in (0,1)$, there exists $\lambda_* = \lambda_*(\mu) \in (1,2)$ such that if any λ-DSS divergence free data $v_0 \in L^{3,\infty}(\mathbb{R}^3)$ with factor $\lambda \in (1, \lambda_*]$ satisfies (1.13)*, then the λ-DSS local Leray solution v is regular in $\mathbb{R}^3 \times (0,\infty)$ with*

$$|v(x,t)| \le \frac{C_3}{\sqrt{t}} \quad \text{in } \mathbb{R}^3 \times (0,\infty),$$

where C_3 is a constant depending on v_0.

Remark. In Corollary 1.6, $\lambda - 1$ has to be sufficiently small and its smallness depends on the ratio parameter μ in (1.13). The situation is similar to [31, Theorem 1.1]: The pointwise estimate is based on regularity theory, which is known only for short time. If $\lambda - 1$ is not small, we cannot expect to use the available regularity theory to prove pointwise estimate everywhere.

The rest of this article is organized as follows. In Section 2 we recall auxiliary results, including the theorems of Caffarelli-Kohn-Nirenberg [7], Kato [20], and the localization of divergence free vector fields. We also present an interior regularity result for the perturbed Stokes equation, which plays a crucial role in the proof of Theorem 1.1. Then we address the local analysis of the Navier-Stokes equations and the proof of Theorem 1.1 in Section 3.

2 Preliminaries

We first recall the following rescaled version of the result of Caffarelli-Kohn-Nirenberg [7, Proposition 1]. It is formulated in the present form in [29, 25], and is the basis for many regularity criteria, see e.g. in [13].

Lemma 2.1. *There are absolute constants ϵ_{CKN} and $C_{CKN} > 0$ with the following property. Suppose (v,π) is a suitable weak solution of* (NS) *with zero force in Q_{r_1}, $r_1 > 0$, with*

$$\frac{1}{r_1^2}\int_{Q_{r_1}} |v|^3 dx\, dt + \frac{1}{r_1^2}\int_{Q_{r_1}} |\pi|^{3/2} dx\, dt \le \epsilon_{CKN},$$

then $v \in L^\infty(Q_{r_1/2})$ and

$$\|v\|_{L^\infty(Q_{r_1/2})} \le \frac{C_{CKN}}{r_1}. \tag{2.1}$$

We next recall the results due to Kato [20] and Giga [10].

Lemma 2.2. *There exists $\epsilon_2 > 0$ such that if $v_0 \in L^3_\sigma(\mathbb{R}^3)$ with $\epsilon = \|v_0\|_{L^3} \le \epsilon_2$, then there is a unique mild solution $v \in L^\infty(0,\infty; L^3(\mathbb{R}^3))$ of* (NS) *with zero force and initial data v_0 that satisfies*

$$\|v\|_{L^\infty_t L^3_x \cap L^5_{t,x}(\mathbb{R}^3\times(0,\infty))} + \sup_{t>0} t^{1/2}\|v(t)\|_{L^\infty(\mathbb{R}^3)} \le C\epsilon. \tag{2.2}$$

The following lemma concerns localization of divergence free vector fields.

Lemma 2.3 (localization)**.** *Let $1 < p < \infty$ and $0 < r < R$. There is a linear map Φ from $V = \{v \in L^p(B_R;\mathbb{R}^3) : \operatorname{div} v = 0\}$ into itself, and a constant $C = C(p, r/R) > 0$ such that for $v \in V$ and $a = \Phi v \in V$, we have $\operatorname{supp} a \subset B_{\frac{1}{2}(r+R)}$, $v = a$ in B_r, and $\|a\|_{L^p(B_R)} \le C\|v\|_{L^p(B_R)}$.*

We will also recall the following lemma, which is proved by Jia and Šverák [16, Lemma 2.1].

Lemma 2.4. *Let f be a nonnegative nondecreasing bounded function defined on $[0,1]$ with the following property: for some constants $0 < \sigma < 1$, $0 < \theta < 1$, $M > 0$, $\beta > 0$, we have*

$$f(s) \le \theta f(t) + \frac{M}{(t-s)^\beta}, \quad \sigma < s < t < 1.$$

Then,

$$\sup_{s\in[0,\sigma]} f(s) \le C(\sigma,\theta,\beta)M,$$

for some positive constant C depending only on σ, θ, β.

We end this section with the following interior result for the perturbed Stokes system. Recall $Q_r = B_r \times (-r^2, 0)$.

Proposition 2.5. *For any $q \in [5,\infty)$, there exists $\delta_0 = \delta_0(q) > 0$ such that the following statement holds. For any $M > 0$, if $G \in L^5(Q_1;\mathbb{R}^{3\times 3})$ with $\|G\|_{L^5(Q_1)} \le M$, $a \in L^5(Q_1)$ with $\operatorname{div} a = 0$, $\|a\|_{L^5(Q_1)} \le \delta_0$, $\xi \in \mathbb{R}^3$, $|\xi| \le 1$, $u \in L^\infty L^2 \cap L^2\dot{H}^1(Q_1)$, $p \in L^{3/2}(Q_1)$,*

$$\|u\|_{L^3(Q_1)} + \|p\|_{L^{3/2}(Q_1)} \le M,$$

and they solve the a-perturbed Stokes equations

$$u_t - \Delta u + (a+\xi)\cdot\nabla u + u\cdot\nabla a + \operatorname{div} G + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \textit{in } Q_1, \tag{2.3}$$

then we have

$$u \in L^q(Q_{1/2}), \quad \|u\|_{L^q(Q_{1/2})} \le C(q)M.$$

This proposition is proved via bootstrap argument based on a localization technique and the linear Stokes estimates; see [19] for the details.

3 Local analysis for the Navier-Stokes equations

In this section we prove Theorem 1.1. The proof is split into 3 subsections.

3.1 Decay estimates for the perturbed Navier-Stokes equation

Let (u,p) be a suitable weak solution of the following a-perturbed Navier-Stokes equations in $Q = B_1 \times (0,T)$, with $a \in L^5(Q)$, $\operatorname{div} a = 0$,

$$u_t - \Delta u + (a+u)\cdot\nabla u + u\cdot\nabla a + \nabla p = 0, \qquad \operatorname{div} u = 0. \tag{3.1}$$

That is, $u \in L^\infty L^2(Q) \cap L^2\dot{H}^1(Q)$, $p \in L^{3/2}(Q)$, the pair solves (3.1) in the distributional sense, and satisfies the *perturbed local energy inequality*: For all non-negative $\phi \in C_c^\infty(Q)$, we have

$$\begin{aligned}
&\int |u|^2\phi(t)\,dx + 2\int_0^t\!\!\int |\nabla u|^2\phi\,dx\,dt \\
&\le \int_0^t\!\!\int |u|^2(\partial_t\phi + \Delta\phi)\,dx\,dt + \int_0^t\!\!\int \left((|u|^2(u+a) + 2pu\right)\cdot\nabla\phi\,dx\,dt \\
&\quad + \int_0^t\!\!\int u_j a_i \partial_j(u_i\phi)\,dx\,dt.
\end{aligned} \tag{3.2}$$

This is equivalent to (1.2) for $v = u + a$ if v is a weak solution of (NS) in Q and a is a strong solution of (NS); see the argument in Subsection 3.3 for details.

Let $z_0 = (x_0, t_0)$ and $Q_r(z_0) = B_r(x_0) \times (t_0 - r^2, t_0)$. We denote

$$\varphi(u,p,r,z_0) := \left(\frac{1}{r^2}\int_{Q_r(z_0)} \left|u - (u)_{Q_r(z_0)}\right|^3 dz\right)^{\frac13} + \left(\frac{1}{r^2}\int_{Q_r(z_0)} \left|p - (p)_{B_r(x_0)}(t)\right|^{3/2} dz\right)^{\frac23}, \tag{3.3}$$

where

$$(u)_{Q_r(z_0)} = \frac{1}{|Q_r(z_0)|}\int_{Q_r(z_0)} u dz, \qquad (p)_{B_r(x_0)}(t) = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} p\,dx.$$

Note that φ is dimension-free in the sense of [7], and its form is invariant under scaling.

Lemma 3.1 (Decay estimate)**.** *For any $\alpha \in (0,1)$, there is a small $\delta_0 > 0$ such that the following holds. Let (u,p) be a suitable weak solution to the perturbed Navier-Stokes equations (3.1) in $Q_r(z)$, with $a \in L^5(Q_r(z))$, $\operatorname{div} a = 0$, $\|a\|_{L^5(Q_r(z))} = \delta \le \delta_0$. Denote $(u)_r = (u)_{Q_r(z)}$. Then, for any $\theta \in (0,1/3)$ there exist $\epsilon = \epsilon(\theta,\alpha) > 0$ and $C = C(\alpha) > 0$ independent of θ such that if*

$$r|(u)_r| \le 1, \quad \varphi(u,p,r,z) + r\,|(u)_r|\,\delta < \epsilon, \tag{3.4}$$

then

$$\theta r\,|(u)_{\theta r}| \le 1, \tag{3.5}$$

$$\varphi(u,p,\theta r,z) \le C\theta^\alpha \left[\varphi(u,p,r,z) + r\,|(u)_r|\,\delta\right]. \tag{3.6}$$

Proof. Take $q \in (5,\infty)$ such that $\alpha < 1 - \frac{5}{q}$ and choose $\delta_0 = \delta_0(q(\alpha))$ according to Proposition 2.5. Since φ and $r(u)_r$ are dimension-free, we may assume $r = 1$. We may also assume $z = 0$ and skip the z-dependence in φ without loss of generality. We first show (3.5). Indeed,

$$\begin{aligned}\theta|(u)_\theta| &\le \theta|(u-(u)_1)_\theta| + \theta|(u)_1| \\ &\le \theta|Q_\theta|^{-\frac13}\|u-(u)_1\|_{L^3(Q_\theta)} + \theta \\ &\le C_3\theta^{-\frac23}\varphi(1) + \theta,\end{aligned} \tag{3.7}$$

with $C_3 = |Q_1|^{-\frac13}$. By (3.4), $\varphi(1) \le \epsilon$, hence $\theta|(u)_\theta| < 1$ if

$$\epsilon \le \frac{\theta^{2/3}}{2C_3}. \tag{3.8}$$

Next we show the decay estimate (3.6). Here we argue by contradion, following a similar argument as given in e.g. [25, Lemma 3.2] and [16, Lemma 2.3]. Since some modification is required, we give the details for completeness. Suppose that this is not the case. Then there exist suitable weak solutions (u_i,p_i) of (3.1), a_i, and ϵ_i with $\lim_{i\to\infty}\epsilon_i = 0$ such that

$$\xi_i = (u_i)_1, \quad |\xi_i| \le 1, \quad \|a_i\|_{L^5(Q_1)} \le \delta_0, \quad \operatorname{div} a_i = 0,$$

$$\varphi(u_i,p_i,1) + |\xi_i|\|a_i\|_{L^5(Q_1)} = \epsilon_i,$$

$$\varphi(u_i,p_i,\theta) \ge C_2\theta^\alpha\epsilon_i.$$

Here $C_2 > 0$ is a large constant to be chosen later. Setting $v_i = (u_i - \xi_i)/\epsilon_i$ and $q_i = (p_i - (p_i)_1(t))/\epsilon_i$, it follows that

$$\|v_i\|_{L^3(Q_1)} + \|q_i\|_{L^{\frac32}(Q_1)} + \frac{|\xi_i|}{\epsilon_i}\|a_i\|_{L^5(Q_1)} = 1,$$

$$\left(\frac{1}{\theta^2}\int_{Q_\theta}|v_i-(v_i)_{Q_\theta}|^3\,dz\right)^{\frac13} + \left(\frac{1}{\theta^2}\int_{Q_\theta}|q_i-(q_i)_{B_\theta}(t)|^{\frac32}\,dz\right)^{\frac23} \ge C_2\theta^\alpha, \tag{3.9}$$

and (v_i,q_i) satisfies

$$\partial_t v_i - \Delta v_i + (\epsilon_i v_i + a_i + \xi_i)\cdot\nabla v_i + \left(v_i + \frac{\xi_i}{\epsilon_i}\right)\cdot\nabla a_i + \nabla q_i = 0, \quad \operatorname{div} v_i = 0.$$

Denote

$$E_i(r) = \operatorname{ess}\sup_{-r^2<t<0}\int_{B_r}\frac{|v_i|^2}{2}dx + \int_{-r^2}^0\int_{B_r}|\nabla v_i|^2\,dxdt.$$

By the local energy inequality for (3.1), the calculation in [16, page 242] shows that, for $3/4 < r_1 < r_2 < 1$,

$$E_i(r_1) \le \frac{C}{(r_2 - r_1)^2} + (C\|a_i\|_{L^5(Q_1)} + \frac{1}{2})E_i(r_2),$$

By Lemma 2.4, if $\|a_i\|_{L^5(Q_1)} \le \delta_0$ is sufficiently small, we have $E_i(3/4) < C$ for all i.

By the uniform bound $E_i(3/4) < C$ for all i, there exist $(v, q) \in (L^3 \times L^{3/2})(Q_{3/4})$, $\xi \in \mathbb{R}^3$, and $a, G \in L^5(Q_{3/4})$ such that (if necessary, subsequence can be taken)

$$v_i \longrightarrow v \quad \text{strongly in} \quad L^3(Q_{3/4}), \qquad \xi_i \longrightarrow \xi,$$

$$q_i \rightharpoonup q \quad \text{weakly in} \quad L^{\frac{3}{2}}(Q_{3/4}), \qquad a_i \rightharpoonup a \quad \text{weakly in} \quad L^5(Q_{3/4}),$$

$$\frac{\xi_i}{\epsilon_i} \otimes a_i \rightharpoonup G \quad \text{weakly in} \quad L^5(Q_{3/4}),$$

as $i \to \infty$. Furthermore, (v, q) solves the linear perturbed Stokes system in $Q_{3/4}$

$$\partial_t v - \Delta v + \xi \cdot \nabla v + a \cdot \nabla v + v \cdot \nabla a + \operatorname{div} G + \nabla q = 0, \quad \operatorname{div} v = 0.$$

Due to Proposition 2.5, it follows that $v \in L^q(Q_{1/2})$, $q > 5$, for the exponent q chosen at the beginning of the proof. Thus, by the strong convergence of v_i to v in $L^3(Q_{3/4})$, we have for sufficiently large i

$$\left(\frac{1}{\theta^2}\int_{Q_\theta} |v_i - (v_i)_\theta|^3 \, dz\right)^{\frac{1}{3}} \le C\theta^{1-\frac{5}{q}} \le C\theta^\alpha. \tag{3.10}$$

On the other hand, by the pressure equation, we decompose $q_i = q_i^R + q_i^H$ such that

$$q_i^R = (-\Delta)^{-1}\operatorname{div}\operatorname{div}\left([\epsilon v_i \otimes v_i + v_i \otimes a_i + a_i \otimes v_i]\chi_{B_{\frac{3}{4}}}\right).$$

Here $\chi_{B_{\frac{3}{4}}}$ is the characteristic function of $B_{\frac{3}{4}}$. Since v_i converges strongly to v in $L^3(Q_{3/4})$ and a_i converges weakly to a in $L^5(Q_{3/4})$, the Calderón-Zygmund estimate implies that q_i^R converges strongly to q^R in $L^{\frac{3}{2}}(Q_{3/4})$, where q^R is

$$q^R = (-\Delta)^{-1}\operatorname{div}\operatorname{div}\left([v \otimes a + a \otimes v]\chi_{B_{\frac{3}{4}}}\right).$$

We note that $q^R \in L^l(Q_{1/2})$, where $1/l = 1/q + 1/5$. Therefore,

$$\left(\frac{1}{\theta^2}\int_{Q_\theta} |q^R|^{\frac{3}{2}} \, dz\right)^{\frac{2}{3}} \le C\theta^{2-\frac{5}{l}} = C\theta^{1-\frac{5}{q}}.$$

Thus, for large i, we also have

$$\left(\frac{1}{\theta^2}\int_{Q_\theta} |q_i^R|^{\frac{3}{2}} \, dz\right)^{\frac{2}{3}} \le C\theta^{1-\frac{5}{q}}.$$

Since q_i^H is harmonic (in x) in $Q_{3/4}$, we see that

$$\left(\frac{1}{\theta^2}\int_{Q_\theta} |q_i^H - (q_i^H)_{B_\theta}(t)|^{\frac{3}{2}} \, dz\right)^{\frac{2}{3}} \le C\theta^{\frac{5}{3}}.$$

Adding up the above estimates,

$$\left(\frac{1}{\theta^2}\int_{Q_\theta} |q_i - (q_i)_{B_\theta}(t)|^{\frac{3}{2}}\right)^{\frac{2}{3}} \le C\theta^{1-\frac{5}{q}} \le C\theta^\alpha. \tag{3.11}$$

The sum of (3.10) and (3.11) contradicts (3.9) if we take C_2 sufficiently large. This completes the proof. □

3.2 Regularity criterion for the perturbed Navier-Stokes equations

In this subsection we prove the following regularity criterion for perturbed Navier-Stokes equations (3.1). It is an extension of the result [16, Theorem 2.2] for the perturbed term $a \in L^m(Q_1)$ with $m > 5$.

Lemma 3.2 (Regularity criterion)**.** *For any fixed $\beta \in (0,1)$, there exist small constants $\epsilon_1(\beta)$ and $\delta(\beta) > 0$ with the following properties: Let (u,p) be a suitable weak solution to the perturbed Navier-Stokes equations (3.1) in $Q_{3/4}$, with $a \in L^5(Q_{3/4})$, $\operatorname{div} a = 0$, $\|a\|_{L^5(Q_{3/4})} \le \delta$, and*

$$\int_{Q_{3/4}} |u|^3 + |p|^{\frac{3}{2}}\, dz \le \epsilon_1. \tag{3.12}$$

Then we have

$$\sup_{z_0=(x_0,t_0)\in Q_{\frac14}} \sup_{r<\frac14} \frac{1}{r^{2+3\beta}} \int_{Q_r(z_0)\cap Q_{3/4}} |u|^3 + \left|p-(p)_{B_r(x_0)}(t)\right|^{3/2} dz < C(\beta). \tag{3.13}$$

Remark. Our estimate (3.13) does not imply Hölder continuity, but Morrey type regularity. On the other hand, the Hölder continuity was shown by a different method in [1].

Proof. For fixed $\beta \in (0,1)$, choose $\alpha = (1+\beta)/2$ so that $\alpha \in (\beta,1)$, and choose $\theta \in (0,1/3)$ so that the factor $C\theta^\alpha$ in (3.6) is bounded by $\frac12\theta^\beta$, and $\theta^{1-\beta} < \frac12$. In the following we omit the dependence on $z_0 \in Q_{1/4}$ to simplify the notation. Let $B(r) = r\,|(u)_r|$ and $\varphi(r)$ be defined by (3.3). It is proved in (3.7) for $r=1$ that

$$B(\theta r) \le C_3\theta^{-\frac23}\varphi(r) + \theta B(r), \tag{3.14}$$

where $C_3 = |Q_1|^{-1/3}$. The proof for general r is the same. Let

$$\Psi(r) = \varphi(r) + (2C_3)^{-1}\theta^{\frac23+\beta}B(r).$$

We will show by induction that

$$\text{condition (3.4) is valid} \quad \text{and} \tag{3.15}$$

$$\Psi(\theta r) \le \theta^\beta \Psi(r) \tag{3.16}$$

for $r \in I_k = [\frac{\theta^{k+1}}{4}, \frac{\theta^k}{4}]$ with $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let

$$\Psi_k = \sup_{z_0\in Q_{1/4},\ r\in I_k} \Psi(r;z_0), \quad k \in \mathbb{N}_0.$$

By (3.12),

$$\Psi_0 \le C(\beta)\epsilon_1^{1/3} \le \epsilon,$$

if $\epsilon_1 = \epsilon_1(\beta)$ is sufficiently small. In particular, the condition (3.4) is uniformly satisfied for every $z_0 = (x_0,t_0) \in Q_{1/4}$ and $r \in I_0$.

Suppose that (3.15) has been proved for $r \in \cup_{j<k} I_j$ and condition (3.4) is satisfied for $r \in I_k$ for some $k \in \mathbb{N}_0$. By (3.6) of Lemma 3.1 and (3.14) (note Lemma 3.1 is formulated in any scale),

$$\begin{aligned}
\Psi(\theta r) &= \varphi(\theta r) + (2C_3)^{-1}\theta^{\frac23+\beta}B(\theta r)\\
&\le \frac{\theta^\beta}{2}\varphi(r) + \frac{\theta^\beta}{2}\delta B(r) + \frac{\theta^\beta}{2}\varphi(r) + (2C_3)^{-1}\theta^{\frac53+\beta}B(r)\\
&= \theta^\beta\varphi(r) + \theta^\beta\left(C_3\delta\theta^{-\frac23-\beta} + \theta^{1-\beta}\right)(2C_3)^{-1}\theta^{\frac23+\beta}B(r),
\end{aligned}$$

which is bounded by $\theta^\beta\Psi(r)$ if $\delta \le \min\{\delta_0(\alpha), (2C_3)^{-1}\theta^{\frac23+\beta}\}$. This shows (3.16) for $r \in I_k$.

As a result, $\Psi_{k+1} \le \theta^\beta \Psi_k \le \cdots \le \theta^{(k+1)\beta}\Psi_0 \le \theta^{(k+1)\beta}\epsilon$. Hence

$$r|(u)_r| = B(r) \le 2C_3\theta^{-\frac{2}{3}-\beta}\Psi_{k+1} \le 2C_3\theta^{-\frac{2}{3}-\beta}\theta^\beta\epsilon \le 1$$

by (3.8),

$$r|(u)_r|\delta \le 1 \cdot \delta \le \epsilon/2,$$

and

$$\varphi(u, p, r, z_0) \le \Psi_{k+1} \le \theta^\beta \epsilon \le \epsilon/2$$

for $r \in I_{k+1}$. This shows (3.15) for $r \in I_{k+1}$.

By induction, we have shown (3.15), (3.16) for all $r \le 1/4$ and all $z_0 \in Q_{1/4}$. In particular, if $r \in I_k$,

$$\Psi(r, z_0) \le \Psi_k \le \theta^{k\beta}\epsilon \le C\epsilon r^\beta,$$

which implies (3.13). □

3.3 Proof of Theorem 1.1

We now prove Theorem 1.1. Choose $\alpha = 1/2$, $\beta = 1/4$ and choose $\theta > 0$ so small that $\theta^{\alpha-\beta}$, $\theta^{1-\beta}$ and θ^β are sufficiently small in the proof of Lemma 3.2.

By Lemma 2.3, there is $a_0 \in L^3(\mathbb{R}^3)$ with

$$a_0 = v_0 \quad \text{in } B_{3/4}, \quad a_0 = 0 \quad \text{in } B_1^c, \quad \operatorname{div} a_0 = 0, \quad \|a_0\|_{L^3(\mathbb{R}^3)} \le C(3, \frac{3}{4})\|v_0\|_{L^3(B_1)} \le \epsilon_2,$$

where ϵ_2 is the constant in Lemma 2.2. By Lemma 2.2, there is a unique mild solution a of (NS) with zero force and initial data $a(0) = a_0$ that satisfies (2.2). In particular,

$$\|a\|_{L^5(\mathbb{R}^3\times(0,\infty))} \le C\epsilon_2. \tag{3.17}$$

Let π_a be its corresponding pressure. We have $\pi_a = R_iR_ja_ia_j$, and

$$\|\pi_a\|_{L^{5/2}(\mathbb{R}^3\times(0,\infty))} \le C\|a\|^2_{L^5(\mathbb{R}^3\times(0,\infty))} \le C\epsilon_2^2. \tag{3.18}$$

By the maximal regularity for the inhomogeneous Stokes system, we have

$$\nabla a \in L^{5/2}(\mathbb{R}^3 \times (0, \infty)), \quad \nabla \pi_a \in L^{5/3}(\mathbb{R}^3 \times (0, \infty)). \tag{3.19}$$

Let $b_0 = v_0 - a_0$, $b = v - a$, and $\pi_b = \pi - \pi_a$. Denote $T = T_1 \in (0, 1/2)$ to be fixed later. Observe that (b, π_b) is a weak solution of the a-perturbed Navier-Stokes equations (3.1) in $Q = B_1 \times (0, T)$, with $b(x, 0) = b_0(x)$, and $b_0(x) = 0$ in $B_{3/4}$. It is easy to see that (b, π_b) satisfies the perturbed local energy inequality (3.2). By the interpolation, $\|v\|_{L^4_tL^3_x(Q)} \le C\|v\|_{L^\infty_tL^2_x\cap L^2_tH^1_x(Q)}$. Hence the assumption (1.4) leads to

$$\|v\|_{L^3(Q)} \le C\|v\|_{L^4_tL^3_x(Q)}T^{\frac{1}{12}} \le C\sqrt{M}T^{\frac{1}{12}} \tag{3.20}$$

and

$$\|\pi\|_{L^{3/2}(Q)} \le \|\pi\|_{L^2_tL^{3/2}_x(Q)}T^{\frac{1}{6}} \le CMT^{\frac{1}{6}}. \tag{3.21}$$

Thus, taking $T \le \epsilon^4 M^{-6}$ with ϵ sufficiently small, we get

$$\int_0^T \int_{B_1} |b|^3 + |\pi_b|^{\frac{3}{2}}\, dz \le 2C\epsilon \le \epsilon_1, \tag{3.22}$$

where ϵ_1 is the constant in (3.12) of Lemma 3.2.

Extend a, b, and π_b by zero for $t < 0$ and denote $Q_r^T := B_r \times (T - r^2, T)$. By the definition of $b = v - a$ and $b_0(x) = 0$ in $B_{3/4}$ we have $\lim_{t\to 0+} \|b(t)\|_{L^2(B_{3/4})} = 0$. This continuity condition at

$t=0$ together with the bounds (3.20), (3.21) shows that (b,π_b) is a suitable weak solution of (3.1) in $Q^T_{3/4}$ satisfying the perturbed local energy inequality (3.2), and $\frac{3}{4}|(b)_{Q^T_{3/4}}|\le 1$. In particular, (b,π_b) satisfies (3.1) across $t=0$ in the sense of distributions. We now apply Lemma 3.2 to see

$$\sup_{z_0\in Q^T_{\frac14}}\sup_{r<\frac14}\frac{1}{r^{2+3\beta}}\int_{Q_r(z_0)}|b|^3+\left|\pi_b-(\pi_b)_{B_r(x_0)}(t)\right|^{3/2}dz<C.$$

Choose largest $r_1\le\frac14$ satisfying $Cr_1^{3\beta}\le\frac12\epsilon_{\text{CKN}}$. We may also take T so that $T\le r_1^2$, which implies $Q^T_{\frac14}\supset B_{\frac14}\times(0,T)$, and

$$\sup_{z_0\in B_{\frac14}\times(0,T)}\sup_{r<r_1}\frac{1}{r^2}\int_{Q_r(z_0)}|b|^3+\left|\pi_b-(\pi_b)_{B_r(x_0)}(t)\right|^{3/2}dz<Cr^{3\beta}<\frac12\epsilon_{\text{CKN}}. \tag{3.23}$$

For $r\ge r_1$ we have

$$\sup_{z_0\in B_{\frac14}\times(0,T)}\sup_{r\ge r_1}\frac{1}{r^2}\int_{Q_r(z_0)\cap Q}|b|^3\,dz<\frac{1}{r_1^2}C\epsilon<\frac12. \tag{3.24}$$

Applying (3.17), (3.23), and (3.24) to $v=a+b$, we obtain

$$\sup_{z_0\in B_{\frac14}\times(0,T)}\sup_{0<r<\infty}\frac{1}{r^2}\int_{Q_r(z_0)\cap Q}|v|^3\,dz<1. \tag{3.25}$$

Now for any $z_0=(x_0,t_0)\in B_{1/4}\times(0,T)$, take $r=\frac12\sqrt{t_0}$. We have $r<r_1$ and

$$r^2<t<4r^2\quad\text{if }(x,t)\in Q_r(z_0).$$

For this r, let

$$\tilde\pi=\pi_a+\pi_b-(\pi_b)_{B_r(x_0)}(t).$$

Taking ϵ_2 sufficiently small in (3.17), (3.18), and using (3.23) we have

$$\frac{1}{r^2}\int_{Q_r(z_0)}|v|^3+|\tilde\pi|^{3/2}dz<\epsilon_{\text{CKN}}.$$

Since $(v,\tilde\pi)$ is a suitable weak solution of (NS) in $Q_r(z_0)$, by Lemma 2.1, we obtain

$$|v(z_0)|\le\|v\|_{L^\infty(Q_{r/2}(z_0))}\le\frac{C_{\text{CKN}}}{r/2}=\frac{4C_{\text{CKN}}}{\sqrt{t_0}}. \tag{3.26}$$

This completes the proof of Theorem 1.1. □

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