

ON THE MILNOR FIBRATION FOR $f(\mathbf{z})\bar{g}(\mathbf{z})$ II

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1. LOCALLY TAME NON-DEGENERATE COMPLETE INTERSECTION PAIR

1.1. Introduction. Let $f(\mathbf{z})$ and $g(\mathbf{z})$ be holomorphic functions vanishing at the origin. For $h(\mathbf{z}) := f(\mathbf{z})g(\mathbf{z})$, there exists a tubular Milnor fibration $h : E(r, \delta)^* \rightarrow D_\delta^*$ or a spherical Milnor fibration $h/|h| : S_r \setminus K_r \rightarrow S^1$ for small r and $\delta \ll r$ ([15, 11]). Here $E(r, \delta)^* := \{\mathbf{z} \in B_r^{2n} \mid 0 \neq |f(\mathbf{z})| \leq \delta\}$ and $K_r := f^{-1}(0) \cap S_r^{2n-1}$. We consider the mixed function $H(\mathbf{z}, \bar{\mathbf{z}}) := f(\mathbf{z})\bar{g}(\mathbf{z})$ and the existence problem of its Milnor fibration. The link of H is the same as the complex link given by $h(\mathbf{z})$ but the fibration structure along the link of $g = 0$ is conversely oriented. It turns out that such a fibration does not exist for an arbitrary pair. This problem has been studied by several authors but there are not yet satisfactory results ([26, 27, 28, 24]). For a non-degenerate mixed function, it is known that the Milnor fibration exists ([19]). However for $n \geq 3$, H can not be non-degenerate as $f = g = 0$ is a non-isolated singular locus for H . In our previous paper [23], we have shown the existence of Milnor fibrations for H under the assumption that f, g are convenient non-degenerate functions, satisfying the multiplicity condition. A convenient non-degenerate function f has an isolated singularity at the origin. In this paper, we consider the same problem without assuming the convenience. That is, we consider the case that $f = 0$ or $g = 0$ may have non-isolated singularity at the origin.

1.2. Vanishing coordinate subspaces and locally tameness. Let $f(\mathbf{z})$ be a holomorphic function of n complex variables z_1, \dots, z_n which vanishes at the origin. Consider a coordinate subspace $\mathbb{C}^I := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_j = 0, j \notin I\}$ where $I \subset \{1, 2, \dots, n\}$. \mathbb{C}^I is called a *vanishing coordinate subspace* of f if the restriction of f to \mathbb{C}^I is identically zero. The restriction of f is denoted as f^I . We denote the set of vanishing subspaces of f (respectively of g) by \mathcal{V}_f (resp. by \mathcal{V}_g). Let $P = (p_1, \dots, p_n)$ be a semi-positive weight vector. We put $I(P) := \{i \mid p_i = 0\}$. Take a vanishing coordinate subspace \mathbb{C}^I and take an arbitrary semi-positive weight vector $P = (p_1, \dots, p_n)$ such that $I(P) = I$. Then the face function f_P is a weighted homogeneous function of the variables $(z_j)_{j \notin I}$ with a positive degree $d(P; f)$ with respect to the weight vector P .

2000 *Mathematics Subject Classification.* 14J70, 14J17, 32S25.

Key words and phrases. locally tame, non-degenerate, toric multiplicity condition.

Recall that f is non-degenerate if for any strictly positive weight vector P (i.e., $I(P) = \emptyset$), $f_P : \mathbb{C}^{*n} \rightarrow \mathbb{C}$ has no critical points ([16]). We say that the function f (or the hypersurface $V(f) := f^{-1}(0)$) is *locally tame and non-degenerate* if it is non-degenerate and for any vanishing coordinate subspace \mathbb{C}^I , there exists a positive number r_I such that for any weight vector P with $I(P) = I$, f_P is a non-degenerate function of $(z_j)_{j \notin I}$ with the other variables $(z_i)_{i \in I} \in \mathbb{C}^{*I}$ being fixed in the ball $\sum_{i \in I} |z_i|^2 \leq r_I$ ([21, 8]). Put $V(f)^\sharp := \cup_{\mathbb{C}^I \notin \mathcal{V}_f} V(f^I) \cap \mathbb{C}^{*I}$. Recall that $V(f)^\sharp$ is smooth near the origin (Lemma (2.2), [17]).

For the pair of function $\{f, g\}$, consider the following conditions.

- (1) The hypersurfaces $V(f) = f^{-1}(0), V(g) = g^{-1}(0)$ are locally tame and non-degenerate.
- (2) The variety $V(f, g) = \{f = g = 0\}$ is a locally tame non-degenerate complete intersection variety. Namely (2-a) for any strictly positive weight vector P , the variety $\{\mathbf{z} \in \mathbb{C}^{*n} \mid f_P(\mathbf{z}) = g_P(\mathbf{z}) = 0\}$ is a smooth complete intersection variety. (2-b) For any common vanishing coordinate subspace \mathbb{C}^I , there exists a positive number r_I such that for any weight vector P with $I(P) = I$, $\{f_P = g_P = 0\}$ is a non-degenerate complete intersection variety in \mathbb{C}^{*J} with $J = I^c$ and $\mathbf{z}_I \in \mathbb{C}^{*I}$ is fixed in the ball $\sum_{i \in I} |z_i|^2 \leq r_I$.

We say $\{f, g\}$ is a *locally tame non-degenerate pair* if it satisfies only (1) and (2). The pair $\{f, g\}$ is a *disjoint locally tame non-degenerate complete intersection pair* if it satisfies (1), (2-a) and (3) $\{f, g\}$ satisfies the disjointness of vanishing subspaces, i.e. $\mathcal{V}_f \cap \mathcal{V}_g = \emptyset$.

Note that if $\mathbb{C}^I \in \mathcal{V}_f \setminus \mathcal{V}_g$, there exists a positive number r_I such that for any semi-positive weight vector P with $I(P) = I$, $g_P = g^I$ and $f_P = g_P = 0$ is a non-degenerate complete intersection variety.

2. ISOLATEDNESS OF THE CRITICAL VALUE

2.0.1. *Multiplicity condition.* We slightly generalize the multiplicity condition which is introduced in [23]. We say that $H := f\bar{g}$ satisfies *the multiplicity condition* if there exists a good resolution $\pi : X \rightarrow \mathbb{C}^n$ of the holomorphic function $h := fg$ such that

- (i) $\pi : X \setminus \pi^{-1}(V(h)) \rightarrow \mathbb{C}^n \setminus V(h)$ is biholomorphic and the divisor defined by $\pi^*(fg) = 0$ has only normal crossing singularities and the respective strict transforms $\tilde{V}(f)$ of $V(f)$ and $\tilde{V}(g)$ of $V(g)$ are smooth.
- (ii) Put $\pi^{-1}(\mathbf{0}) = \cup_{j=1}^s D_j$ where D_1, \dots, D_s are smooth compact divisors in X . Denote the respective multiplicities of π^*f and π^*g along D_j by m_j and n_j . Then $m_j \neq n_j$ for $j = 1, \dots, s$.

Assume that there exists a regular simplicial cone subdivision Σ^* of the dual Newton diagram $\Gamma^*(fg)$ and let $\hat{\pi} : X \rightarrow \mathbb{C}^n$ be the corresponding admissible toric modification. Let \mathcal{V}^+ be the set of strictly positive vertices of Σ^* . Then

it gives a good resolution of the function fg and the compact exceptional divisors are bijectively correspond to $\{\hat{E}(P) \mid P \in \mathcal{V}^+\}$ ([17]). Recall that the multiplicity of $\hat{\pi}^*f$ and $\hat{\pi}^*g$ along the divisor $\hat{E}(P)$ are given by $d(P, f)$ and $d(P, g)$ respectively. We say that $\hat{\pi}$ satisfies *the toric multiplicity condition for H* if

$$d(P, f) \neq d(P, g), \quad \forall P \in \mathcal{V}^+.$$

For further detail about the toric modification $\hat{\pi} : X \rightarrow \mathbb{C}^n$, we refer to [17].

Lemma 1 (Isolatedness of the critical value, Lemma 3 [23]). *Assume that $\{f, g\}$ is a locally tame and non-degenerate complete intersection pair. Assume that there exists an admissible toric modification $\hat{\pi} : X \rightarrow \mathbb{C}^n$ which satisfies the toric multiplicity condition. Then there exist positive numbers r_1 such that 0 is the unique critical value of H on $B_{r_1}^{2n}$.*

The proof follows by the exact same argument as Lemma 3,[23].

2.0.2. *A sufficient condition for the toric multiplicity condition.* We consider the following truncated cone. Let $h(\mathbf{z}) = \sum_{\nu} a_{\nu} \mathbf{z}^{\nu}$ be a holomorphic function which is not necessarily convenient. Let $\Gamma_+(h)$ be the convex hull of the union $\bigcup_{\nu, a_{\nu} \neq 0} \{\nu + (\mathbb{R}^+)^n\}$ as usual. The Newton boundary $\Gamma(h)$ is defined by the union of compact faces of $\Gamma_+(h)$. To give a sufficient condition for the multiplicity condition, we further consider following.

Definition 2. *We define the set $\Gamma_{++}(h)$ and $\text{Int}\Gamma_{++}(h)$ as*

$$\Gamma_{++}(h) = \{r\nu \mid r \geq 1, \nu \in \Gamma(h)\}, \quad \text{Int}\Gamma_{++}(h) = \{r\nu \mid r > 1, \nu \in \Gamma(h)\}$$

Note that $\Gamma_{++}(h) \subset \Gamma_+(h)$ and the equality holds if and only if h is convenient. The following gives a sufficient condition for the multiplicity condition.

Lemma 3. *Assume $\{f, g\}$ is a locally tame non-degenerate complete intersection pair. Suppose the following condition is satisfied.*

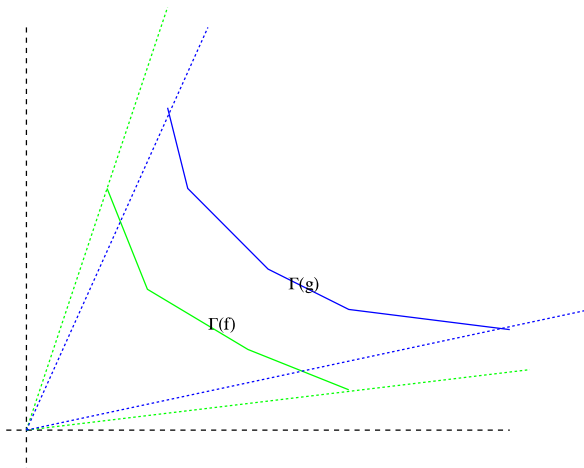
(\sharp): $\Gamma(f) \subset \text{Int}\Gamma_{++}(g)$ or $\Gamma(g) \subset \text{Int}\Gamma_{++}(f)$.

Then the multiplicity condition is satisfied with respect to any admissible toric modification.

See Figure 1 which shows the situation $\text{Int}\Gamma_{++}(f) \supset \Gamma(g)$. The condition (\sharp) is a generalization of Newton multiplicity condition in [23] for non-convenient f and g . We call (\sharp) *the tame Newton multiplicity condition*.

Example 4. 1. *Assume that $f(\mathbf{z})$ (respectively g) is a convenient function and assume that $\Gamma(f) \cap \Gamma(g) = \emptyset$ and $\Gamma(g)$ is above $\Gamma(f)$ (resp. $\Gamma(f)$ is above $\Gamma(g)$). Then the tame Newton multiplicity condition is satisfied.*

2. *Assume $\{f, g\}$ is a locally tame non-degenerate complete intersection pair and let $\hat{\pi} : X \rightarrow \mathbb{C}^n$ is an admissible toric modification. Let \mathcal{V}^+ be the strictly positive vertices of Σ^* . Consider the mapping $\varphi_m : \mathbb{C}^n \rightarrow \mathbb{C}^n$ defined by $\varphi(\mathbf{z}) = (z_1^m, \dots, z_n^m)$ and put $f_m(\mathbf{z}) := \varphi^*f(\mathbf{z})$ and $g_m(\mathbf{z}) := \varphi^*g(\mathbf{z})$.*

FIGURE 1. $\Gamma(g) \subset \text{Int} \Gamma_{++}(f)$

Then there exists a sufficiently large m such that $\hat{\pi} : X \rightarrow \mathbb{C}^n$ satisfies the toric multiplicity condition for $f_m \bar{g}$ and $f \bar{g}_m$ respectively. This follows from the canonical equality $d(P, f_m) = m d(P, f)$ and $d(P, g_m) = m d(P, g)$ and the stability of the dual Newton diagrams $\Gamma^*(f) = \Gamma^*(f_m)$, $\Gamma^*(g) = \Gamma^*(g_m)$.

3. Let $f(\mathbf{z}) = c_1 z_1^a + \cdots + c_n z_n^a$. Let $g(\mathbf{z})$ be any locally tame non-degenerate function. Then $\{f, g\}$ is a locally tame non-degenerate pair for generic coefficients c_1, \dots, c_n and satisfies the Newton multiplicity condition if $a = 1$. If $a > 1$, $\{f, g \prod_{i=1}^n z_i^a\}$ satisfies the Newton multiplicity condition, as $\Gamma_{++}(f) \supset \text{Int} \Gamma_{++}(g)$.

Remark 5. If f is locally tame and non-degenerate and if \mathbb{C}^I is not a vanishing coordinate subspace for f , f^I is also locally tame and non-degenerate as a function on \mathbb{C}^I . See the argument in Proposition (1.5), Chapter III [17]. Locally tameness has been defined for mixed functions (Definition 2.7, [8]). If a holomorphic function $f(\mathbf{z})$ is locally tame, it is also locally tame as a mixed function.

3. FIBRATION PROBLEM FOR FUNCTION $f\bar{g}$

We study the existence problem for the Milnor fibration of the mixed function $H(\mathbf{z}, \bar{\mathbf{z}}) := f(\mathbf{z})\bar{g}(\mathbf{z})$ in a more general situation. In this paper, we do not assume the convenience of f and g and therefore $V(f)$ or $V(g)$ may have non-isolated singularities at the origin. There are also interesting

works from more general viewpoint in Parameswaran and Tibar [25, 24] and Araujo dos Santos, Ribeiro and Tibar [5] where authors consider the case of critical values being not isolated.

3.1. Canonical stratification. We assume that $\{f, g\}$ is a locally tame non-degenerate complete intersection pair. Consider the hypersurface $V(fg) = V(f) \cup V(g)$. Note that the mixed hypersurface $V(f\bar{g})$ is equal to $V(fg)$ as real algebraic varieties. We consider the following canonical stratification \mathcal{S} of \mathbb{C}^{*n} which also give a stratification of $V(fg)$. Put $V^{*I}(f) = V(f^I) \cap \mathbb{C}^{*I}$ if \mathbb{C}^I is not a vanishing coordinate subspace. Here $\mathbb{C}^{*I} = \{(z_i) \in \mathbb{C}^I \mid \forall z_i \neq 0, i \in I\}$. We first define a stratification \mathcal{S}^I of \mathbb{C}^{*I} as follows.

$$\begin{aligned} & \{\mathbb{C}^{*I} \setminus (V^{*I}(f) \cup V^{*I}(g)), V^{*I}(f)', V^{*I}(g)', V^{*I}(f) \cap V^{*I}(g)\}, \text{ if } f^I \neq 0, g^I \neq 0 \\ & \{\mathbb{C}^{*I} \setminus V^{*I}(f), V^{*I}(f)\}, \text{ if } g^I \equiv 0, f^I \neq 0 \\ & \{\mathbb{C}^{*I} \setminus V^{*I}(g), V^{*I}(g)\}, \text{ if } f^I \equiv 0, g^I \neq 0 \\ & \{\mathbb{C}^{*I}\}, \text{ if } f^I \equiv 0, g^I \equiv 0. \end{aligned}$$

and we define $\mathcal{S} = \cup_I \mathcal{S}^I$. If $\{f, g\}$ is a disjoint locally tame non-degenerate pair, the last case does not exist. Here $V^{*I}(f)' = V^{*I}(f) \setminus V^{*I}(g)$ and $V^{*I}(g)' = V^{*I}(g) \setminus V^{*I}(f)$. $V^I(f)$ is empty only if f^I is a monomial.

We call \mathcal{S} the canonical toric stratification of $V(fg) = V(f\bar{g})$. Note that \mathcal{S} is a complex analytic stratification.

3.2. Transversality and Thom's a_f -regularity. We use the notation $V(H, \mathbf{z}) := H^{-1}(H(\mathbf{z}))$ hereafter. Another key condition for the existence of the Milnor fibration is the transversality of the nearby fibers $H^{-1}(\eta)$, $\eta \neq 0$ and the sphere S_r^{2n-1} . Assume that 0 is the unique critical value of H in $B_{r_1}^{2n}$.

Transversality of nearby fibers: For any pair $r_2 \leq r_1$, there exists a positive number δ such that for any r , $r_2 \leq r \leq r_1$ and non-zero η with $|\eta| \leq \delta$, $H^{-1}(\eta)$ and S_r^{2n-1} intersect transversely. This condition follows if H satisfies the Thom's a_f -regularity (See for example, Proposition 11, [21]). Recall that H satisfies a_f -condition at the origin if there exists a stratification \mathcal{S} of $H^{-1}(0) \cap B_{r_1}^{2n}$ for some $r_1 > 0$ such that for any sequence \mathbf{q}_ν , $\nu = 1, 2, \dots$ which converges $\mathbf{q}_0 \in M$, $M \in \mathcal{S}$ and $\mathbf{q}_0 \neq \mathbf{0}$, the limit of the tangent space $T_{\mathbf{q}_\nu} V(H, \mathbf{q}_\nu)$ (if it exists) includes the tangent space of M at \mathbf{q}_0 .

Theorem 6. *Assume that either (i) $\{f, g\}$ is a locally tame non-degenerate complete intersection pair which satisfies also the tame Newton multiplicity condition (\sharp) or (ii) $\{f, g\}$ is a disjoint locally tame non-degenerate complete intersection pair. In the case (ii), we assume also that H has a unique critical value 0 in a ball $B_{r_1}^{2n}$. Then $H = f\bar{g}$ satisfies a_f -regularity.*

Note that in case (i), the tame Newton multiplicity condition guarantees the isolatedness of the critical value of H . For the proof, we consider the canonical toric stratification \mathcal{S} on $V(fg)$. We choose $r_0, r_1 \geq r_0 > 0$ sufficiently small so that for any $r \leq r_0$, the canonical toric strata are smooth in B_r^{2n} and any sphere S_ρ^{2n-1} with $0 < \rho \leq r_0$ meets transversally with every strata of \mathcal{S} of positive dimension. We use Curve selection lemma (see [15, 10]). Suppose we have a real analytic curve $\mathbf{z}(t)$, $0 \leq t \leq 1$ such that $\mathbf{z}(0) = \mathbf{a} \in V(H) \cap B_{r_0}^{2n}$, $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{z}(t) \in \mathbb{C}^n \setminus V(H)$ for $t > 0$. Put $K = \{i \mid z_i(t) \neq 0\}$ and write the expansion as

$$\begin{aligned} z_i(t) &= \alpha_i t^{p_i} + (\text{higher terms}), \alpha_i \neq 0, i \in K \\ &\equiv 0, i \notin K \end{aligned}$$

Let $M \in \mathcal{S}^I$ be the stratum which contains \mathbf{a} . We have to show that the limit of the tangent space of the fiber $V(H, \mathbf{z}(t))$ at $\mathbf{z}(t)$ for $t \rightarrow 0$ contains the tangent space of the stratum M at \mathbf{a} . The restriction of f, g and H on \mathbb{C}^K satisfy also the locally tame non-degenerate assumption. As the argument for the proof is exactly the same, we assume for simplicity that $K = \{1, \dots, n\}$ hereafter. That is, we assume that $\mathbf{z}(t) \in \mathbb{C}^{*n}$ for $t \neq 0$ and $\mathbf{z}(0) = \mathbf{a}$. Put $P = (p_1, \dots, p_n)$ and

$$(1) \quad \begin{cases} I := \{i \mid p_i = 0\}, \\ J = I^c = \{1, \dots, n\} \setminus I, \quad \mathbf{w} := (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^{*n}. \end{cases}$$

Note that $p_i = 0$ if and only if $i \in I$. Thus $\mathbf{a} = \mathbf{w}_I$ and $0 \neq \|\mathbf{a}\| \leq r_0$. We will show that

$$\lim_{t \rightarrow 0} T_{\mathbf{z}(t)} V(H, \mathbf{z}(t)) \supset T_{\mathbf{a}} M.$$

We use the key property that the tangent space of the level hypersurface $V(H, \mathbf{z}(t))$ at $\mathbf{z}(t)$ contains the intersection of two tangent spaces of the level complex hypersurfaces $V(f, \mathbf{z}(t))$ and $V(g, \mathbf{z}(t))$ by Proposition 14, [23]. Here we are assuming that r_0 is sufficiently small so that 0 is the only critical value for f and g on $B_{r_0}^{2n}$. We divide the situation into three cases.

- (a) $\mathbb{C}^I \notin \mathcal{V}_f \cup \mathcal{V}_g$, i.e. $f^I \neq 0, g^I \neq 0$.
- (b) $\mathbb{C}^I \in \mathcal{V}_f$ and $\mathbb{C}^I \notin \mathcal{V}_g$, i.e. $f^I \equiv 0, g^I \neq 0$.
- (b)' $\mathbb{C}^I \in \mathcal{V}_g$ and $\mathbb{C}^I \notin \mathcal{V}_f$, i.e. $f^I \neq 0, g^I \equiv 0$.
- (c) $\mathbb{C}^I \in \mathcal{V}_f \cap \mathcal{V}_g$, i.e. $f^I \equiv 0, g^I \equiv 0$.

As (b) and (b)' is symmetric, it is enough to consider three cases (a), (b) and (c).

We first consider the case (a). The case (a) can be divided into two subcases:

- (a-1) $\mathbf{a} \in M = V^{*I}(f) \cap V^{*I}(g)$.
- (a-2) $\mathbf{a} \in V^{*I}(f)' = V^{*I}(f) \setminus V^{*I}(g)$, or $\mathbf{a} \in V^{*I}(g)' = V^{*I}(g) \setminus V^{*I}(f)$.

In the case (a-1), \mathbf{a} is a non-singular point of $\mathbf{a} \in V(f, g)$. As the tangent space $T_{\mathbf{z}(t)} V(f, \mathbf{z}(t))$ converges to $T_{\mathbf{a}} V(f)$ which includes $T_{\mathbf{a}} V^{*I}(f)$ and $T_{\mathbf{z}(t)} V(g, \mathbf{z}(t))$ converges to $T_{\mathbf{a}} V(g)$ which includes $T_{\mathbf{a}} V^{*I}(g)$ and $T_{\mathbf{a}} M =$

$T_{\mathbf{a}}V^{*I}(f) \cap T_{\mathbf{a}}V^{*I}(g)$ by the Newton non-degeneracy assumption, the assertion follows from Proposition 14, [23].

In the case (a-2), $\mathbf{a} \in V^{*I}(f)'$ or $\mathbf{a} \in V^{*I}(g)'$, \mathbf{a} is a non-singular point of $V(H)$ and the assertion is obvious from the continuity of the tangent space.

Consider the case (b). Thus we assume that $\mathbb{C}^I \in \mathcal{V}_f \setminus \mathcal{V}_g$. By the local tameness assumption, the limit of the normalized holomorphic gradient vector $\lim_{t \rightarrow 0} \overline{\partial f}(\mathbf{z}(t)) / \|\overline{\partial f}(\mathbf{z}(t))\|$ along $\mathbf{z}(t)$ is a vector in \mathbb{C}^J . Here $J = \{1, \dots, n\} \setminus I$. (Recall $\partial f(\mathbf{z}) = (\frac{\partial f(\mathbf{z})}{\partial z_1}, \dots, \frac{\partial f(\mathbf{z})}{\partial z_n})$.) Thus the limit of the tangent space of $V(f, \mathbf{z}(t))$ contains \mathbb{C}^I by the local tameness assumption. There are two subcases.

- (b-1) $\mathbf{a} \in V^{*I}(g)$, or
- (b-2) $\mathbf{a} \in \mathbb{C}^{*I} \setminus V^{*I}(g)$.

Note that $M = V^{*I}(g)$ in the case (b-1) and $M = \mathbb{C}^{*I} \setminus V^{*I}(g)$ in the case (b-2) respectively. In the case of (b-1), the limit of the normalized vector of $\overline{\partial f}(\mathbf{z}(t))$ is a vector in \mathbb{C}^J by the local tameness assumption of f . Thus the limit of $T_{\mathbf{z}(t)}V(f, \mathbf{z}(t))$ includes \mathbb{C}^I . On the other hand, as $\overline{\partial g}(\mathbf{a})$ is non-zero, $T_{\mathbf{a}}V(g)$ is transverse to \mathbb{C}^I at \mathbf{a} . Thus for any sufficiently small t , they are transverse and the limit of the intersection of two tangent space of the tangent space of $V(f, \mathbf{z}(t))$ and $V(g, \mathbf{z}(t))$ contains $T_{\mathbf{a}}V^{*I}(g)$.

Now we consider the case (b-2). We claim that the limit of the tangent space $T_{\mathbf{z}(t)}V(H, \mathbf{z}(t))$ includes \mathbb{C}^I , the tangent space of the stratum $M = \mathbb{C}^{*I} \setminus V^{*I}(g)$ at \mathbf{a} . First we prepare a sublemma.

Sublemma 7. *Let f be a holomorphic function and write $f(z) = k(\mathbf{z}, \bar{\mathbf{z}}) + i\ell(\mathbf{z}, \bar{\mathbf{z}})$ where $k = \Re f$, $\ell = \Im f$. Then we have $\bar{\partial}k = \frac{1}{2}\overline{\partial f}$ and $\bar{\partial}\ell = \frac{i}{2}\overline{\partial f}$. In particular, two gradient vectors $\bar{\partial}k$ and $\bar{\partial}\ell$ are linearly dependent over \mathbb{C} but linearly independent over \mathbb{R} at a non-critical point \mathbf{z} of f .*

The assertion follows from the identities:

$$\overline{\partial}k = \bar{\partial}k, \quad \overline{\partial}\ell = \bar{\partial}\ell, \quad \bar{\partial}f = \bar{\partial}k + i\bar{\partial}\ell = 0, \quad \partial f = \partial k + i\partial\ell.$$

Put $p_{\min} = \min\{p_j \mid j \notin I\}$. First we can write

Lemma 8. *The orders of $\bar{\partial}\Re f(\mathbf{z}(t))$ and $\bar{\partial}\Im f(\mathbf{z}(t))$ are equal to the order of $\overline{\partial f}(\mathbf{z}(t))$. Put $s = \text{order } \overline{\partial f}(\mathbf{z}(t))$. Then s and strictly less than $d(P; f) - p_{\min}$. We can write further as follows.*

$$\begin{aligned} \overline{\partial f}(\mathbf{z}(t)) &= \mathbf{v}t^s + (\text{higher terms}), \quad \exists \mathbf{v} \in \mathbb{C}^J \\ \bar{\partial}\Re f(\mathbf{z}(t)) &= \frac{1}{2}\mathbf{v}t^s + (\text{higher terms}) \\ \bar{\partial}\Im f(\mathbf{z}(t)) &= \frac{i}{2}\mathbf{v}t^s + (\text{higher terms}) \end{aligned}$$

In particular, $\lim_{t \rightarrow 0} T_{\mathbf{z}(t)}V(f, \mathbf{z}(t))$ is the complex orthogonal of \mathbf{v} .

Now we are ready to analyze the case (b-2). Note that the limit of normalized gradient vector $\overline{\partial f}(\mathbf{z}(t))$ is $\mathbf{v}/\|\mathbf{v}\|$. For a vector \mathbf{v} , let $\mathbf{v}^{\perp \mathbb{C}}$ be the

subspace of \mathbb{C}^n which are complex orthogonal to \mathbf{v} . Namely $\mathbf{v}^{\perp\mathbb{C}} = \{\mathbf{w} \in \mathbb{C}^n \mid (\mathbf{w}, \mathbf{v}) = 0\}$. Now we claim

Assertion 9. *Assume $\mathbf{a} \in \mathbb{C}^{*I} \setminus V^{*I}(g)$. Then $\lim_{t \rightarrow 0} T_{\mathbf{z}(t)}V(H, \mathbf{z}(t)) = \lim_{t \rightarrow 0} T_{\mathbf{z}(t)}V(f, \mathbf{z}(t))$.*

Proof. Put $b := \bar{g}(\mathbf{a})$ and write $b = b_1 + ib_2$ with $b_1, b_2 \in \mathbb{R}$. First we use the equalities:

$$\Re(H) = \Re(f)\Re(\bar{g}) - \Im(f)\Im(\bar{g}), \quad \Im(H) = \Re(f)\Im(\bar{g}) + \Im(f)\Re(\bar{g}).$$

Then the gradient vectors are given as

$$\begin{aligned} \bar{\partial}\Re(H)(\mathbf{z}(t)) &= (\bar{\partial}\Re(f)\Re(\bar{g}))(\mathbf{z}(t)) + (\Re(f)\bar{\partial}\Re(\bar{g}))(\mathbf{z}(t)) \\ &\quad - (\bar{\partial}\Im(f)\Im(\bar{g}))(\mathbf{z}(t)) - (\Im(f)\bar{\partial}\Im(\bar{g}))(\mathbf{z}(t)) \\ &\equiv b_1\bar{\partial}\Re(f)(\mathbf{z}(t)) - b_2\bar{\partial}\Im(f)(\mathbf{z}(t)) \text{ modulo } (t^{s+1}) \\ &\equiv \frac{\mathbf{v}\bar{b}}{2}t^s \text{ modulo } (t^{s+1}) \\ \bar{\partial}\Im(H)(\mathbf{z}(t)) &= (\bar{\partial}\Re(f)\Im(\bar{g}))(\mathbf{z}(t)) + (\Re(f)\bar{\partial}\Im(\bar{g}))(\mathbf{z}(t)) \\ &\quad + \bar{\partial}\Im(f)\Re(\bar{g})(\mathbf{z}(t)) + \Im(f)\bar{\partial}\Re(\bar{g})(\mathbf{z}(t)) \\ &\equiv b_2(\bar{\partial}\Re(f)(\mathbf{z}(t)) + b_1\bar{\partial}\Im(f)(\mathbf{z}(t)) \text{ modulo } (t^{s+1}) \\ &\equiv \frac{(i\mathbf{v}\bar{b})}{2}t^s \text{ modulo } (t^{s+1}) \end{aligned}$$

and therefore the normalized vector of these gradient vectors $\bar{\partial}\Re(H)(\mathbf{z}(t))$ and $\bar{\partial}\Im(H)(\mathbf{z}(t))$ converges to the vectors

$$\frac{\mathbf{v}\bar{b}}{\|\mathbf{v}\bar{b}\|}, \quad i \frac{\mathbf{v}\bar{b}}{\|\mathbf{v}\bar{b}\|}$$

respectively. This implies the limit of the tangent space $T_{\mathbf{z}(t)}V(H, \mathbf{z}(t))$ is the real orthogonal of the real 2-dimensional subspace span by these two vectors, that is nothing but the complex subspace $\mathbf{v}^{\perp\mathbb{C}}$ which is equal to the limit of $T_{\mathbf{z}(t)}V(f, \mathbf{z}(t))$. The proof of the assertion for (b-2) is now completed. The case $\{f, g\}$ is a disjoint tame non-degenerate complete intersection pair is now proved.

Now we consider the last case (c) $\mathbb{C}^I \in \mathcal{V}_f \cap \mathcal{V}_g$. Recall that $\mathbf{w} = (\alpha_1, \dots, \alpha_n)$. We divide the situation into three subcases.

- (c-1) $f_P(\mathbf{w}) = g_P(\mathbf{w}) = 0$.
- (c-2) $f_P(\mathbf{w}) = 0$ and $g_P(\mathbf{w}) \neq 0$ or (c-2)' $f_P(\mathbf{w}) \neq 0$ and $g_P(\mathbf{w}) = 0$.
- (c-3) $f_P(\mathbf{w}) \neq 0, g_P(\mathbf{w}) \neq 0$.

We restate the assertion as the following lemma.

Lemma 10. *Assume that $\mathbb{C}^I \in \mathcal{V}_f \cap \mathcal{V}_g$. Then the limit of the tangent space $T_{\mathbf{z}(t)}V(H, \mathbf{z}(t))$ includes \mathbb{C}^I as a subspace.*

Proof. First assume that $f_P(\mathbf{w}) = g_P(\mathbf{w}) = 0$. Put $\overline{\partial f}(\mathbf{z}(t)) = (u_1(t), \dots, u_n(t))$ and $\overline{\partial g}(\mathbf{z}(t)) = (v_1(t), \dots, v_n(t))$. We can write as

$$\begin{aligned} u_j(t) &= \frac{\overline{\partial f_P}}{\partial z_j}(\mathbf{w})t^{d(P;f)-p_j} + (\text{higher terms}) \\ v_j(t) &= \frac{\overline{\partial g_P}}{\partial z_j}(\mathbf{w})t^{d(P;g)-p_j} + (\text{higher terms}). \end{aligned}$$

Put o_f and o_g be the orders of $\overline{\partial f}(\mathbf{z}(t))$ and $\overline{\partial g}(\mathbf{z}(t))$ respectively. That is $o_f = \min\{\text{ord}_t u_i(t) \mid i = 1, \dots, n\}$ and $o_g = \min\{\text{ord}_t v_i(t) \mid i = 1, \dots, n\}$. Then the limit of $\overline{\partial f}(\mathbf{z}(t))$ and $\overline{\partial g}(\mathbf{z}(t))$ up to scalar multiplications are represented respectively by

$$(2) \quad \lim_{t \rightarrow 0} \frac{1}{t^{o_f}} \overline{\partial f}(\mathbf{z}(t)), \quad \lim_{t \rightarrow 0} \frac{1}{t^{o_g}} \overline{\partial g}(\mathbf{z}(t)).$$

We denote these limit vectors as $\lim_{t \rightarrow 0}^{(n)} \overline{\partial f}(\mathbf{z}(t))$ and $\lim_{t \rightarrow 0}^{(n)} \overline{\partial g}(\mathbf{z}(t))$. If these two limits are linearly independent over \mathbb{C} , the intersection

$$T_{\mathbf{z}(t)}V(f, \mathbf{z}(t)) \cap T_{\mathbf{z}(t)}V(g, \mathbf{z}(t))$$

converges to the the complex orthogonal subspace to these two limit vectors. That is,

$$\langle \overline{\partial f}(\mathbf{z}(t)), \overline{\partial g}(\mathbf{z}(t)) \rangle^{\perp_{\mathbb{C}}} \mapsto \langle \lim_{t \rightarrow 0}^{(n)} \overline{\partial f}(\mathbf{z}(t)), \lim_{t \rightarrow 0}^{(n)} \overline{\partial g}(\mathbf{z}(t)) \rangle^{\perp_{\mathbb{C}}}.$$

The problem happens if these two limits are linearly dependent. We use a similar argument as the one which is used in the proof of Theorem 20, [21] or Theorem 3.14, [8] to solve this problem. For the simplicity of the argument, we assume that $J = \{1, \dots, m\}$ and $I = \{m+1, \dots, n\}$ and we assume that

$$p_1 \geq p_2 \geq \dots \geq p_m > 0, p_{m+1} = \dots = p_n = 0.$$

Note that $p_{\min} = p_m$ under the above assumption and

$$\text{ord } u_j(t) \geq d(P; f) - p_j, \quad \text{ord } v_j(t) \geq d(P; g) - p_j, \quad j = 1, \dots, m$$

while for $j \geq m+1$,

$$\text{ord } u_j(t) \geq d(P; f), \quad \text{ord } v_j(t) \geq d(P; g), \quad j = m+1, \dots, n.$$

Now we consider first (c-1): $f_P(\mathbf{w}) = g_P(\mathbf{w}) = 0$. By the locally tame non-degeneracy assumption, there exists $1 \leq a, b \leq m, a \neq b$ so that we have

$$(3) \quad \det \begin{pmatrix} \frac{\overline{\partial f_P}}{\partial z_a}(\mathbf{w}) & \frac{\overline{\partial f_P}}{\partial z_b}(\mathbf{w}) \\ \frac{\overline{\partial g_P}}{\partial z_a}(\mathbf{w}) & \frac{\overline{\partial g_P}}{\partial z_b}(\mathbf{w}) \end{pmatrix} \neq 0.$$

Here we assume that $a \neq b$ but we do not assume that $a < b$. In particular,

$$\begin{aligned} o_f &\leq \max\{d(P; f) - p_a, d(P; f) - p_b\} \leq d(P; f) - p_m, \\ o_g &\leq \max\{d(P; g) - p_a, d(P; g) - p_b\} \leq d(P; g) - p_m. \end{aligned}$$

For simplicity, we may assume that $o_f \leq o_g$ and consider

$$\ell_0 := \min\{j \mid \text{ord } u_j(t) = o_f\}, \quad m_0 := \min\{j \mid \text{ord } v_j(t) = o_g\}.$$

We call ℓ_0, m_0 the leading indices of $\overline{\partial f}(\mathbf{z}(t))$ and $\overline{\partial g}(\mathbf{z}(t))$.

Case 1. Assume that $\ell_0 \neq m_0$. Then the two limit gradient vectors given by (2) are already linearly independent. There are nothing to do further.

Case 2. Assume that $\ell_0 = m_0$. Then we take a monomial function $\rho(t) = ct^{o_g - o_f}$, $c \in \mathbb{C}$ and replace $\partial g(\mathbf{z}(t))$ by

$$\mathbf{v}^{(1)}(t) = \overline{\partial g}(\mathbf{z}(t)) - \rho(t)\overline{\partial f}(\mathbf{z}(t)).$$

We choose a constant c so that $\text{ord } v_{m_0}^{(1)}(t) > o_g$. We put $\text{ord } v_{m_0}^{(1)}(t) = \infty$ if $v_{m_0}^{(1)}(t) \equiv 0$. Here $v_{m_0}^{(1)}(t)$ is the m_0 -th component of $\mathbf{v}^{(1)}(t)$. Note that the two dimensional complex subspace $W = \langle \overline{\partial f}(\mathbf{z}(t)), \overline{\partial g}(\mathbf{z}(t)) \rangle$ generated by $\{\overline{\partial f}(\mathbf{z}(t)), \overline{\partial g}(\mathbf{z}(t))\}$ is the same with subspace $\langle \overline{\partial f}(\mathbf{z}(t)), \mathbf{v}^{(1)}(t) \rangle$ generated by $\{\overline{\partial f}(\mathbf{z}(t)), \mathbf{v}^{(1)}(t)\}$. Thus their complex orthogonal subspaces are also equal. We continue this operation

$$\overline{\partial g} \rightarrow \mathbf{v}^{(1)} \rightarrow \dots \rightarrow \mathbf{v}^{(k)}$$

until the leading index of $\mathbf{v}^{(k)}$ changes. Note that the k -times operation $\partial g(\mathbf{z}(t)) \rightarrow \mathbf{v}^{(k)}(t)$ is given as

$$\mathbf{v}^{(k)}(t) = \overline{\partial g}(\mathbf{z}(t)) - \rho_k(t)\overline{\partial f}(\mathbf{z}(t))$$

where $\rho_k(t)$ is a polynomial of variable t whose lowest degree is $o_g - o_f$. By (3), we may assume that $\partial g_P / \partial z_a(\mathbf{w}) \neq 0$. Note that $\text{ord } v_{m_0}^{(\nu)}(t)$ is strictly increasing as long as $\nu \leq k - 1$ and $\text{ord } \mathbf{v}^{(\nu)}(t) = \text{ord } v_{m_0}^{(\nu)}(t)$. Let us look the components a, b which is given by

$$v_\tau^{(k)}(t) = \frac{\overline{\partial g}}{\partial z_\tau}(\mathbf{z}(t)) - \rho_k(t) \frac{\overline{\partial f}}{\partial z_\tau}(\mathbf{z}(t)), \quad \tau = a, b.$$

Assertion 11. *One of the following inequalities holds.*

$$\text{ord } v_a^{(k)}(t) \leq d(P; g) - p_a, \quad \text{or} \quad \text{ord } v_b^{(k)}(t) \leq d(P; g) - p_b.$$

Proof. Assume that $\text{ord } v_a^{(k)}(t) > d(P; g) - p_a$. We will show that $\text{ord } v_b^{(k)}(t) \leq d(P; g) - p_b$. As the first term of $\rho_k(t) \frac{\overline{\partial f}}{\partial z_a}(\mathbf{z}(t))$ kill the first term of $\frac{\overline{\partial g}}{\partial z_a}(t)$, the order of $\rho_k(t) \frac{\overline{\partial f}}{\partial z_a}(t)$ is equal to $d(P; g) - p_a$. There are two cases to be considered.

(A) $\frac{\partial f_P}{\partial z_a}(\mathbf{w}) \neq 0$ or (B) $\frac{\partial f_P}{\partial z_a}(\mathbf{w}) = 0$.

Assume the case (A). Then we have $\text{ord } \rho_k(t) = d(P; g) - d(P; f) = o_g - o_f$ and which implies that

$$\text{ord } \rho_k(t) \frac{\overline{\partial f_P}}{\partial z_b}(\mathbf{z}(t)) \geq (d(P; f) - p_b) + (d(P; g) - d(P; f)) = d(P; g) - p_b.$$

Then putting λ be the coefficient of $t^{d(P;g)-d(P;f)}$ in $\rho_k(t)$, we have

$$\begin{aligned} 0 &\neq \det \begin{pmatrix} \overline{\frac{\partial f_P}{\partial z_a}}(\mathbf{w}) & \overline{\frac{\partial f_P}{\partial z_b}}(\mathbf{w}) \\ \overline{\frac{\partial g_P}{\partial z_a}}(\mathbf{w}) & \overline{\frac{\partial g_P}{\partial z_b}}(\mathbf{w}) \end{pmatrix} \\ &= \det \begin{pmatrix} \overline{\frac{\partial f_P}{\partial z_a}}(\mathbf{w}) & \overline{\frac{\partial f_P}{\partial z_b}}(\mathbf{w}) \\ \overline{\frac{\partial g_P}{\partial z_a}}(\mathbf{w}) + \lambda \overline{\frac{\partial f_P}{\partial z_a}}(\mathbf{w}) & \overline{\frac{\partial g_P}{\partial z_b}}(\mathbf{w}) + \lambda \overline{\frac{\partial f_P}{\partial z_b}}(\mathbf{w}) \end{pmatrix} \\ &= \det \begin{pmatrix} \overline{\frac{\partial f_P}{\partial z_a}}(\mathbf{w}) & \overline{\frac{\partial f_P}{\partial z_b}}(\mathbf{w}) \\ 0 & \overline{\frac{\partial g_P}{\partial z_b}}(\mathbf{w}) + \lambda \overline{\frac{\partial f_P}{\partial z_b}}(\mathbf{w}) \end{pmatrix} \end{aligned}$$

and thus $\overline{\frac{\partial g_P}{\partial z_b}}(\mathbf{w}) + \lambda \overline{\frac{\partial f_P}{\partial z_b}}(\mathbf{w}) \neq 0$ by (3), and $v_b^{(k)} = d(P;g) - p_b$.

Consider the case (B) now. Then $\text{ord } \rho_k(t) < d(P;g) - d(P;f)$. By (3), $\overline{\frac{\partial f_P}{\partial z_b}}(\mathbf{w}) \neq 0$ and $\text{ord } v^{(k)}(t)_b < d(P;g) - p_b$. Thus in both cases, under the above operation, $\text{ord } v^{(k)}(t) \leq d - p_m$. \square

The above argument implies that the number k of operations is bounded by $k \leq d(P;g) - p_m - o_g$. At the last operation, the leading index of $\mathbf{v}^{(k)}(t)$ is different from m_0 and the limit vector of $\mathbf{v}^{(k)}(t)$ and $\overline{\partial f}(\mathbf{z}(t))$ are linearly independent and they are in the subspace \mathbb{C}^J . As

$T_{\mathbf{z}(t)}V(f, \mathbf{z}(t)) \cap T_{\mathbf{z}(t)}V(g, \mathbf{z}(t))$ is the complex orthonormal subspace of the two dimensional subspace $\langle \overline{\partial f}(\mathbf{z}(t)), \overline{\partial g}(\mathbf{z}(t)) \rangle_{\mathbb{C}}$ and it is equal to the complex orthonormal subspace of $\langle \overline{\partial f}(\mathbf{z}(t)), \mathbf{v}^{(k)}(t) \rangle_{\mathbb{C}}$, the limit of $T_{\mathbf{z}(t)}V(f, \mathbf{z}(t)) \cap T_{\mathbf{z}(t)}V(g, \mathbf{z}(t))$ includes the vanishing subspace \mathbb{C}^I . As $T_{\mathbf{z}(t)}V(H, \mathbf{z}(t))$ includes $T_{\mathbf{z}(t)}V(f, \mathbf{z}(t)) \cap T_{\mathbf{z}(t)}V(g, \mathbf{z}(t))$ as a subspace, $\lim_{t \rightarrow 0} T_{\mathbf{z}(t)}V(H, \mathbf{z}(t)) \supset \mathbb{C}^I$. Thus the proof of case (c-1) is done.

Now we consider the case (c-2): $f_P(\mathbf{w}) = 0, g_P(\mathbf{w}) \neq 0$. Consider the hypersurface $V := \{(\mathbf{z}_J \in \mathbb{C}^{*J} \mid f_P(\mathbf{z}) = 0, \mathbf{z}_I = \mathbf{a}\}$. By the locally tameness assumption, V is a non-singular hypersurface in \mathbb{C}^{*I} . Let us consider the restriction $g_P : V \rightarrow \mathbb{C}$. As g_P is a weighted homogeneous polynomial function of weight $P_J = (p_1, \dots, p_m)$ and V is invariant under the associated \mathbb{C}^* action on \mathbb{C}^J , g_P has no non-zero critical value on V . Namely $\overline{\partial} g_P(\mathbf{w})$ is non-zero and linearly independent with $\overline{\partial} f_P(\mathbf{w})$. Thus there is a pair $a \leq b \leq m$ which satisfies (3). Thus the rest of the argument is the exact same as above and $\lim_{s \rightarrow 0} T_{\mathbf{z}(s)}V(H, \mathbf{z}(s)) \supset \mathbb{C}^I$. The case (c-2)': $f_P(\mathbf{w}) \neq 0$ and $g_P(\mathbf{w}) = 0$ is treated similarly.

Now we consider the case (c-3): $f_P(\mathbf{w}), g_P(\mathbf{w}) \neq 0$. In this case, we need the assumption that f, g satisfies the tame Newton multiplicity condition. Let $\deg_P f = d_f$ and $\deg_P g = d_g$. Put $d_r := d_f + d_g$ and $d_p = d_f - d_g$. Then the tame Newton multiplicity condition implies $d_p \neq 0$. The mixed function $H_P = f_P \bar{g}_P : \mathbb{C}^J \rightarrow \mathbb{C}$ is a strongly mixed weighted homogeneous polynomial which satisfies $H(\rho e^{i\theta} \circ \mathbf{z}) = \rho^{d_r} e^{d_p \theta} H(\mathbf{z})$ and thus 0 is the only critical value. Thus $\overline{\partial} \Re H_P(\mathbf{w}), \overline{\partial} \Im H_P(\mathbf{w})$ are linearly independent over \mathbb{R} and we can proceed the same argument as above replacing $\overline{\partial} f, \overline{\partial} g$

to $\bar{\partial}\Re H(\mathbf{z}(t)), \bar{\partial}\Im H(\mathbf{z}(t))$ to conclude the real two dimensional subspace $\langle \bar{\partial}\Re H(\mathbf{z}(t)), \bar{\partial}\Im H(\mathbf{z}(t)) \rangle_{\mathbb{R}}$ has a limit which is a real 2-dimensional subspace of \mathbb{C}^J . Thus $\lim_{t \rightarrow 0} T_{\mathbf{z}(t)}V(H, \mathbf{z}(t)) \supset \mathbb{C}^I$. See the proof of Theorem 20, [21] for further detail. \square

Now the proof of Theorem 6 is completed. \square

By Proposition 11, [21], we get the transversality assertion:

Corollary 12. *Let $\{f, g\}$ be as in Theorem 6. Then there exists a positive number r_0 such that for any $r_1, 0 < r_1 \leq r_0$, there exists a positive number $\delta(r_1)$ so that for any $\eta \neq 0$ with $|\eta| \leq \delta(r_1)$ and any $\rho, r_1 \leq \rho \leq r_0$, the nearby fiber $H^{-1}(\eta)$ is non-singular in $B_{r_0}^{2n}$ and intersects transversely with the sphere S_{ρ}^{2n-1} .*

3.3. Existence of a tubular Milnor fibration. By Lemma 1, Theorem 6 and Corollary 12, we apply Ehresmann's fibration theorem ([35]) to obtain:

Theorem 13. *Assume that $\{f, g\}$ satisfies the same assumption as in Theorem 6. Then there exists a positive number ε and a sufficiently small $\delta \ll \varepsilon$ such that*

$$H = f\bar{g} : E(\varepsilon; \delta)^* \rightarrow D_{\delta}^*$$

is a locally trivial fibration where $E(\varepsilon, \delta)^* := \{\mathbf{z} \mid 0 \neq |H(\mathbf{z})| \leq \delta, \|\mathbf{z}\| \leq \varepsilon\}$ and $D_{\delta}^* := \{\zeta \in \mathbb{C} \mid 0 \neq |\zeta| \leq \delta\}$,

By Corollary 12, the fibration does not depend on the choice of ε and δ . For a disjoint locally tame non-degenerate pair $\{f, g\}$, applying the argument of 2 of Example 4, we have:

Corollary 14. *Assume that $\{f, g\}$ is a disjoint locally tame non-degenerate complete intersection pair. Fix an admissible toric modification $\hat{\pi} : X \rightarrow \mathbb{C}^n$. Take $m > 0$ large so that $\{f_m, g\}$ and $\{f, g_m\}$ satisfy the toric multiplicity condition with $\hat{\pi}$. Then changing the coefficients of f_m or g_m slightly if necessary, $\{f_m, g\}$ and $\{f, g_m\}$ are locally tame non-degenerate complete intersections respectively for which $\hat{\pi} : X \rightarrow \mathbb{C}^n$ satisfies the toric multiplicity condition. Thus $H_m := f_m\bar{g}$ and $K := f\bar{g}_m$ have tubular Milnor fibrations.*

For the definition f_m , see Example 4.

4. SPHERICAL MILNOR FIBRATION

In this section, we study the existence of the spherical Milnor fibration. For a fixed small $r > 0$, we consider the mapping $\varphi : S_r^{2n-1} \setminus K \rightarrow S^1$ where $K = V(H) \cap S_r^{2n-1}$ and $\varphi(\mathbf{z}) = H(\mathbf{z})/|H(\mathbf{z})|$.

Lemma 15 (Lemma 10, [23]). *We assume $\{f, g\}$ is a weak locally tame non-degenerate complete intersection pair satisfying the assumption in Theorem 6. Then there exists a positive number r_3 so that $\varphi : S_r^{2n-1} \setminus K \rightarrow S^1$ has no critical points for any $r, 0 < r \leq r_3$.*

In the proof of Lemma 17 below, we will simultaneously reprove Lemma 15. Using Lemma 15 and the transversality property of the fibers $H^{-1}(\eta)$, $0 \neq |\eta| \leq \delta$ and the sphere S_r^{2n-1} (Corollary 12), we obtain the following.

Theorem 16. *Assume $\{f, g\}$ is a weak locally tame non-degenerate complete intersection pair as in Theorem 6. For a sufficiently small r , $\varphi : S_r^{2n-1} \setminus K \rightarrow S^1$ is a locally trivial fibration.*

Proof. Consider the neighborhood of K defined by $N(K) := \{\mathbf{z} \in S_r^{2n-1} \setminus K \mid |H(\mathbf{z})| \leq \delta\}$. Corollary 12 says that three vectors $\mathbf{z}, \mathbf{v}_1(\mathbf{z}), \mathbf{v}_2(\mathbf{z})$ are linearly independent over \mathbb{R} on $N(K)$. For the definition of $\mathbf{v}_1, \mathbf{v}_2$, see the next section. Construct a vector field \mathcal{V} on $S_r^{2n-1} \setminus K$ such that $\Re(\mathcal{V}(\mathbf{z}), \mathbf{v}_2(\mathbf{z})) = 1$ and furthermore if $\mathbf{z} \in N(K)$, it also satisfies $\Re(\mathcal{V}(\mathbf{z}), \mathbf{v}_1(\mathbf{z})) = 0$. Then along the integral curves of \mathcal{V} , the argument of $H(\mathbf{z})$ is monotonic increase and the absolute value of H is constant when it enters in the neighborhood $N(K)$. Thus the integral curves exist for any time interval. For the local triviality, we use the integration of \mathcal{V} . \square

4.1. Equivalence of tubular and spherical Milnor fibrations. In this section, we consider the equivalence problem of two Milnor fibrations. Let us recall two vector fields on the complement of $V(H)$ which are defined as follows ([19]).

$$\begin{aligned} \mathbf{v}_1 &= \overline{\partial \log H} + \bar{\partial} \log H = \frac{\bar{\partial} f}{\bar{f}} + \frac{\bar{\partial} g}{\bar{g}} \\ \mathbf{v}_2 &= i(\overline{\partial \log H} - \bar{\partial} \log H) = i \left(\frac{\bar{\partial} f}{\bar{f}} - \frac{\bar{\partial} g}{\bar{g}} \right). \end{aligned}$$

$\mathbf{v}_1, \mathbf{v}_2$ are real orthogonal. Let $\mathbf{z}(t)$ be a real analytic curve in $\mathbb{C}^n \setminus V(H)$. Then we have

$$\begin{aligned} & \frac{d}{dt} \log H(\mathbf{z}(t)) \\ &= \frac{1}{f(\mathbf{z}(t))} \sum_{i=1}^n \frac{\partial f}{\partial z_i}(\mathbf{z}(t)) \frac{dz_i(t)}{dt} + \frac{1}{\bar{g}(\mathbf{z}(t))} \sum_{i=1}^n \overline{\frac{\partial g}{\partial z_i}(\mathbf{z}(t))} \frac{dz_i(t)}{dt} \\ &= \frac{1}{2} \left(\frac{d\mathbf{z}(t)}{dt}, \mathbf{v}_1(\mathbf{z}(t)) - i\mathbf{v}_2(\mathbf{z}(t)) \right) + \frac{1}{2} \overline{\left(\frac{d\mathbf{z}(t)}{dt}, \mathbf{v}_1(\mathbf{z}(t)) + i\mathbf{v}_2(\mathbf{z}(t)) \right)} \\ &= \Re \left(\frac{d\mathbf{z}(t)}{dt}, \mathbf{v}_1(\mathbf{z}(t)) \right) + i \Re \left(\frac{d\mathbf{z}(t)}{dt}, \mathbf{v}_2(\mathbf{z}(t)) \right). \end{aligned}$$

Thus we $\mathbf{v}_1(\mathbf{z})$ and $\mathbf{v}_2(\mathbf{z})$ are gradient vectors of $\Re \log H(\mathbf{z}) = \log |H(\mathbf{z})|$ and $\Im \log H(\mathbf{z}) = i \arg H(\mathbf{z})$. They are defined on $\mathbb{C}^n \setminus V(H)$. A key lemma is the following.

Lemma 17. *Assume that $\{f, g\}$ is as in Theorem 6. There exists a positive number r_0 such that for any $\mathbf{z} \in B_{r_0}^{2n} \setminus V(H)$, either three vectors*

$\mathbf{z}, \mathbf{v}_1(\mathbf{z}), \mathbf{v}_2(\mathbf{z})$ are linearly independent over \mathbb{R} or they are linearly dependent and the relation takes the following form:

$$\mathbf{z} = \lambda \mathbf{v}_1(\mathbf{z}) + \mu \mathbf{v}_2(\mathbf{z}), \lambda, \mu \in \mathbb{R},$$

where λ is positive.

Proof. Assume that there exists a real analytic curve $\mathbf{z}(t)$ in $\mathbb{C}^n \setminus V(H)$ and real valued rational functions $\lambda(t), \mu(t)$ such that

$$(4) \quad \mathbf{z}(t) = \lambda(t) \mathbf{v}_1(\mathbf{z}(t)) + \mu(t) \mathbf{v}_2(\mathbf{z}(t))$$

and $\mathbf{z}(0) = \mathbf{0}$. If $\mu(t) \equiv 0$, the assertion follows from Corollary 3.4, [15]. Thus we may assume that $\mu(t) \not\equiv 0$. Let $I = \{j \mid z_j(t) \not\equiv 0\}$. As f^I, g^I satisfies the same assumption, we assume for simplicity that $I = \{1, \dots, n\}$. Thus $\mathbf{z}(t) \in \mathbb{C}^{*n}$ for $t \neq 0$. Consider their Taylor or Laurent expansions

$$\begin{aligned} f(\mathbf{z}(t)) &= \gamma t^{m_f} + (\text{higher terms}), \gamma \in \mathbb{C}^* \\ g(\mathbf{z}(t)) &= \beta t^{m_g} + (\text{higher terms}), \beta \in \mathbb{C}^* \\ z_i(t) &= a_i t^{p_i} + (\text{higher terms}), a_i \in \mathbb{C}^* \\ \lambda(t) &= \lambda_0 t^{\nu_1} + (\text{higher terms}), \lambda_0 \in \mathbb{R} \\ \mu(t) &= \mu_0 t^{\nu_2} + (\text{higher terms}), \mu_0 \in \mathbb{R}^*. \end{aligned}$$

In the proof, we reprove Lemma 15. Thus $\lambda_0 = 0$ only if $\lambda(t) \equiv 0$ and in that case, we understand $\nu_1 = +\infty$. If this is the case, $\mathbf{z}(t)$ is a critical point of $\varphi : S_\tau^{2n-1} \setminus K_\tau$ where $\tau = \|\mathbf{z}(t)\|$ and K_τ is the link of $H^{-1}(0)$ in this sphere. Put $\ell := \min\{d(f; P) - m_f, d(g; P) - m_g\}$ and let us define

$$\varepsilon_f = \begin{cases} 1, & \text{if } d(P; f) - m_f = \ell \\ 0, & \text{if } d(P; f) - m_f > \ell \end{cases}$$

$$\varepsilon_g = \begin{cases} 1, & \text{if } d(P; g) - m_g = \ell \\ 0, & \text{if } d(P; g) - m_g > \ell \end{cases}$$

As $\text{ord } f(\mathbf{z}(t)) \geq d(P; f)$, $\text{ord } g(\mathbf{z}(t)) \geq d(P; g)$, we have that $\ell \leq 0$. Put $\mathbf{v}_k(\mathbf{z}(t)) = (v_k^1(t), \dots, v_k^n(t))$ for $k = 1, 2$. Observe that

$$(5) \quad v_1^j(t) = \left(\frac{\overline{f_{P,j}(\mathbf{a})}}{\bar{\gamma}} \varepsilon_f + \frac{\overline{g_{P,i}(\mathbf{a})}}{\bar{\beta}} \varepsilon_g \right) t^{\ell-p_j} + \dots$$

$$(6) \quad v_2^j(t) = i \left(\frac{\overline{f_{P,j}(\mathbf{a})}}{\bar{\gamma}} \varepsilon_f - \frac{\overline{g_{P,j}(\mathbf{a})}}{\bar{\beta}} \varepsilon_g \right) t^{\ell-p_j} + \dots$$

Put

$$\begin{aligned} P(\mathbf{a}) &:= (p_1 a_1, \dots, p_n a_n), \quad p_{\min} = \min\{p_j \mid j \in I\}, \\ J &= \{j \mid p_j = p_{\min}\}, \quad \nu_0 = \min\{\nu_1, \nu_2\} \end{aligned}$$

and put $\delta_i = 1$ or 0 according to $\nu_i = \nu_0$ or $\nu_i > \nu_0$ respectively for $i = 1, 2$. By (4), we get

$$\begin{aligned} a_j t^{p_j} &+ \dots \\ &= \lambda_0 \left(\frac{\overline{f_{P,j}(\mathbf{a})}}{\bar{\gamma}} \varepsilon_f + \frac{\overline{g_{P,i}(\mathbf{a})}}{\bar{\beta}} \varepsilon_g \right) t^{\nu_1 + \ell - p_j} + \dots \\ &+ \mu_0 i \left(\frac{\overline{f_{P,j}(\mathbf{a})}}{\bar{\gamma}} \varepsilon_f - \frac{\overline{g_{P,j}(\mathbf{a})}}{\bar{\beta}} \varepsilon_g \right) t^{\nu_2 + \ell - p_j} + \dots \\ &= e_j t^{\nu_0 + \ell - p_j} + \dots \end{aligned}$$

where

$$e_j = \left\{ \lambda_0 \delta_1 \left(\frac{\overline{f_{P,j}(\mathbf{a})}}{\bar{\gamma}} \varepsilon_f + \frac{\overline{g_{P,i}(\mathbf{a})}}{\bar{\beta}} \varepsilon_g \right) + \mu_0 \delta_2 i \left(\frac{\overline{f_{P,j}(\mathbf{a})}}{\bar{\gamma}} \varepsilon_f - \frac{\overline{g_{P,j}(\mathbf{a})}}{\bar{\beta}} \varepsilon_g \right) \right\}.$$

If $\nu_0 + \ell - 2p_{\min} > 0$, we get a contradiction $a_j = 0$, $j \in J$. Thus $\nu_0 + \ell - 2p_{\min} \leq 0$. Consider the vectors

$$\begin{aligned} \mathbf{v}_1^{(0)} &= (w_1^1, \dots, w_1^n), \quad w_1^j = \frac{\overline{f_{P,j}(\mathbf{a})}}{\bar{\gamma}} \varepsilon_f + \frac{\overline{g_{P,j}(\mathbf{a})}}{\bar{\beta}} \varepsilon_g \\ \mathbf{v}_2^{(0)} &= (w_2^1, \dots, w_2^n), \quad w_2^j = i \left(\frac{\overline{f_{P,j}(\mathbf{a})}}{\bar{\gamma}} \varepsilon_f - \frac{\overline{g_{P,j}(\mathbf{a})}}{\bar{\beta}} \varepsilon_g \right) \end{aligned}$$

Assume that $\nu_0 + \ell - 2p_{\min} < 0$. By (7), we get

$$(7) \quad \lambda_0 \delta_1 w_1^j + \mu_0 \delta_2 w_2^j = 0, \quad j = 1, \dots, n.$$

If $\ell < 0$, $\varepsilon_f f_P(\mathbf{a}) = 0$ and $\varepsilon_g g_P(\mathbf{a}) = 0$. The above equality gives a contradiction to the non-degeneracy condition either for $V(f)$ if $\varepsilon_f = 1, \varepsilon_g = 0$, or for $V(g)$ if $\varepsilon_f = 0, \varepsilon_g = 1$ or for the intersection variety $V(f, g)$ if $\varepsilon_f = \varepsilon_g = 1$.

Assume $\ell = 0$. Then $\nu_0 < 2p_{\min}$, $\gamma = f_P(\mathbf{a})$ and $\beta = g_P(\mathbf{a})$. We consider the equality

$$(8) \quad \sum_{j \in J} \frac{dz_j(t)}{dt} z_j(t) = \sum_{j \in J} \frac{dz_j(t)}{dt} \left(\overline{\lambda(t)v_1^j(t)} + \overline{\mu(t)v_2^j(t)} \right).$$

The left hand side has order $2p_{\min} - 1$ as

$$\sum_{j \in J} \frac{z_j(t)}{dt} z_j(t) = \sum_{j \in J} p_j |a_j|^2 t^{2p_{\min} - 1} + \dots$$

Using (7) and Euler equality, we see that the leading term of the right hand is $t^{\nu_0 - 1}$ which has the coefficient

$$\lambda_0 \delta_1 (d(P; f) + d(P; g)) + i \mu_0 \delta_2 (d(P; f) - d(P; g)) \neq 0.$$

The coefficient is non-zero. (If $\delta_1 = 0$, we use the Newton multiplicity condition to see the imaginary part is non-zero.) Thus the order is strictly

smaller than $2p_{min} - 1$, which is a contradiction. Thus the case $\nu_0 + \ell - 2p_{min} < 0$ does not occur. Thus the following equality holds:

$$\nu_0 + \ell - 2p_{min} = 0.$$

(7) implies the following equality.

$$\lambda_0 \delta_1 w_1^j + \mu_0 \delta_2 w_2^j = \begin{cases} a_j & j \in J \\ 0, & j \notin J. \end{cases}$$

We consider the equality (8) again.

The left side of (7) has order $2p_{min} - 1$ with the coefficient $\sum_{j \in J} p_{min} |a_j|^2 > 0$. The right side has order $2p_{min} - 1$ and the coefficient is given through Euler equality as

$$\begin{aligned} \lambda_0 \delta_1 \left(\frac{d(P; f) f_P(\mathbf{a})}{\gamma} \varepsilon_f + \frac{d(P; g) g_P(\mathbf{a})}{\beta} \varepsilon_g \right) \\ + \mu_0 \delta_2 i \left(\frac{d(P; f) f_P(\mathbf{a})}{\gamma} \varepsilon_f - \frac{d(P; g) g_P(\mathbf{a})}{\beta} \varepsilon_g \right) \end{aligned}$$

If $\ell < 0$, $f_P(\mathbf{a}) \varepsilon_f = g_P(\mathbf{a}) \varepsilon_g = 0$ and the above coefficient is zero. Thus we get a contradiction. Thus the only possible case is $\ell = 0$ and therefore

$$f_P(\mathbf{a}), g_P(\mathbf{a}) \neq 0, \nu_0 = 2p_{min}.$$

We observe also $\delta_1 \neq 0$, as otherwise the coefficient is purely imaginary. Thus we should have

$$\ell = 0, \nu_1 \leq \nu_2, \gamma = f_P(\mathbf{a}), \beta = g_P(\mathbf{a}).$$

The leading coefficients of (8) gives the equality:

$$\sum_{j \in J} p_{min} |a_j|^2 = \lambda_0 (d(P; f) + d(P; g)) + i \delta_2 \mu_0 (d(P; f) - d(P; g)).$$

Thus taking the real part of this equality, we conclude that $\lambda_0 > 0$. This also proves $\lambda(t) \equiv 0$ does not occur as $\lambda_0 \neq 0$. This gives another proof of Corollary 12. \square

Now we are ready to prove the equivalence theorem.

Theorem 18. *Assume that $\{f, g\}$ is a locally tame non-degenerate complete intersection pair as in Theorem 6. Consider the tubular and spherical Milnor fibrations*

$$\begin{aligned} H : \partial E(r, \delta)^* &\rightarrow S_\delta^1 \\ \varphi : S_r^{2n-1} \setminus K &\rightarrow S^1. \end{aligned}$$

These two fibrations are equivalent. Here we use the notations

$$\partial E(r, \delta)^* := \{\mathbf{z} \in B_r^{2n} \mid |H(\mathbf{z})| = \delta\}, K = S_r^{2n-1} \cap V(H).$$

Proof. Let δ be sufficiently small and put $\partial N(K) := \{\mathbf{z} \in S_r^{2n-1} \mid |H(\mathbf{z})| < \delta\}$. By the transversality, $N(K)$ is contractible to K and $N(K) \setminus K$ is diffeomorphic to $\partial N(K) \times (0, 1)$ and $\varphi : S_r^{2n-1} \setminus N(K) \rightarrow S^1$ is equivalent to the spherical fibration $\varphi : S_r^{2n-1} \setminus K \rightarrow S^1$. Note that vectors $\mathbf{v}_1, \mathbf{v}_2$ are real orthogonal. Take a locally finite open covering $\mathcal{U} = \{U_\alpha, \alpha \in A\} \cup \{V_\beta, \beta \in B\}$ of $B_r^{2n} \cap \{\mathbf{z} \mid |H(\mathbf{z})| \geq \delta\}$ as follows. Each U_α, V_β are open disk with center p_α, p_β . Secondly in each $U_\alpha, \{\mathbf{z}, \mathbf{v}_1(\mathbf{z}), \mathbf{v}_2(\mathbf{z})\}$ are linearly independent over \mathbb{R} , while in V_β, p_β can be written as

$$p_\beta = \lambda \mathbf{v}_1(p_\beta) + \mu \mathbf{v}_2(p_\beta), \quad \lambda > 0$$

and we take the radius of V_β is small enough so that $\Re(\mathbf{z}, \mathbf{v}_1(\mathbf{z})) > 0$ for any $\mathbf{z} \in V_\beta$. (There might exist a point $\mathbf{z} \in V_\beta$ where $\mathbf{z}, \mathbf{v}_1(\mathbf{z}), \mathbf{v}_2(\mathbf{z})$ are linearly independent.) Construct a vector field \mathbf{w}_α on U_α so that

$$\Re(\mathbf{w}_\alpha(\mathbf{z}), \mathbf{v}_2(\mathbf{z})) = 0, \Re(\mathbf{w}_\alpha(\mathbf{z}), \mathbf{v}_1(\mathbf{z})) = 1, \Re(\mathbf{w}_\alpha(\mathbf{z}), \mathbf{z}) = 1, \quad \forall \mathbf{z} \in U_\alpha.$$

Such a vector field is called a Milnor vector field in [5, 6]. Along any integration curve $\mathbf{z}(t)$ starting a point $p \in \partial E(r, \delta)^*$, $\arg H(\mathbf{z}(t))$ is constant and $|H(\mathbf{z}(t))|, \|\mathbf{z}(t)\|$ are strictly increasing. This curve arrives at $\mathbf{z}(s(p)) \in S_r^{2n-1} \setminus N(K)$ for a finite time $s(p) > 0$. Using this integration and the correspondence $p \mapsto \mathbf{z}(s(p))$, we can construct a diffeomorphism $\psi : \partial E(r, \delta)^* \rightarrow S_r^{2n-1} - N(K)$ and ψ gives the commutative diagram:

$$\begin{array}{ccc} \partial E(r, \delta)^* & \xrightarrow{\psi} & S_r^{2n-1} - N(K) \\ \downarrow H & & \downarrow \varphi \\ S_\delta^1 & \xrightarrow{\iota} & S^1 \end{array}$$

where $\iota(\eta) = \eta/\delta$. Thus ψ gives an isomorphism of the two fibrations. \square

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