Optimal execution strategies with generalized price impact in a discrete–time setting*

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Abstract

This paper examines a discrete-time optimal trade execution problem with generalized price impacts. We extend a model recently discussed in Ohnishi and Shimoshimizu (2019), which consider price impacts of (aggregate) random trade execution orders posed by noise–traders as well as a large trader. Although Ohnishi and Shimoshimizu (2019) assume that trading volumes submitted by noise–traders are serially independent, this paper allows a Markovian dependence.

Our new problem is formulated as a Markov decision process with state variables including the last noise–traders' orders. Over a finite horizon, the large trader with Constant Absolute Risk Aversion (CARA) von Neumann-Morgenstern (vN–M) utility function is assumed to maximize the expected utility from the final wealth. By applying the backward induction method of dynamic programming, we characterize the optimal value function and optimal trade execution strategy, and conclude that the trade execution strategy is a time–dependent affine function of three state variables: the remained trade execution volume of the large trader, (so–called) the residual effects of past price impacts caused by both of the large trader and other noise–traders, and the new state variable, i.e., the last trade execution orders submitted by noise–traders. This model enables us to investigate how the execution strategies and trade performances of a large trader are affected by the orders posed by noise–traders.

1 Introduction

The researches concerned with "optimal execution problem" play a fundamental role in analyzing a security market as the so–called "high–frequency trading (HFT)" or algorithmic trading have emerged in these decades. In a real marketplace, there are some kind of institutional traders called "large traders" who have a great influence on the market or the market price of traded assets (i.e. price impacts) through their own (large) submission. If large traders execute orders by splitting their large orders into small pieces, they can mitigate the price impact through their trades. On the other hand, executing slowly indicates that one is more likely to be exposed to the price fluctuation risk. From these facts, large traders have to recognize the price impacts as "liquidity risk" and construct an execution strategy considering both liquidity risk and volatility risk.

In this paper, we theoretically investigate an execution problem associated with the interaction among large traders and non–large traders (noise–traders) in a discrete–time setting. The pioneering work [2] address the optimization problem of minimizing the expected execution cost in a discrete–time framework via a dynamic programming approach. Notwithstanding a valuable insight into the execution problem, their model fails to take into account any attitudes toward risk. Subsequently,

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[1] derive an optimal execution strategy by considering both the execution cost and volatility risk, which entails the analysis with a mean–variance approach. As for the application of stochastic dynamic programming for optimal execution problems, [16] and [17] construct models with the residual effect of the transient price impact which dissipates over the trading time window. These works solve an optimization problem of maximizing an expected utility payoff from the final wealth at the maturity via a method of dynamic programming and derive an optimal execution strategy in a deterministic and nonrandomized class.

A multitude of researches focusing on execution problems formulate the market model without price impacts caused by the submission of noise–traders on the execution price. As mentioned in [23], however, the small trades have relatively by far larger impacts on the price than that of large trades from a statistical point of view. Thus, following this result, one should take into account a price impact caused by noise–traders when constructing a price impact model. [4] incorporates the price impact caused by other traders into the construction of the midprice process by describing the market order–flow through a general Markov process and derive a closed–form strategy for a large trader. They also show that the optimal execution strategies are different from the one obtained in [1] when noise–traders cause a price impact and coincide with the one obtained in [1] when noise–traders are assumed to have no influence on the midprice. This analysis is based on assuming the temporary and permanent price impacts and not the transient price impact. Our model considers the transient price impact through the "residual effect" of the past execution (caused by both a large trader and noise–traders) on a risky asset price. This setting enables us to analyze how residual effects affect the execution strategy of a large trader.

The paper [18], [21], and [22] theoretically investigate how the existence of other traders affects the execution strategy and trade performance of a large trader through the following two models: a single–large–trader Markov decision model, and a two–large–trader Markov game model. These models then yield the optimal execution strategy and an equilibrium execution strategy at a Markov perfect equilibrium. These kinds of investigation reveal that both strategies are not necessarily deterministic, although a multitude of researches show that optimal and equilibrium execution strategies often become deterministic. Incorporating the price impact caused by noise–traders into the price impact modeling is the novelty of the research in [22]. The formulation of an execution problem as a game model is also a significant factor in analyzing how the existence of other large traders affects the execution strategy of a large trader.

In this paper we consider a maximization problem concerned with the expected utility maximization of a large trader in a discrete–time setting. Constructing a Bellman equation (or dynamic programming equation) yields an optimal execution strategy in the form of a discrete–time trading strategy. Our finding is that the optimal execution strategy strongly depends on the aggregate volume submitted by noise–traders: that is, the optimal execution strategy becomes an affine function of trading volume posed by noise–traders as well as remaining volume (which the large trader has to execute in the remainder of the time horizon) and the residual effect of the past execution volumes, and not on the wealth process or the price dynamics. We also discover that this execution strategy is not always deterministic as shown in [18], [21], and [22]. Our focus is placed on the formulation of the noise–traders' submission as a sequence of random variables which has a Markov dependency.

This paper proceeds as follows. In Section 2, we construct a market model that characterizes the generalized price impact model by defining the price impact caused by the aggregate volume posed by noise–traders. Then, we solve the maximization problem of the expected utility of a risk–averse large trader with Constant Absolute Risk Aversion (CARA) von Neumann-Morgenstern (vN-M) utility (or negative exponential utility) from the wealth at the maturity. This leads to an optimal execution strategy. Finally, Section 3 concludes.

2 Price Impact Model with Non–Large Traders Effects

In the discrete time framework $t \in \{1, \ldots, T, T+1\} (T \in \mathbb{Z}_+ := \{1, 2, \ldots\})$, we assume that one large trader in a financial market must purchase $\mathfrak{Q}(\in \mathbb{R})$ volume of one risky asset by the time $T+1$. We also suppose that he/she has a Constant Absolute Risk Aversion (CARA) von Neumann-Morgenstern (vN-M) utility (or negative exponential) utility function with the absolute risk aversion rate $\gamma > 0$.

2.1 Market Model

Firstly, $q_t (\in \mathbb{R})$ represents large amount of orders submitted by the large trader at time $t \in \{1, \ldots, T\}$. Then, we denote by $\overline{Q}_t$ the remained execution volume, that is, the number of shares remained to purchase by the large trader at time $t \in \{1, \ldots, T, T+1\}$.[1] This assumption yields $\overline{Q}_1 = \mathfrak{Q}$ and

$$\overline{Q}_{t+1} = \overline{Q}_t - q_t, \quad t = 1, \ldots, T. \tag{2.1}$$

We assume that the market price (or quoted price) of the risky asset at time $t \in \{1, \ldots, T, T+1\}$ is set as P_t. Since the large trader has a great influence on the risky asset's price through his/her large amount of orders when executing the transaction, the execution price at time $t \in \{1, \ldots, T, T+1\}$ becomes not P_t but $\widehat{P}_t$ with the additive execution cost. In the rest of this paper, we assume that submitting one unit of (large) order at time $t \in \{1, \ldots, T\}$ causes the instantaneous price impact denoted as $\lambda_t (\in \mathbb{R})$. We also assume that the aggregate trading volume posed by noise–traders also has some impact on the execution price. κ_t represents the price impact per unit at time $t \in \{1, \ldots, T\}$ caused by noise–traders. The aggregate trading volume submitted by noise–traders at time $t \in \{1, \ldots, T\}$ is assumed to be a sequence of random variables v_t which has a Markovian dependence and follows a normal distribution with the following mean and variance:

$$v_0 \equiv 0; \quad v_1 \sim N(a_1^v, \sigma_1^v); \quad v_{t+1}|_{v_t} \sim N\left(a_{t+1}^v - b_{t+1}^v v_t, (\sigma_t^v)^2\right), \quad t = 1, \ldots, T-1. \tag{2.2}$$

Note that a_{t+1}^v, b_{t+1}^v, and σ_t^v are deterministic functions of time t. Then, the dynamics of v_t can be written with an i.i.d. sequence of normally distributed random variables $(\omega_t)_{t \in \{1,\ldots,T\}}$ as follows:

$$v_0 \equiv 0; \quad v_1 = a_1^v + \sigma_1^v \omega_1; \quad v_{t+1} = (a_{t+1}^v - b_{t+1}^v v_t) + \sigma_{t+1}^v \omega_{t+1}, \quad t = 1, \ldots, T-1, \tag{2.3}$$

where $\omega_t \sim N(0,1)$ for all $t \in \{1, \ldots, T\}$. By definition, if $a_t^v \equiv a^v$ and $b_t \equiv b^v$ (that is, a_{t+1}^v and b_{t+1} are time–independent) for all $t \in \{1, \ldots, T\}$ and $-1 < b^v < 0$, then v_t follows an AR (1) process. This case corresponds to a (time–)discretized version of an ergodic Ornstein–Uhlenbeck (OU) process. [8] considers an optimal execution problem in a continuous time setting under the assumption that the aggregate trading volume posed by noise–traders takes the form of an OU process. It must be noted that a lot of empirical researches highlight the importance of taking the autocorrelattion of order flow into account (see [12], [13], [6], [7]). Moreover, if $b_t^v = b^v \equiv -1$ for all $t \in \{1, \ldots, T\}$, then v_t follows a unit root process. In the case where $b_t^v = b^v < -1$, v_t follows an a (time–)discretized version of an non–ergodic OU process. This case reflects the situation that the noise–traders' buying (selling) at time $t \in \{1, \ldots, T\}$ incurrs further "tendency" of buying (selling) by noise–traders at time $t+1$. We can also confirm that if $b_t^v < 0$ for all $t \in \{1, \ldots, T\}$, larger v_t yields stochastically larger v_{t+1}, while if $b_t^v > 0$, larger v_t yields stochastically smaller v_{t+1}. Furthermore, in the case where $b_t = 0$ for all $t \in \{1, \ldots, T\}$, the model is reduced to [18] and [22].

In the sequel of this paper, the buy–trade and sell–trade of a large trader are supposed to induce the same (instantaneous) price impact. Assuming this would be inconsistent with the situation observed in a real market. However, we can justify this assumption from the statistical analysis of market data in [4] and [5].[2] The argument below is based on the model setting discussed in [18],

[1] The positive $\overline{Q}_t$ for $t \in \{1, \ldots, T, T+1\}$ stand for the acquisition and negative $\overline{Q}_t$ the liquidation of the risky asset. This setting allows us to establish a similar setup for a selling problem of a large trader.

[2] In their works, they estimate the permanent and temporary price impact by conducting a linear regression of price changes on net order–flow using trading data obtained from Nasdaq. According to this estimation and the relevant statistics show that the linear assumption of the price impact is compatible with the real stock market and that the price impact caused by both buy and sell trades are thought of as same from the viewpoint of statistical analysis.

[21], and [22]. We define the execution price in the form of a linear price impact model as follows:

$$\widehat{P}_t = P_t + (\lambda_t q_t + \kappa_t v_t), \quad t = 1, \dots, T. \tag{2.4}$$

We next define the residual price impact of past price at time $t \in \{1, \dots, T, T+1\}$ represented by R_t by means of the following exponential decay kernel function $G(t)$ of time $t \in \{1, \dots, T, T+1\}$:

$$G(t) := \mathrm{e}^{-\rho t}, \quad t = 1, \dots, T, T+1. \tag{2.5}$$

Using a deterministic price reversion rate α_t ($\in [0,1]$) and deterministic resilience speed ρ ($\in [0, \infty)$), the dynamics of the residual effect of past price impact R_t is given as follows:

$$\begin{aligned} R_1 &= 0; \\ R_{t+1} &= \sum_{k=1}^{t} (\lambda_k q_k + \kappa_k v_k)\, \alpha_k \mathrm{e}^{-\rho((t+1)-k)} \\ &= \mathrm{e}^{-\rho} \sum_{k=1}^{t-1} (\lambda_k q_k + \kappa_k v_k)\, \alpha_k \mathrm{e}^{-\rho(t-k)} + (\lambda_t q_t + \kappa_t v_t)\, \alpha_t \mathrm{e}^{-\rho} \\ &= [R_t + (\lambda_t q_t + \kappa_t v_t)\, \alpha_t]\, \mathrm{e}^{-\rho}, \quad t = 1, \dots, T. \end{aligned} \tag{2.6}$$

Eq. (2.6) shows that R_t has a Markov property in this setting, i.e., R_{t+1} depends only on R_t and the transient price impact $(\lambda_t q_t + \kappa_t v_t)\alpha_t \mathrm{e}^{-\rho}$. The Markov property arises thanks to the assumption of the exponantial decay kernel. Here we also define by the independent random variables ε_t at time $t \in \{1, \dots, T\}$ the effect of the public news/information about the economic situation between t and $t+1$, and assume that ε_t follows a normal distribution with mean μ_t^{ε} and variance $(\sigma_t^{\varepsilon})^2$, i.e.,

$$\varepsilon_t \sim N\left(\mu_t^{\varepsilon}, (\sigma_t^{\varepsilon})^2\right), \quad t = 1, \dots, T. \tag{2.7}$$

We suppose that the two stochastic processes, v_t and ε_t for $t \in \{1, \dots, T\}$ are mutually independent for convenience.

As for the "fundamental price" of the risky asset at time $t \in \{1, \dots, T\}$, denoted by P_t^f, we consider the formulation carefully . Since the residual effect of the past execution dissipates over the course of the trading horizon, we define $P_t - R_t$ as the fundamental price of the risky asset. By the definition of ε_t and by assuming that permanent price impact is represented by $(\lambda_k q_k + \kappa_k v_k)(1-\alpha_t)$, we can set the fundamental price $P_t^f := P_t - R_t$ with a permanent price impact as follows:

$$\begin{aligned} P_{t+1}^f &= P_{t+1} - R_{t+1} \\ &= P_t - R_t + (\lambda_t q_t + \kappa_t v_t)(1 - \alpha_t) + \varepsilon_t \\ &= P_t^f + (\lambda_t q_t + \kappa_t v_t)(1 - \alpha_t) + \varepsilon_t, \quad t = 1, \dots, T. \end{aligned} \tag{2.8}$$

This relation indicates that the permanent price impact caused by large traders and noise–traders and the public news or information about an economic situation are assumed to affect the fundamental price. This assumption also reveals that the permanent price impact may give a non–zero trend to the fundamental price, even if the mean of ε_t is zero for all $t \in \{1, \dots, T\}$. According to Eq. (2.4), (2.6), and (2.8), the dynamics of market price or the relation between P_{t+1} and P_t are described as

$$\begin{aligned} P_{t+1} &= P_t + (R_{t+1} - R_t) + (\lambda_t q_t + \kappa_t v_t)(1 - \alpha_t) + \varepsilon_t \\ &= P_t - (1 - \mathrm{e}^{-\rho})R_t + (\lambda_t q_t + \kappa_t v_t)\{\alpha_t \mathrm{e}^{-\rho} + (1 - \alpha_t)\} + \varepsilon_t, \quad t = 1, \dots, T. \end{aligned} \tag{2.9}$$

Remark 2.1. In this context, $(\lambda_t q_t + \kappa_t v_t)(1 - \alpha_t)$, $(\lambda_t q_t + \kappa_t v_t)\alpha_t$, and $(\lambda_t q_t + \kappa_t v_t)\alpha_t \mathrm{e}^{-\rho}$ represent the permanent impact, temporary impact, and transient impact, respectively. Moreover, if $\rho \to \infty$, the residual effect of past price impact becomes zero for all $t \in \{1, \ldots, T\}$ since $R_1 = 0$ and from Eq. (2.6)

$$\lim_{\rho \to \infty} R_{t+1} = \lim_{\rho \to \infty} [R_t + (\lambda_t q_t + \kappa_t v_t)\alpha_t]\mathrm{e}^{-\rho} = 0, \quad t = 1, \ldots, T, \tag{2.10}$$

and therefore,

$$\begin{aligned} P_{t+1} &= P_t - (1 - \mathrm{e}^{-\rho})r_t + (\lambda_t q_t + \kappa_t v_t)\{\alpha_t \mathrm{e}^{-\rho} + (1 - \alpha_t)\} + \varepsilon_t, \\ &= P_t + (\lambda_t q_t + \kappa_t v_t)(1 - \alpha_t) + \varepsilon_t, \quad t = 1, \ldots, T, \end{aligned} \tag{2.11}$$

that is, we have a permanent impact model. Also, if $\alpha_t = 1$, the model is reduced to a transient impact model. Also, if $\kappa_t = 0$ or $\sigma_{v_t} = 0$, the model is reduced to [17].

From the definition of the execution price, the wealth process w_t satisfies

$$W_{t+1} = W_t - \widehat{P}_t q_t = W_t - \{P_t + (\lambda_t q_t + \kappa_t v_t)\}\, q_t, \quad t = 1, \ldots, T. \tag{2.12}$$

2.2 Formulation as a Markov Decision Process

In this subsection, we formulate the large trader's problem as a discrete–time Markov decision process. In a discrete–time window $t \in \{1, \ldots, T, T+1\}$, we define the state of the decision process at time $t \in \{1, \ldots, T, T+1\}$ as 5–tuple and denote it as

$$s_t = (W_t, P_t, \overline{Q}_t, R_t, v_{t-1})^\top \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} =: S. \tag{2.13}$$

For $t \in \{1, \ldots, T\}$, an allowable action chosen at state s_t is an execution volume $q_t \in \mathbb{R} =: A$ so that the set A of admissible actions is independent of the current state s_t.

When an action q_t is chosen in a state s_t at time $t \in \{1, \ldots, T\}$, a transition to a next state

$$s_{t+1} = (W_{t+1}, P_{t+1}, \overline{Q}_{t+1}, R_{t+1}, v_t)^\top \in S \tag{2.14}$$

occurs according to the law of motion which we have precisely described in the previous subsection. We symbolically describe the transition by a (Borel measurable) system dynamics function h_t (: $S \times A \times (\mathbb{R} \times \mathbb{R}) \longrightarrow S$):

$$s_{t+1} = h_t(s_t, q_t, (\omega_t, \varepsilon_t)), \quad t = 1, \ldots, T. \tag{2.15}$$

A utility payoff (or reward) arises only in a terminal state s_{T+1} at the end of horizon $T+1$ as

$$g_{T+1}(s_{T+1}) := \begin{cases} -\exp\{-\gamma W_{T+1}\} & \text{if } \overline{Q}_{T+1} = 0; \\ -\infty & \text{if } \overline{Q}_{T+1} \neq 0, \end{cases} \tag{2.16}$$

where $\gamma > 0$ represents the risk aversion. The term $-\infty$ means a hard constraint enforcing the large trader to execute all of the remaining volume $\overline{Q}_T$ at the maturity T, that is, $q_T = \overline{Q}_T$.

If we define a (history–independent) one–stage decision rule f_t at time $t \in \{1, \ldots, T\}$ by a Borel measurable map from a state $s_t \in S = \mathbb{R}^5$ to an action

$$q_t = f_t(s_t) \in A = \mathbb{R}, \tag{2.17}$$

then a Markov execution strategy π is defined as a sequence of one–stage decision rules

$$\pi := (f_1, \ldots, f_t, \ldots, f_T). \tag{2.18}$$

We denote the set of all Markov execution strategies as Π_{M}. Further, for $t \in \{1, \ldots, T\}$, we define the sub–execution strategy after time t of a Markov execution strategy $\pi = (f_1, \ldots, f_t, \ldots, f_T) \in \Pi_{\mathrm{M}}$ as $\pi_t := (f_t, \ldots, f_T)$, and the entire set of π_t as $\Pi_{\mathrm{M},t}$.

By definition (2.16), the value function under an execution strategy π becomes an expected utility payoff arising from the terminal wealth W_{T+1} of the large trader with the absolute risk aversion γ:

$$V_1^\pi\big[s_1\big] = \mathbb{E}_1^\pi\Big[g_{T+1}(s_{T+1})\Big|s_1\Big] = \mathbb{E}_1^\pi\Big[-\exp\big\{-\gamma W_{T+1}\big\}\cdot 1_{\{\overline{Q}_{T+1}=0\}} + (-\infty)\cdot 1_{\{\overline{Q}_{T+1}\neq 0\}}\Big|s_1\Big], \tag{2.19}$$

where 1_A is the indicator function of the event A and, for $t \in \{1,\dots,T\}$, $\mathbb{E}_t^\pi$ is a conditional expectation given a condition at time t under π.

Then, for $t \in \{1,\dots,T,T+1\}$ and $s_t \in S$, we further let

$$V_t^\pi\big[s_t\big] = \mathbb{E}_t^\pi\Big[g_{T+1}(s_{T+1})\Big|s_t\Big] = \mathbb{E}_t^\pi\Big[-\exp\big\{-\gamma W_{T+1}\big\}\cdot 1_{\{\overline{Q}_{T+1}=0\}} + (-\infty)\cdot 1_{\{\overline{Q}_{T+1}\neq 0\}}\Big|s_t\Big] \tag{2.20}$$

be the expected utility payoff at time t under the strategy π. It should be noted that the expected utility payoff $V_t^\pi\big[s_t\big]$ depends on the Markov execution policy $\pi = (f_1,\dots,f_t,\dots,f_T)$ only through the sub–execution policy $\pi_t := (f_t,\dots,f_T)$ after time t.

Now, we define the optimal value function as follows:

$$V_t\big[s_t\big] = \sup_{\pi\in\Pi_{\mathrm{M}}} V_t^\pi\big[s_t\big], \quad s_t \in S, \quad t = 1,\dots,T,T+1. \tag{2.21}$$

From the principle of optimality, the optimality equation (Bellman equation, or dynamic programming equation) becomes

$$V_t\big[s_t\big] = \sup_{q_t\in\mathbb{R}} \mathbb{E}\Big[V_{t+1}\big[h_t\left(s_t, q_t, (\omega_t, \varepsilon_t)\right)\big]\Big|s_t\Big], \quad s_t \in S, \quad t = 1,\dots,T,T+1. \tag{2.22}$$

2.3 Dynamics of the Optimal Execution

The optimal dynamic execution strategy π is acquired by solving the above equation (2.22) backwardly in time t from maturity T.

Theorem 2.1 (Optimal Value Function and Optimal Execution Strategy)**.**

1. The optimal execution volume at time $t \in \{1,\dots,T,T+1\}$, denoted as q_t^*, becomes an affine function of the aggregate volume submitted by noise–traders at time $t-1$ as well as the remaining execution volume $\overline{Q}_t$ and the cumulative residual effect R_t:

$$q_t^* = f_t\left(W_t, P_t, \overline{Q}_t, R_t, v_{t-1}\right) = a_t + b_t\overline{Q}_t + c_tR_t + d_tv_{t-1}, \quad t = 1,\dots,T. \tag{2.23}$$

2. The optimal value function $V_t\big[s_t\big]$ at time $t \in \{1,\dots,T,T+1\}$ is represented as follows:

$$\begin{aligned} V_t\big[W_t, P_t, \overline{Q}_t, R_t, v_{t-1}\big] = -\exp\Big\{-\gamma\Big[&W_t - P_t\overline{Q}_t + G_t\overline{Q}_t^2 + H_t\overline{Q}_t + I_t\overline{Q}_tR_t + J_tR_t^2 + L_tR_t \\ &+ M_t\overline{Q}_tv_{t-1} + N_tR_tv_{t-1} + X_tv_{t-1}^2 + Y_tv_{t-1} + Z_t\Big]\Big\}, \end{aligned} \tag{2.24}$$

where a_t, b_t, c_t, d_t; $G_t, H_t, I_t, J_t, L_t, M_t, N_t, X_t, Y_t, Z_t$ for $t \in \{1,\dots,T,T+1\}$ are deterministic functions of time t which are dependent on the problem parameters, and can be computed backwardly in time t from maturity T.

See the proof of this theorem in Appendix A.

From the above theorem, we find that the optimal execution volume q_t^* for $t \in \{1,\dots,T\}$ depend on the state $s_t = (W_t, P_t, \overline{Q}_t, R_t, v_{t-1})^\top$ of the decision process through the total volume submitted by noise–traders at the previous time v_{t-1} in addition to the remaining execution volume $\overline{Q}_t$ and the cumulative residual effect R_t, and not through the wealth W_t or market price P_t. Not only does our analysis show that the optimal execution strategy becomes a stochastic one but also it reveals that the orders posed by noise–traders both directly and indirectly affect the execution strategy of the large trader. This is our contribution to the field of the optimal execution problem. Thus, we have the following facts.

Corollary 2.1. If the trading volumes submitted by noise–traders v_t for $t \in \{1, \ldots, T\}$, are deterministic, the optimal execution volumes q_t^* at time $t \in \{1, \ldots, T\}$ also become deterministic functions of time, which means that the optimal execution strategy is a one in a class of the static (and non–randomized) execution strategy.

A great number of researches focusing on the execution problem of a single large trader yields an optimal execution strategy in a deterministic class, although these researches do not yield an optimal execution strategy which reflects the (direct) effect caused by the noise–traders' trading volume.

2.4 In the Case with Target Close Order

In this subsection, we consider a model with closing price. The time framework $t \in \{1, \ldots, T, T+1\}$ is same in the model mentioned above. However, we add an assumption that a large trader can execute his/her remaining execution volume at time $T+1$, i.e., $\overline{Q}_{T+1}$, with closing price P_{T+1}. We further assume that the trading at time $T+1$ impose the large trader to pay the additive cost χ_{T+1} per unit of the remaining volume.

According to the above settings, the value function at the maturity becomes

$$V_{T+1}\big[s_{T+1}\big] = -\exp\Big\{-\gamma\Big[W_{T+1} - (P_{T+1} + \chi_{T+1}\overline{Q}_{T+1})\overline{Q}_{T+1}\Big]\Big\}. \tag{2.25}$$

Then, the following theorem holds.

Theorem 2.2 (Optimal Value Function and Optimal Execution Strategy in the Case with Target Close Order)**.**

1. The optimal execution volume at time $t \in \{1, \ldots, T, T+1\}$, denoted as q_t^*, becomes an affine function of the aggregate volume submitted by noise–traders at time $t-1$ as well as the remaining execution volume $\overline{Q}_t$ and the cumulative residual effect R_t:

$$q_t^* = f_t\left(W_t, P_t, \overline{Q}_t, R_t, v_{t-1}\right) = a_t^* + b_t^*\overline{Q}_t + c_t^* R_t + d_t^* v_{t-1}, \quad t = 1, \ldots, T. \tag{2.26}$$

2. The optimal value function $V_t\big[s_t\big]$ at time $t \in \{1, \ldots, T, T+1\}$ takes the form as follows:

$$\begin{aligned} V_t\big[W_t, P_t, \overline{Q}_t, R_t, v_{t-1}\big] = -\exp\Big\{-\gamma\Big[&W_t - P_t\overline{Q}_t + G_t^*\overline{Q}_t^2 + H_t^*\overline{Q}_t + I_t^*\overline{Q}_t R_t + J_t^* R_t^2 + L_t^* R_t \\ &+ M_t^*\overline{Q}_t v_{t-1} + N_t^* R_t v_{t-1} + X_t^* v_{t-1}^2 + Y_t^* v_{t-1} + Z_t^*\Big]\Big\}, \end{aligned} \tag{2.27}$$

where $a_t^*, b_t^*, c_t^*, d_t^*;\ G_t^*, H_t^*, I_t^*, J_t^*, L_t^*, M_t^*, N_t^*, X_t^*, Y_t^*, Z_t^*$ for $t \in \{1, \ldots, T, T+1\}$ are deterministic functions of time t which are dependent on the problem parameters, and can be computed backwardly in time t from maturity T.

We omit the proof of this theorem since it is very similar to that of Theorem 2.1.

3 Conclusion

We constructed, in a (finite) discrete–time framework, a model focusing on a single large trader. The large trader maximizes the expected Constant Absolute Risk Aversion (CARA) von Neumann-Morgenstern (vN-M) utility which arises from his/her wealth at the end of the trading epoch in a market with noise–traders. The most important result which emerged from this research is as follows: the aggregate execution volume of noise–traders has both direct and indirect impacts on the execution strategy of the large trader.

In the above models, we have assumed that that the price reversion rate and the resilience speed are deterministic. This assumption makes the fundamental price of the risky asset observable for large traders before the trading time. The fundamental value of a risky asset is, however, unobservable and uncertain in a real market. Therefore, we can evolve the model built in this paper as an incomplete state information model, which leads to an analysis in a more realistic situation of the marketplace. Developing an incomplete state information model of single– or multiple–large traders will contribute to some developments of a study involved in a trading market.

Adding to the possibility of extending our research to an incomplete state information model, there would be room for formulating an execution problem as a stochastic game of multiple large traders. In our current research, we assume that there is a single large trader in a security market, although many large traders participate in a real market. The assumption may be relaxed by assuming that multiple large traders influence the execution price with each other. This makes us capable of formulating the problem as a stochastic game played by multiple large traders. The formulation is, however, intractable in terms of obtaining an analytical solution. Thus, there might be room for searching for a more tractable model of an execution problem concerned with multiple large traders which yields a semi–analytical or, if possible, an analytical solution.

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Appendix

A Proof of Theorem 2.1

We derive the optimal execution volume q_t^* at time $t \in \{1, \dots, T\}$ by backward induction method of dynamic programming from the maturity T. From the assumption that the large trader must unwind all the remainder of his/her position at time $t = T$,

$$\overline{Q}_{T+1} = \overline{Q}_T - q_T = 0, \tag{A.1}$$

must hold, which yields $\overline{Q}_T = q_T$. Then, for $t = T$, with the relation of the moment–generating function of v_t,

$$\mathbb{E}\Big[\exp\big\{\gamma\kappa_T\overline{Q}_T v_T\big\}\Big] = \exp\big\{\gamma\kappa_T\overline{Q}_T(a_{v_T} - b_{v_T}v_{T-1}) + \frac{1}{2}\gamma^2\kappa_T^2\overline{Q}_T^2\sigma_{v_T}^2\big\}, \tag{A.2}$$

Eq. (2.22), or the Bellman equation at time $t = T$, becomes

$$\begin{aligned}
V_T[s_T] &= \sup_{q_T\in\mathbb{R}} \mathbb{E}\Big[V_{T+1}[W_{T+1}, P_{T+1}, \overline{Q}_{T+1}, R_{T+1}, v_T]\Big| W_T, P_T, \overline{Q}_T, R_T, v_{T-1}\Big] \\
&= \sup_{q_T\in\mathbb{R}} \mathbb{E}\Big[-\exp\{-\gamma W_{T+1}\}\Big| W_T, P_T, \overline{Q}_T, R_T, v_{T-1}\Big] \\
&= \sup_{q_T\in\mathbb{R}} \mathbb{E}\Big[-\exp\{-\gamma[W_T - [P_T + (\lambda_T q_T + \kappa_T v_{T-1})]\, q_T]\}\Big| W_T, P_T, \overline{Q}_T, R_T, v_T\Big] \\
&= \mathbb{E}\Big[-\exp\{-\gamma[W_T - [P_T + (\lambda_T \overline{Q}_T + \kappa_T v_T)]\, \overline{Q}_T]\}\Big| W_T, P_T, \overline{Q}_T, R_T, v_{T-1}\Big] \\
&= -\exp\Big\{-\gamma\Big[W_T - P_T\overline{Q}_T + G_T\overline{Q}_T^2 + H_T\overline{Q}_T + M_T\overline{Q}_T v_{T-1}\Big]\Big\}, \qquad \text{(A.3)}
\end{aligned}$$

where

$$G_T := -\lambda_T - \frac{1}{2}\gamma\kappa_T^2(\sigma_T^v)^2 (< 0); \quad H_T := -\kappa_T a_T^v; \quad M_T := \kappa_T b_T^v.$$

For $t = T-1$, we have

$$\begin{aligned}
&V_{T-1}[s_{T-1}] \\
&= \sup_{q_{T-1}\in\mathbb{R}} \mathbb{E}\Big[V_T[s_T]\Big| s_{T-1}\Big] \\
&= \sup_{q_{T-1}\in\mathbb{R}} \mathbb{E}\Big[-\exp\Big\{-\gamma\Big[W_T - P_T\overline{Q}_T + G_T\overline{Q}_T^2 + H_T\overline{Q}_T + M_T\overline{Q}_T v_T\Big]\Big\}\Big| s_{T-1}\Big] \\
&= \sup_{q_{T-1}\in\mathbb{R}} \mathbb{E}\Big[-\exp\Big\{-\gamma\Big[W_{T-1} - \{P_{T-1} + (\lambda_{T-1}q_{T-1} + \kappa_{T-1}v_{T-1})\}q_{T-1} \\
&\quad - \{P_{T-1} - (1-\mathrm{e}^{-\rho})R_{T-1} + (\lambda_{T-1}q_{T-1} + \kappa_{T-1}v_{T-1})\{\alpha_{T-1}\mathrm{e}^{-\rho} + (1-\mathrm{e}^{-\rho})\} + \varepsilon_{T-1}\}(\overline{Q}_{T-1} - q_{T-1}) \\
&\quad + G_T(\overline{Q}_{T-1} - q_{T-1})^2 + M_T(\overline{Q}_{T-1} - q_{T-1})v_{T-1}\Big]\Big\}\Big| s_{T-1}\Big] \\
&= \sup_{q_{T-1}\in\mathbb{R}} -\exp\Big\{-\gamma\Big[-A_{T-1}q_{T-1}^2 + (B_{T-1}\overline{Q}_{T-1} + C_{T-1}R_{T-1} + D_{T-1}v_{T-2} + F_{T-1})\,q_{T-1} \\
&\quad + W_{T-1} - P_{T-1}\overline{Q}_{T-1} + \Big\{G_T - \frac{1}{2}\gamma(\alpha^{T-1})^2\kappa_{T-1}^2(\sigma_{T-1}^v)^2 - \frac{1}{2}\gamma(\sigma_{T-1}^\varepsilon)^2\Big\}\overline{Q}_{T-1}^2 \\
&\quad + (H_T - \alpha^{T-1}\kappa_{T-1}a_{T-1}^v + \gamma\alpha^{T-1}\kappa_{T-1}M_T(\sigma_{T-1}^v)^2 - \mu_{T-1}^\varepsilon)\,\overline{Q}_{T-1} + (1-\mathrm{e}^{-\rho})\overline{Q}_{T-1}R_{T-1} \\
&\quad + \kappa_{T-1}\alpha^{T-1}b_{T-1}^v\overline{Q}_{T-1}v_{T-2} + M_T b_{T-1}^v v_{T-2} - \Big(M_T - \frac{1}{2}\gamma M_T^2\sigma_{T-1}^v\Big)\Big]\Big\}, \qquad \text{(A.4)}
\end{aligned}$$

with the following relation:

$$\begin{aligned}
\alpha^{T-1} &:= \alpha_{T-1}\mathrm{e}^{-\rho} + (1-\alpha_{T-1}); \\
A_{T-1} &:= (1-\alpha^{T-1})\lambda_{T-1} - G_T + \frac{1}{2}\gamma(1-\alpha^{T-1})^2\kappa_{T-1}^2(\sigma_T^v)^2 + \frac{1}{2}\gamma(\sigma_{T-1}^\varepsilon)^2; \\
B_{T-1} &:= -\alpha^{T-1}\lambda_{T-1} - 2G_T - \gamma\alpha^{T-1}(1-\alpha^{T-1})\kappa_{T-1}^2(\sigma_T^v)^2 + \gamma(\sigma_{T-1}^\varepsilon)^2; \\
C_{T-1} &:= -(1-\mathrm{e}^{-\rho}); \\
D_{T-1} &:= (1-\alpha^{T-1})\kappa_{T-1}b_{T-1}^v; \\
F_{T-1} &:= -H_T - (1-\alpha^{T-1})\kappa_{T-1}a_{T-1}^v + \gamma(1-\alpha^{T-1})\kappa_{T-1}M_T(\sigma_{T-1}^v)^2 + \mu_{\varepsilon_{T-1}}.
\end{aligned}$$

Finding the optimal execution volume q_{T-1}^* which attains the supremum of Eq. (A.4) is equivalent to finding the one which yields the maximum of the following function $K_{T-1}(q_{T-1})$ defined as

$$\begin{aligned}
K_{T-1}(q_{T-1}) &:= -A_{T-1}q_{T-1}^2 + \left(B_{T-1}\overline{Q}_{T-1} + C_{T-1}R_{T-1} + D_{T-1}v_{T-1} + F_{T-1}\right) q_{T-1} \\
&\quad + W_{T-1} - P_{T-1}\overline{Q}_{T-1} + \left\{G_T - \frac{1}{2}\gamma(\alpha^{T-1})^2\kappa_{T-1}^2(\sigma_{T-1}^v)^2 - \frac{1}{2}\gamma(\sigma_{T-1}^\varepsilon)^2\right\}\overline{Q}_{T-1}^2 \\
&\quad + \left(H_T - \alpha^{T-1}\kappa_{T-1}a_{T-1}^v + \gamma\alpha^{T-1}\kappa_{T-1}M_T(\sigma_{T-1}^v)^2 - \mu_{T-1}^\varepsilon\right)\overline{Q}_{T-1} + (1-\mathrm{e}^{-\rho})\overline{Q}_{T-1}R_{T-1} \\
&\quad + \kappa_{T-1}\alpha^{T-1}b_{T-1}^v\overline{Q}_{T-1}v_{T-2} - M_T b_{T-1}^v v_{T-2} + \left(M_T - \frac{1}{2}\gamma M_T^2\sigma_{T-1}^v\right), \qquad \text{(A.5)}
\end{aligned}$$

since both Eq. (A.4) and Eq. (A.5) are concave functions with respect to q_{T-1}. Thus, by completing the square of K_{T-1}, we obtain the optimal execution volume q_{T-1}^* as

$$\begin{aligned}
q_{T-1}^* &= \frac{B_{T-1}\overline{Q}_{T-1} + C_{T-1}R_{T-1} + D_{T-1}v_{T-1} + F_{T-1}}{2A_{T-1}} \qquad \text{(A.6)}\\
&\quad \left(=: a_{T-1} + b_{T-1}\overline{Q}_{T-1} + c_{T-1}r_{T-1} + d_{T-1}v_{T-1}\right).
\end{aligned}$$

Thus, the optimal value function at time $T-1$ becomes a functional form as follows:

$$\begin{aligned}
&V_{T-1}\left[s_{T-1}\right] \\
&= -\exp\Big\{-\gamma\big[W_{T-1} - P_{T-1}\overline{Q}_{T-1} + G_{T-1}\overline{Q}_{T-1}^2 + H_{T-1}\overline{Q}_{T-1} + I_{T-1}\overline{Q}_{T-1}R_{T-1} + J_{T-1}R_{T-1}^2 + L_{T-1}R_{T-1} \\
&\quad + M_{T-1}\overline{Q}_{T-1}v_{T-1} + N_{T-1}R_{T-1}v_{T-1} + X_{T-1}v_{T-1}^2 + Y_{T-1}v_{T-1} + Z_{T-1}\big]\Big\}, \qquad \text{(A.7)}
\end{aligned}$$

where

$$\begin{aligned}
G_{T-1} &:= G_T - \frac{1}{2}\gamma(\alpha^{T-1})^2\kappa_{T-1}^2(\sigma_{T-1}^v)^2 - \frac{1}{2}\gamma(\sigma_{T-1}^\varepsilon)^2 + \frac{B_{T-1}^2}{4A_{T-1}};\\
H_{T-1} &:= H_T - \alpha^{T-1}\kappa_{T-1}a_{T-1}^v + \gamma\alpha^{T-1}\kappa_{T-1}M_T(\sigma_{T-1}^v)^2 - \mu_{T-1}^\varepsilon + \frac{B_{T-1}F_{T-1}}{2A_{T-1}};\\
I_{T-1} &:= (1-\mathrm{e}^{-\rho}) + \frac{B_{T-1}C_{T-1}}{2A_{T-1}}; \quad J_{T-1} := \frac{C_{T-1}^2}{4A_{T-1}}; \quad L_{T-1} := \frac{C_{T-1}F_{T-1}}{2A_{T-1}};\\
M_{T-1} &:= \kappa_{T-1}\alpha^{T-1}b_{T-1}^v + \frac{B_{T-1}D_{T-1}}{2A_{T-1}}; \quad N_{T-1} := \frac{C_{T-1}D_{T-1}}{2A_{T-1}}; \quad X_{T-1} := \frac{D_{T-1}^2}{4A_{T-1}};\\
Y_{T-1} &:= -M_T b_{T-1}^v + \frac{D_{T-1}F_{T-1}}{2A_{T-1}}; \quad Z_{T-1} := M_T - \frac{1}{2}\gamma M_T^2\sigma_{T-1}^v + \frac{F_{T-1}^2}{4A_{T-1}}. \qquad \text{(A.8)}
\end{aligned}$$

For $t \in \{T-2, \dots, 1\}$, we can assume from the above results that, at time $t+1$, the optimal value function has the following functional form:

$$\begin{aligned}
V_{t+1}\left[s_{t+1}\right] = -\exp\Big\{-\gamma\Big[&W_{t+1} - P_{t+1}\overline{Q}_{t+1} + G_{t+1}\overline{Q}_{t+1}^2 + H_{t+1}\overline{Q}_{t+1} + I_{t+1}\overline{Q}_{t+1}R_{t+1} + J_{t+1}R_{t+1}^2 + L_{t+1}R_{t+1} \\
&+ M_{t+1}\overline{Q}_{t+1}v_t + N_{t+1}rRt + 1v_{t-1} + X_{t+1}v_{t-1}^2 + Y_{t+1}v_{t-1} + Z_{t+1}\Big]\Big\}. \qquad \text{(A.9)}
\end{aligned}$$

Then, we can obtain the following calculation by substituting the dynamics of $w_t, p_t, \overline{Q}_t, r_t$ into the equation above:

$$
\begin{aligned}
V_t[s_t] &= \sup_{q_t\in\mathbb{R}} \mathbb{E}\Big[-\exp\Big\{-\gamma\Big[W_{t+1} - P_{t+1}\overline{Q}_{t+1} \\
&\quad + G_{t+1}\overline{Q}^2_{t+1} + H_{t+1}\overline{Q}_{t+1} + I_{t+1}\overline{Q}_{t+1}R_{t+1} + J_{t+1}R^2_{t+1} + L_{t+1}R_{t+1} + Z_{t+1}\Big]\Big\}\Big| W_t, P_t, \overline{Q}_t, R_t, v_{t-1}\Big] \\
&= \sup_{q_t\in\mathbb{R}} -\exp\Big\{-\gamma\Big[-A_t q_t^2 + (B_t\overline{Q}_t + C_tR_t + D_tv_{t-1} + F_{t-1})q_t + W_t - P_t\overline{Q}_t \\
&\quad + \Big[G_{t+1} - \frac{1}{2\{1+2\gamma\zeta_t(\sigma_t^v)^2\}}\gamma\eta_t^2(\sigma_t^v)^2 - \frac{1}{2}\gamma(\sigma_t^\varepsilon)^2\Big]\overline{Q}_t^2 + \Big[H_{t+1} + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\eta_t a_t^v \\
&\quad - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\gamma\theta_t\phi_t(\sigma_t^v)^2 - \mu_t^\varepsilon\Big]\overline{Q}_t + \Big[(1-\mathrm{e}^{-\rho}) + \mathrm{e}^{-\rho}I_{t+1} - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\gamma\eta_t\theta_t(\sigma_t^v)^2\Big]\overline{Q}_tR_t \\
&\quad + \Big[\mathrm{e}^{-2\rho}J_{t+1} - \frac{1}{2\{1+2\gamma\zeta_t(\sigma_t^v)^2\}}\gamma\theta_t^2(\sigma_t^v)^2\Big]R_t^2 + \Big[\mathrm{e}^{-\rho}L_{t+1} + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\theta_t a_t^v \\
&\quad - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\gamma\theta_t\phi_t(\sigma_t^v)^2\Big]R_t - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\delta_t b_t^v\overline{Q}_tv_{t-1} - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\theta_t b_t^v R_t v_{t-1} \\
&\quad + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\zeta_t(b_t^v)^2v_{t-1}^2 - \Big[\frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\phi_t b_t^v + \frac{2}{1+2\gamma\zeta_t(\sigma_t^v)^2}\zeta_t a_t^v b_t^v\Big]v_{t-1} \\
&\quad + \Big[Z_{t+1} + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\phi_t a_t^v - \frac{1}{2\{1+2\gamma\zeta_t(\sigma_t^v)^2\}}\gamma\phi_t^2(\sigma_t^v)^2 + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\zeta_t(a_t^v)^2 + x_t\Big]\Big\},
\end{aligned}
\tag{A.10}
$$

where

$$
\begin{aligned}
&\alpha^t := \alpha_t\mathrm{e}^{-\rho} + (1-\alpha_t); \quad \zeta_t := \kappa_t^2\alpha_t^2\mathrm{e}^{-2\rho}J_{t+1} + \kappa_t\alpha_t\mathrm{e}^{-\rho}N_{t+1} + X_{t+1}; \\
&\delta_t := (\alpha^t - 1)\kappa_t - \kappa_t\alpha_t\mathrm{e}^{-\rho}I_{t+1} + 2\lambda_t\kappa_t\alpha_t^2\mathrm{e}^{-2\rho}J_{t+1} - M_{t+1} + \lambda_t\alpha_t\mathrm{e}^{-\rho}N_{t+1}; \\
&\eta_t := -\kappa_t\alpha^t + \kappa_t\alpha_t\mathrm{e}^{-\rho}I_{t+1} + M_{t+1}; \quad \theta_t := 2\kappa_t\alpha_t\mathrm{e}^{-\rho}J_{t+1} + \mathrm{e}^{-\rho}N_{t+1}; \quad \phi_t := \kappa_t\alpha_t\mathrm{e}^{-\rho}L_{t+1} + Y_t; \\
&x_t := -\frac{1}{\gamma}\log\frac{1}{\sqrt{1+2R\zeta_t(\sigma_t^v)^2}},
\end{aligned}
\tag{A.11}
$$

and

$$
\begin{aligned}
A_t &:= (1-\alpha^t)\lambda_t - G_{t+1} + \lambda_t\alpha_t\mathrm{e}^{-\rho}I_{t+1} - \lambda_t^2\alpha_t^2\mathrm{e}^{-2\rho}J_{t+1} + \frac{1}{2\{1+2\gamma\zeta_t(\sigma_t^v)^2\}}\gamma\delta_t^2(\sigma_t^v)^2 + \frac{1}{2}\gamma(\sigma_t^\varepsilon)^2; \\
B_t &:= -\lambda_t\alpha^t - 2G_{t+1} + \lambda_t\alpha_t\mathrm{e}^{-\rho}I_{t+1} - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\gamma\delta_t\eta_t(\sigma_t^v)^2 + \gamma(\sigma_t^\varepsilon)^2; \\
C_t &:= -(1-\mathrm{e}^{-\rho}) - \mathrm{e}^{-\rho}I_{t+1} + 2\lambda_t\alpha_t\mathrm{e}^{-2\rho}J_{t+1} - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\gamma\delta_t\theta_t(\sigma_t^v)^2; \\
D_t &:= -\frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\gamma\delta_t b_t^v; \\
F &:= -H_{t+1} + \lambda_t\alpha_t\mathrm{e}^{-\rho}L_{t+1} + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\gamma\delta_t a_t^v - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\gamma\delta_t\phi_t(\sigma_t^v)^2 + \mu_t^\varepsilon.
\end{aligned}
\tag{A.12}
$$

To find the optimal execution volume q_t^* at time $t \in \{T-2, \dots, 1\}$ which satisfy (A.10), we only have to calculate the same derivation at time $t = T-1$, that is, completing the square of

$$
\begin{aligned}
K_t(q_t) &:= -A_t q_t^2 + (B_t\overline{Q}_t + C_t R_t + D_t v_{t-1} + F_t) q_t + W_t - P_t \overline{Q}_t \\
&+ \Big[G_{t+1} - \frac{1}{2\{1+2\gamma\zeta_t(\sigma_t^v)^2\}}\gamma\eta_t^2(\sigma_t^v)^2 - \frac{1}{2}\gamma(\sigma_t^\varepsilon)^2\Big]\overline{Q}_t^2 + \Big[H_{t+1} + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\eta_t a_t^v \\
&- \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\gamma\theta_t\phi_t(\sigma_t^v)^2 - \mu_t^\varepsilon\Big]\overline{Q}_t + \Big[(1-\mathrm{e}^{-\rho}) + \mathrm{e}^{-\rho} I_{t+1} - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\gamma\eta_t\theta_t(\sigma_t^v)^2\Big]\overline{Q}_t R_t \\
&+ \Big[\mathrm{e}^{-2\rho} J_{t+1} - \frac{1}{2\{1+2\gamma\zeta_t(\sigma_t^v)^2\}}\gamma\theta_t^2(\sigma_t^v)^2\Big] R_t^2 + \Big[\mathrm{e}^{-\rho} L_{t+1} + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\theta_t a_t^v \\
&- \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\gamma\theta_t\phi_t(\sigma_t^v)^2\Big] R_t - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\delta_t b_t^v \overline{Q}_t v_{t-1} - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\theta_t b_t^v R_t v_{t-1} \\
&+ \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\zeta_t (b_t^v)^2 v_{t-1}^2 - \Big[\frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\phi_t b_t^v + \frac{2}{1+2\gamma\zeta_t(\sigma_t^v)^2}\zeta_t a_t^v b_t^v\Big] v_{t-1} \\
&+ \Big[Z_{t+1} + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\phi_t a_t^v - \frac{1}{2\{1+2\gamma\zeta_t(\sigma_t^v)^2\}}\gamma\phi_t^2(\sigma_t^v)^2 + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\zeta_t (a_t^v)^2 + x_t\Big],
\end{aligned}
\tag{A.13}
$$

which yields the optimal execution volume q_t^* at time $t \in \{T-2, \ldots, 1\}$:

$$
q_t^* \left(:= f(s_t) = \frac{B_t\overline{Q}_t + C_t R_t + D_t v_{t-1} + F_t}{A_t} \right) = a_t + b_t \overline{Q}_t + c_t R_t + d_t v_{t-1}, \tag{A.14}
$$

where

$$
a_t := \frac{F_t}{A_t}; \quad b_t := \frac{B_t}{A_t}; \quad c_t := \frac{C_t}{A_t}; \quad d_t := \frac{D_t}{A_t}. \tag{A.15}
$$

Then, by substituting this into Eq. (A.10), the optimal value function at time $t \in \{T-2, \ldots, 1\}$ has a functional form as follows:

$$
\begin{aligned}
V_t[s_t] &= -\exp\Big\{ -\gamma\Big[W_t - P_t\overline{Q}_t \\
&\quad + \Big[G_{t+1} - \frac{1}{2\{1+2\gamma\zeta_t(\sigma_t^v)^2\}}\gamma\eta_t^2(\sigma_t^v)^2 - \frac{1}{2}\gamma(\sigma_t^\varepsilon)^2\Big]\overline{Q}_t^2 + \Big[H_{t+1} + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\eta_t a_t^v \\
&\quad - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\gamma\theta_t\phi_t(\sigma_t^v)^2 - \mu_t^\varepsilon\Big]\overline{Q}_t + \Big[(1-\mathrm{e}^{-\rho}) + \mathrm{e}^{-\rho} I_{t+1} - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\gamma\eta_t\theta_t(\sigma_t^v)^2\Big]\overline{Q}_t R_t \\
&\quad + \Big[\mathrm{e}^{-2\rho} J_{t+1} - \frac{1}{2\{1+2\gamma\zeta_t(\sigma_t^v)^2\}}\gamma\theta_t^2(\sigma_t^v)^2\Big] R_t^2 + \Big[\mathrm{e}^{-\rho} L_{t+1} + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\theta_t a_t^v \\
&\quad - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\gamma\theta_t\phi_t(\sigma_t^v)^2\Big] R_t - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\delta_t b_t^v \overline{Q}_t v_{t-1} - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\theta_t b_t^v R_t v_{t-1} \\
&\quad + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\zeta_t (b_t^v)^2 v_{t-1}^2 - \Big[\frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\phi_t b_t^v + \frac{2}{1+2\gamma\zeta_t(\sigma_t^v)^2}\zeta_t a_t^v b_t^v\Big] v_{t-1} \\
&\quad + \Big[Z_{t+1} + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\phi_t a_t^v - \frac{1}{2\{1+2\gamma\zeta_t(\sigma_t^v)^2\}}\gamma\phi_t^2(\sigma_t^v)^2 + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\zeta_t (a_t^v)^2 \\
&\quad + x_t + \frac{(B_t\overline{Q}_t + C_t r_t + D_t v_{t-1} + F_t)^2}{4A_t}\Big]\Big]\Big\} \\
&= -\exp\Big\{ -\gamma\Big[W_t - P_t\overline{Q}_t + G_t\overline{Q}_t^2 + H_t\overline{Q}_t + I_t\overline{Q}_t R_t + J_t R_t^2 + L_t R_t \\
&\quad + M_t\overline{Q}_t v_{t-1} + N_t R_t v_{t-1} + X_t v_{t-1}^2 + Y_t v_{t-1} + Z_t\Big]\Big\},
\end{aligned}
\tag{A.16}
$$

where

$$
\begin{aligned}
G_t &:= G_{t+1} - \frac{1}{2\{1+2\gamma\zeta_t(\sigma_t^v)^2\}}\gamma\eta_t^2(\sigma_t^v)^2 - \frac{1}{2}\gamma(\sigma_t^\varepsilon)^2 + \frac{B_t^2}{4A_t};\\
H_t &:= H_{t+1} + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\eta_t a_t^v - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\gamma\eta_t\phi_t(\sigma_t^v)^2 - \mu_t^\varepsilon + \frac{B_tF_t}{2A_t};\\
I_t &:= (1-\mathrm{e}^{-\rho}) + \mathrm{e}^{-\rho}I_{t+1} - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\gamma\eta_t\theta_t(\sigma_t^v)^2 + \frac{B_tC_t}{2A_t};\\
J_t &:= \mathrm{e}^{-2\rho}J_{t+1} - \frac{1}{2\{1+2\gamma\zeta_t(\sigma_t^v)^2\}}\gamma\theta_t^2(\sigma_t^v)^2 + \frac{C_t^2}{4A_t};\\
L_t &:= \mathrm{e}^{-\rho}L_{t+1} + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\theta_t a_t^v - \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\gamma\theta_t\phi_t(\sigma_t^v)^2 + \frac{C_tF_t}{2A_t};\\
M_t &:= -\frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\delta_t b_t^v + \frac{B_tD_t}{2A_t};\\
N_t &:= -\frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\theta_t b_t^v + \frac{C_tD_t}{2A_t};\\
X_t &:= \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\zeta_t(b_t^v)^2 + \frac{D_t^2}{4A_t};\\
Y_t &:= -\frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\phi_t b_t^v - \frac{2}{1+2\gamma\zeta_t(\sigma_t^v)^2}\zeta_t a_t^v b_t^v + \frac{D_tF_t}{2A_t};\\
Z_t &:= Z_{t+1} + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\phi_t a_t^v - \frac{1}{2\{1+2\gamma\zeta_t(\sigma_t^v)^2\}}\gamma\phi_t^2(\sigma_t^v)^2 + \frac{1}{1+2\gamma\zeta_t(\sigma_t^v)^2}\zeta_t(a_t^v)^2\\
&\quad + x_t + \frac{F_t^2}{4A_t}.
\end{aligned}
\tag{A.17}
$$

□