Risk-Sensitive Expectation and Coherent Risk Measures Derived from Utility Functions

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1. Introduction

Risk-sensitive expectation is given by

$$f^{-1}(E(f(\cdot))), \tag{1}$$

where f and f^{-1} are decision maker's utility function and its inverse function and $E(\cdot)$ is an expectation (Howard and Matheson [3]). Eq. (1) estimates risky events through utility functions. *Coherent risk measures* have been studied to improve the criterion of risks with worst scenarios (Artzner et al. [2]): For example, conditional value-at-risks, expected shortfall (Rockafellar and Uryasev [5], Tasche [6]). Kusuoka [4] gave a spectral representation for coherent risk measures. Further Yoshida [7] has introduced a *spectral weighted average value-at-risk* as the best coherent risk measure derived from decision maker's utility functions. This paper discusses risk-sensitive decision making, which will be useful for artificial intelligence's quick and responsible reasoning, based on the concepts of Yoshida [7, 10] and presentation documents in RIMS 2019.

2. Coherent risk measure derived from risk averse utility

- Let P be a non-atomic probability on a sample space Ω.
- We deal with the following *random variables*:

$$\mathcal{X} = \left\{ X : \Omega \mapsto (-\infty, \infty) \;\middle|\; \begin{array}{l} X \text{ has a continuous distribution function} \\ x \mapsto F_X(x) = P(X < x) \text{ and there exists} \\ \text{an open interval } I(\neq \emptyset) \text{ such that} \\ F_X : I \mapsto (0,1) \text{ is strictly increasing and onto} \end{array} \right\}.$$

- *Value-at-risk* at a probability $p(\in (0,1])$ is given by the percentile of the distribution F_X, i.e.

$$\mathrm{VaR}_p(X) = \sup\{x \in I \mid F_X(x) \le p\} = F_X^{-1}(p) \tag{2}$$

for $p \in (0,1)$ and $\mathrm{VaR}_1(X) = \sup I$, where F_X^{-1} is the inverse function of F_X.

- *Average value-at-risk* at a probability $p(\in (0,1])$ is given by

$$\mathrm{AVaR}_p(X) = \frac{1}{p}\int_0^p \mathrm{VaR}_q(X)\, dq. \tag{3}$$

Definition 1 (Artzner at al. [2]). A map $\rho : \mathcal{X} \mapsto (-\infty, \infty)$ is called a *coherent risk measure* if it satisfies the following (i) – (iv):

(i) $\rho(X) \geq \rho(Y)$ for $X, Y \in \mathcal{X}$ satisfying $X \leq Y$. (*monotonicity*)

(ii) $\rho(cX) = c\rho(X)$ for $X \in \mathcal{X}$ and $c \in (0, \infty)$. (*positive homogeneity*)

(iii) $\rho(X + c) = \rho(X) - c$ for $X \in \mathcal{X}$ and $c \in (-\infty, \infty)$. (*translation invariance*)

(iv) $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for $X, Y \in \mathcal{X}$. (*sub-additivity*)

- In this paper we use a law invariant, comonotonically additive, continuous coherent risk measure ρ.

- For a probability $p(\in (0,1])$ and a non-increasing right-continuous function $\lambda : [0,1] \mapsto [0,\infty)$ satisfying $\int_0^1 \lambda(q)\, dq = 1$, we define *a weighted average value-at-risk* with weighting λ on $(0,p)$ by

$$\mathrm{AVaR}_p^\lambda(X) = \int_0^p \mathrm{VaR}_q(X)\, \lambda(q)\, dq \Big/ \int_0^p \lambda(q)\, dq. \tag{4}$$

  Then λ is called a *risk spectrum*.

Lemma 1 (Kusuoka [4], Yoshida [7]). *Let $\rho : \mathcal{X} \mapsto (-\infty, \infty)$ be a law invariant, comonotonically additive, continuous coherent risk measure. Then there exists a risk spectrum λ such that*

$$\rho(X) = -\mathrm{AVaR}_1^\lambda(X) \tag{5}$$

for $X \in \mathcal{X}$. Further, $-\mathrm{AVaR}_p^\lambda$ is a coherent risk measure on $\mathcal{X}$ for $p \in (0,1)$.

- For the family $\mathcal{X}$, we assume the following (i) and (ii):

  (i) There exists a strictly increasing function $\kappa : (0,1) \mapsto (-\infty, \infty)$ such that

$$\mathrm{VaR}_p(X) = \mu + \kappa(p)\, \sigma, \quad p \in (0,1] \tag{6}$$

  for the means μ and the standard deviations σ of random variables $X \in \mathcal{X}$.

  (ii) There exists a probability density function

$$\psi : (\mu, \sigma)(\in (-\infty, \infty) \times [0, \infty)) \mapsto [0, \infty)$$

  for the means μ and the standard deviations σ of random variables $X \in \mathcal{X}$.

- From (4) and (6) we have

$$\mathrm{AVaR}_p^\lambda(X) = \mu + \kappa^\lambda(p)\,\sigma, \tag{7}$$

where

$$\kappa^\lambda(p) = \int_0^p \kappa(q)\,\lambda(q)\,dq \Big/ \int_0^p \lambda(q)\,dq.$$

- Let $f : I \mapsto (-\infty, \infty)$ be a C^2-class *risk averse utility function* satisfying $f' > 0$ and $f'' \leq 0$ on I, where I is an open interval.

Lemma 2 (Yoshida [7]). *A risk spectrum λ which minimizes the distance between the non-linear risk-sensitive form and weighted average value-at-risk (4):*

$$\sum_{X \in \mathcal{X}} \left(f^{-1}\left(\frac{1}{p} \int_0^p f(\mathrm{VaR}_q(X))\,dq \right) - \mathrm{AVaR}_p^\lambda(X) \right)^2 \tag{8}$$

for $p \in (0, 1]$ is given by

$$\lambda(p) = e^{-\int_p^1 C(q)\,dq} C(p), \qquad p \in (0, 1] \tag{9}$$

with a component function C in [7, Theorem 2] if λ is non-increasing,

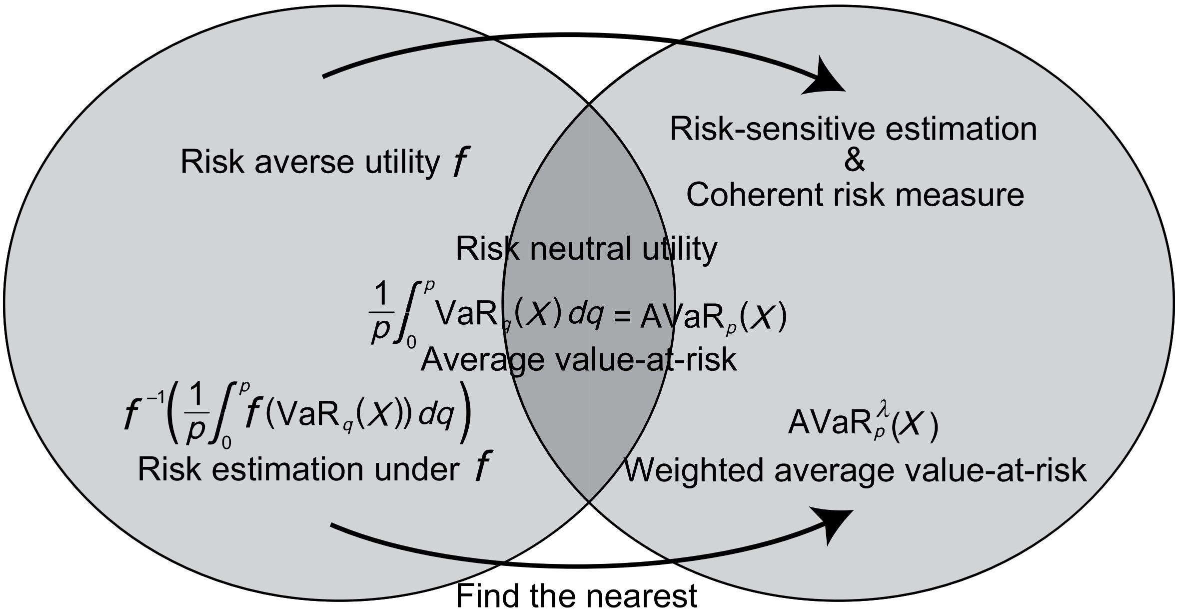

Fig. 1. Risk-sensitive estimation and coherent risk measures derived from risk averse utility f.

Remark. Regarding Eq, (8),

- $f^{-1}\left(\frac{1}{p}\int_0^p f(\mathrm{VaR}_q(X))\,dq\right)$ is the *risk-sensitive estimation of* X *through utility* f.
- $-\mathrm{AVaR}_p^\lambda(\cdot)$ is a *coherent risk measure* with risk spectrum λ.
- $\mathrm{AVaR}_p^\lambda(X)$ is the *weighted average value-at-risk* such that
  - $\mathrm{AVaR}_p^\lambda(X)$ can *inherit decision maker's risk averse sense of utility* f, using *risk spectrum* λ as a weight on $(0,p)$.
  - $\mathrm{AVaR}_p^\lambda(X)$ has *a kind of linear properties* like *positively homogeneity* and *translation invariance* in Definition 1(ii)(iii).

Example 1. Let a domain $I=(-\infty,\infty)$ and let f be a *risk neutral function*

$$f(x)=a\,x+b$$

for $x\in(-\infty,\infty)$ with constants $a(>0)$ and $b(\in(-\infty,\infty))$.

- Its optimal risk spectrum in Lemma 2 is $\lambda(p)=1$ with $C(p)=\frac{1}{p}$.
- The corresponding weighted average value-at-risk (4) is reduced to the *average value-at-risk* (3):

$$\mathrm{AVaR}_p^\lambda(X)=\mathrm{AVaR}_p(X)=\frac{1}{p}\int_0^p \mathrm{VaR}_q(X)\,dq \quad \text{and} \quad \mathrm{AVaR}_1(X)=E(X)$$

  for $X\in\mathcal{X}$ and $p\in(0,1]$.

Example 2. Let a domain $I=(-\infty,\infty)$ and let a *risk averse exponential utility function*

$$f(x)=\frac{1-e^{-\tau x}}{\tau}$$

for $x\in(-\infty,\infty)$ with a constant $\tau(>0)$.

- $-\frac{f''}{f'}=\tau$ is *Arrow's absolute risk averse index* (Aroow [1]).
- Its optimal risk spectrum in Lemma 2 is given by

$$\lambda(p)=e^{-\int_p^1 C(q)\,dq}C(p), \qquad p\in(0,1],$$

  where the component function C is given by

$$C(p)=\frac{1}{p}\cdot\frac{\displaystyle\int_0^\infty\left(1-\frac{1}{\frac{1}{p}\int_0^p e^{\tau\sigma(\kappa(p)-\kappa(q))}\,dq}\right)\sigma^n e^{-\frac{\sigma^2}{2}}\,d\sigma}{\displaystyle\int_0^\infty \log\left(\frac{1}{p}\int_0^p e^{\tau\sigma(\kappa(p)-\kappa(q))}\,dq\right)\sigma^n e^{-\frac{\sigma^2}{2}}\,d\sigma}.$$

Let $\mathcal{X}$ be a family of random variables X which have a *normal distribution* with a density function

$$w(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

for $x \in (-\infty, \infty)$, where μ and σ are the mean and standard deviation of random variables $X \in \mathcal{X}$

- Define an increasing function $\kappa : (0,1) \mapsto (-\infty, \infty)$ by an inverse function

  $$\kappa(p) = G^{-1}(p)$$

  for $p \in (0,1)$, where G is the cumulative distribution function of the *standard normal distribution*

  $$G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}}\, dz$$

  $(x \in (-\infty, \infty))$.

- Then we have value-at-risk

  $$\mathrm{VaR}_p(X) = \mu + \kappa(p)\, \sigma$$

  for $X \in \mathcal{X}$.

Suppose $\mathcal{X}$ has a distribution function ψ:

$$\psi(\mu, \sigma) = \phi(\mu) \cdot \frac{2^{1-n/2}}{\Gamma(n/2)}\, \sigma^{n-1} e^{-\frac{\sigma^2}{2}}$$

for $(\mu, \sigma) \in (-\infty, \infty) \times [0, \infty)$, where $\phi(\mu)$ is some probability distribution and $\frac{2^{1-n/2}}{\Gamma(n/2)}\, \sigma^{n-1} e^{-\frac{\sigma^2}{2}}$ is a *chi distribution* with degree of freedom n. Then we have Figs. 2-4.

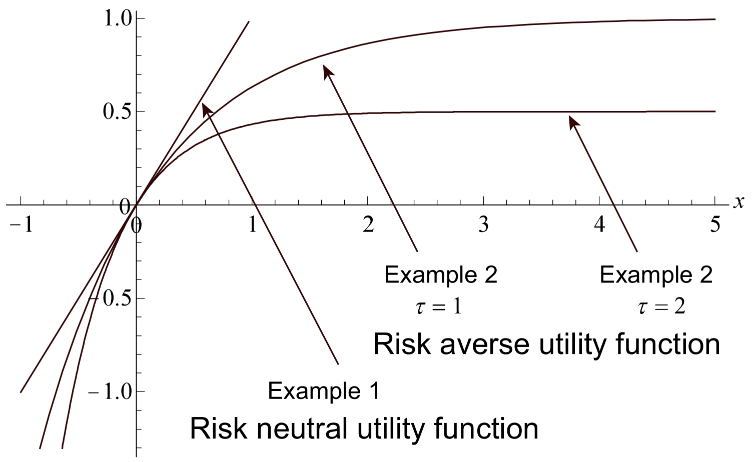

Fig. 2. Utility functions $f(x)$.

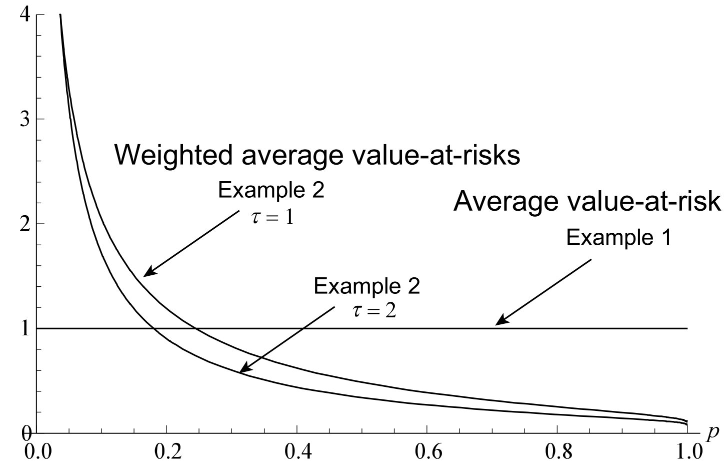

Fig. 3. Risk spectra $\lambda(p)$.

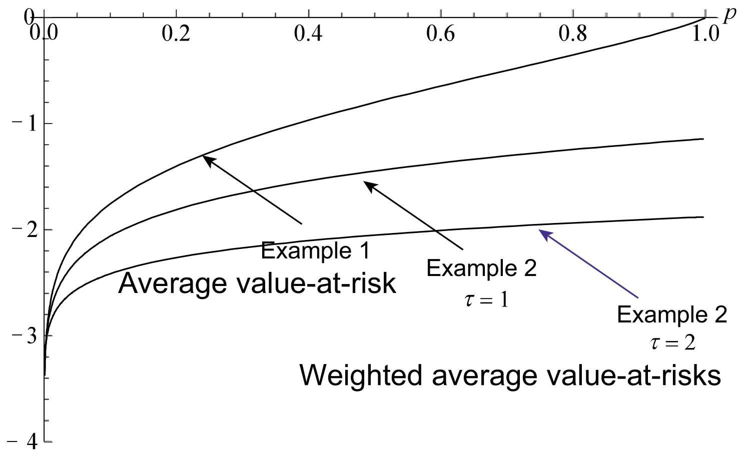

Fig. 4. Functions $\kappa^{\lambda}(p)$.

3. Risk-sensitive decision making with risk constraints

Let ρ be a coherent risk measure in Lemma 1 and let f be a C^2-class risk averse utility functions in the previous section. Let δ be a positive constant. Then we investigate the following problem.

Problem 1. Maximize the risk-sensitive expected reward

$$f^{-1}(E(f(X^{\pi}))) \tag{10}$$

with respect to strategies π under a risk constraint

$$\rho(X^{\pi}) \leq \delta. \tag{11}$$

Hence we estimate the downside risks on $(0, p)$. From Lemmas 1 and 2, there exist risk spectra λ and ν such that

$$f^{-1}(E(f(\cdot))) = f^{-1}\left(\int_0^1 \mathrm{VaR}_q(f(\cdot))\,dq\right) = f^{-1}\left(\int_0^1 f(\mathrm{VaR}_q(\cdot))\,dq\right) \approx \mathrm{AVaR}_1^{\lambda}(\cdot),$$

$$\rho(\cdot) = -\mathrm{AVaR}_p^{\nu}(\cdot).$$

Thus we discuss the following optimization instead of Problem 1.

Problem 2 Maximize weighted average value-at-risks

$$\mathrm{AVaR}_1^{\lambda}(X^{\pi}) = E(X^{\pi}) + \kappa^{\lambda}(1)\cdot\sigma(X^{\pi}) \tag{12}$$

with respect to strategies π under risk constraints

$$\mathrm{AVaR}_p^{\nu}(X^{\pi}) = E(X^{\pi}) + \kappa^{\nu}(p)\cdot\sigma(X^{\pi}) \geq -\delta. \tag{13}$$

- Problem 2 is easier to solve in actual cases than Problem 1 because we calculate only $E(X^{\pi})$ and $\sigma(X^{\pi})$ when we have prepared constants $\kappa^{\lambda}(1)$ and $\kappa^{\nu}(p)$.

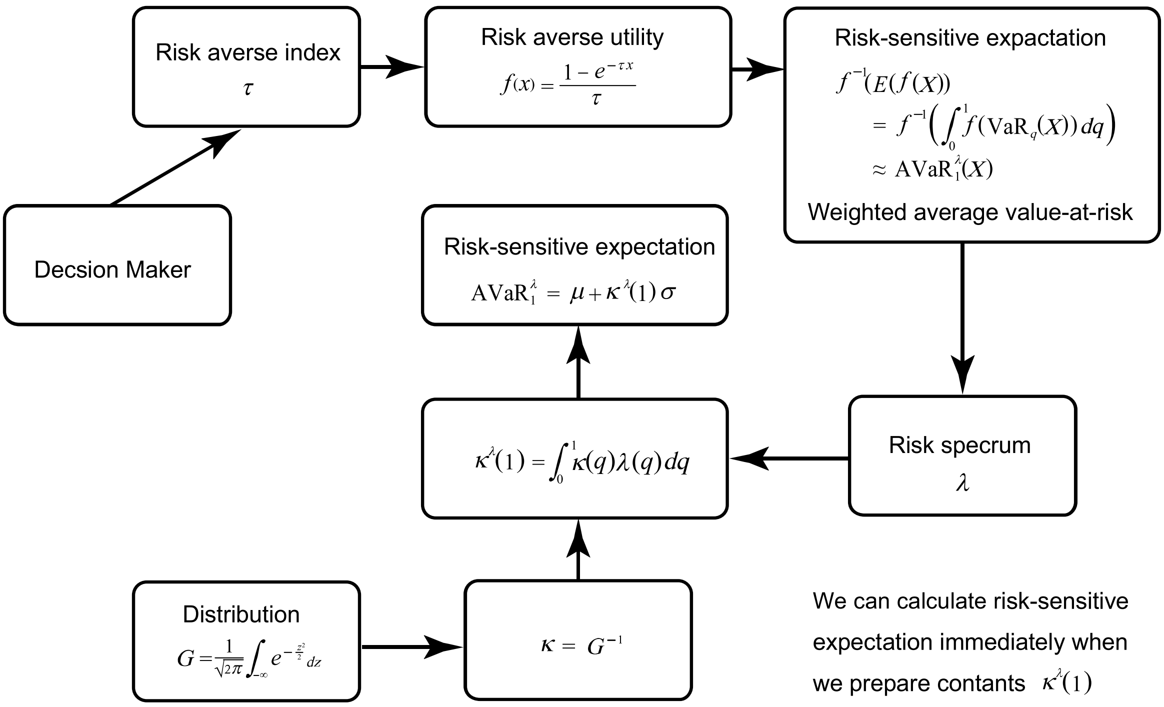

Fig. 5. Risk-sensitive estimation under utility function f.

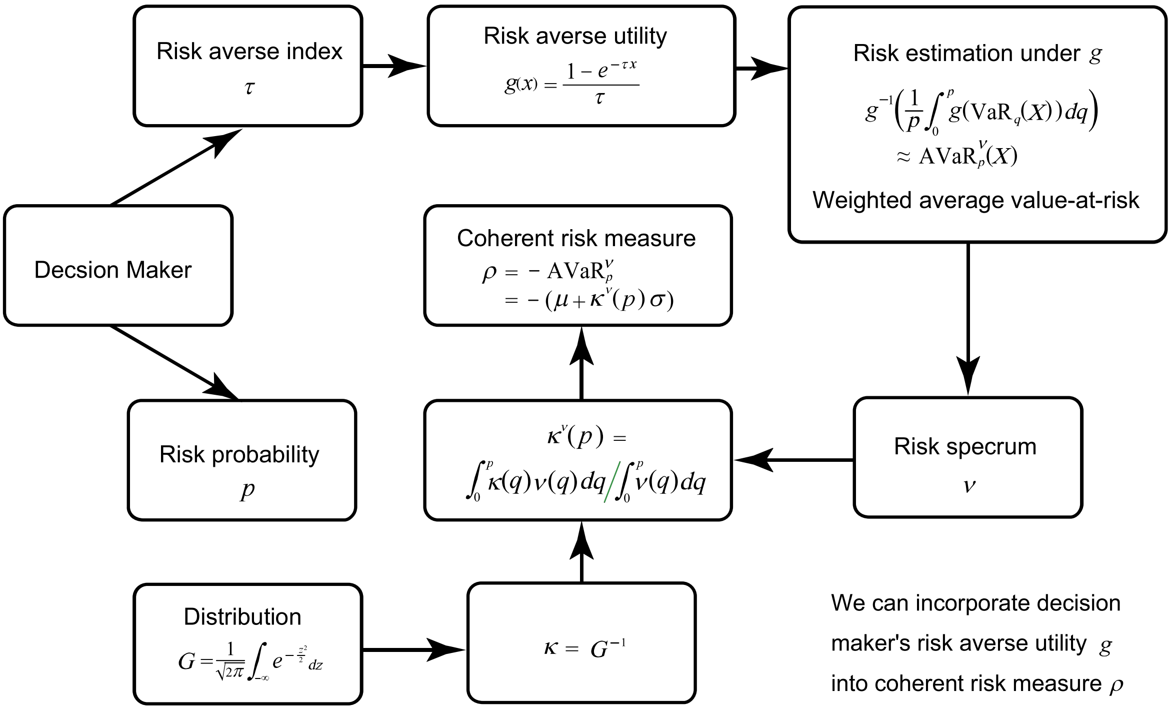

Fig. 6. Coherent risk measure under utility function f.

Using Lemma 2, we can incorporate the decision maker's risk averse attitude into coherent risk measures as weighting for average value-at-risks. As we have seen in Example 2, risk-sensitive estimations are approximated by weighted average risks with the best spectrum λ for with utility f, and the coherent risk measures ρ is also given by weighted average risks with the best spectrum ν for with utility g in the same manner. If we prepare constants $\kappa^\lambda(1)$ and $\kappa^\nu(p)$ once from κ, λ and ν like Figs. 5 and 6, we can calculate risk-sensitive estimation φ and coherent risk values ρ immediately respectively. This kind of quick risk-sensitive decision making will be applicable to reasonable and high-speed computing with artificial intelligence reasoning, for example, stock trading, auto driving and so on.

4. Application to decision making

Yoshida [7] has introduced a *spectral weighted average value-at-risk* as the best coherent risk measure derived from decision maker's utility functions. Using this derived coherent risk measure, In dynamic Markov decision models, Yoshida [9] has discussed risk-sensitive running rewards by dynamic programming, and Yoshida [10] has investigated risk-sensitive terminal rewards by multi-parameter optimization, Yoshida [8] has developed their availability in high-speed computing. Yoshida [11, 12] has also applied it to portfolio selection in finance.

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