Rotationally Symmetric Relations of Standard Normal Distribution Using Right Triangles, Circles, and Squares

- Ordinary Differential Equations, Pythagorean Theorem, Equilateral Triangles, and Golden Ratio -

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1. Introduction

We have confirmed various geometric characterizations of a standard normal distribution in this paper. Until now, we found the two types of differential equations about that and these inverse Mills ratio [1] in 2017. However, we have also noticed one of them such as a second order linear differential equation was misspelled in the paper, RIMS Kôkyûroku 2078-10 [1], since November 2018 [2,3]. Therefore, to search for our correct mathematical and scientific proceedings, we would like to inform the readers of these things sincerely and our new added ideas [4] included our latest works in 2019 [5-9] in this paper.

By the way, we are significantly interested in the following formulation about the twenty-seven percent rule. That is,

$$\phi(\lambda) = 2\lambda\Phi(-\lambda), \quad \because \lambda = 0.612003, \tag{1.1}$$

where $\phi(\cdot)$ means a probability density function of standard normal distribution and $\Phi(\cdot)$ is its cumulative distribution function. Karl Pearson found a probability point $\lambda = 0.612003$ about one hundred years ago [10]. Kelley [11,12], Mosteller [13], Cox [14] and many other researchers [15-22] also dealt with this probability point λ. We can also have proposed our other previous approaches using λ [23-28].

In addition to these contributions, the first author got four important advices by Prof. Hidemasa Yoshimura, Prof. Yuki Inoue, Prof. Koji Kamejima at Osaka Institute of Technology, and some other researchers. Prof. Yoshimura gave us the beauty of small difference from viewpoints of design about our work [1]. From his advice, we can have the opportunity to investigate some tendencies around the probability point $\lambda = 0.612003$. Prof. Inoue pointed out a similar tendency of increasing curves of inverse Mills ratio about our latest work [9]. After that, we happened to have the equivalent tendencies when we tried to search for other tendencies. Prof. Kamejima taught that several folds of probability points of standard normal distribution should be interesting [1]. To show rotationally symmetric relations, the folds of probability points play a significantly important role to solve the keys of geometric characterizations. From other researchers, we are questioned if our proposed differential equations [8] are related to some laws of physics. We guess that the question might be useful and a meaningful comment if we think of several logistic movements and these diagrams with some accelerations.

In this paper, we have confirmed several characterizations based on rotationally symmetric relations with above mentioned viewpoints.

First, we reconfirm the geometric characterizations between second order differential equations of integral forms of cumulative distributions [1-5,28-36], misspelled differential equations [1-4,35], and Bernoulli differential equations [37-39] of inverse Mills ratios [40-45] with reflections, rotations, and superpositions in section 2. At the same time, we explain that a second derivative should play a vital role of acceleration which are related to physics simply.

Second, we show that a mandala shape [9,46,47] by standard normal distributions and inverse Mills ratios geometrically by using Greek Pythagorean theorem [48-53] and ancient Egyptian drawing styles [54] with our modified intercept forms about our previous work [6] from the viewpoints of symmetry [55] in the same section 2. Especially, since we are interested in the history of mathematics and sciences by Stuart's textbooks [51-53] and these symmetries [55], we would like to synthesis standard normal distribution, inverse Mills ratio, Pythagorean theorem, several curves about differential equations, and ancient Egyptian drawings as a unique work mathematically and historically [56,57].

Third, in section 3, we deal with rotationally symmetric relations of standard normal distribution by the probability point $\lambda = 0.612003$ and other points k. We also show the parametric equations and the optimal values with proportions $\Phi(k)$: $\Phi(-k)$ on the probability point k described as some illustrations. Moreover, we would like to show that several types of parametric equations both second order linear differential equations and Bernoulli differential equations are related to the folds of probability density.

Fourth, in section 4, we would like to introduce two mathematically beautiful harmonies. One is that a probability point about equilateral triangles or a regular hexagon are tied to that of the right triangles using a concept of the geometric mean which is related to $\phi(k)$ and $\phi(k) + 2k\Phi(k)$. We guess that this should be really like the relation between $\phi(k)$ and $2k\Phi(-k)$ $(\because k = \lambda)$. Therefore, we have searched for tendencies of several interested probability points and confirmed the geometric characterizations. The other is that the equation $\Phi(k)(1 + \Phi(k)^2)^{1/2} = 1$ might have a geometrically important probability point. We guess that the tendencies should have the possibilities of various spirals. Furthermore, we also reconsider the similar tendencies from European and Oriental historical cultures about the relations close to theology between circles and squares throughout our mistakes to modifications.

In section 5, we deal with some geometric relations of two-dimensional correlated standard normal distribution about regression analysis and principal component analysis synthetically.

2. Geometric Superpositions and Intercept Forms about Integrals of Standard Normal Distribution

If we think of a differential equation and various initial conditions as follows. That is

$$h''(u) + uh'(u) - h(u) = 0, \tag{2.1}$$
$$h(u) = C_{01}\big(\phi(u) + u\Phi(u)\big) + C_{02}u + C_{03},$$

where C_{01}, C_{02}, and C_{03} are constants. $\phi(\cdot)$ is a probability density function of standard normal distribution and $\Phi(\cdot)$ is its cumulative distribution function. Some examples of them are shown such as

$$h(u) = C_1 h_P(u) + C_2 h_N(u) + C_3, \tag{2.2}$$
$$h_P(u) = \phi(u) + u\Phi(u), \quad h_P'(u) = \Phi(u), \quad h_P''(u) = \phi(u), \quad (\text{Case: } C_{01} = 1, C_{02} = 0, C_{03} = 0),$$
$$h_N(u) = \phi(u) - u\Phi(-u), \quad h_N'(u) = -\Phi(-u), \quad h_N''(u) = \phi(u). \quad (\text{Case: } C_{01} = 1, C_{02} = -1, C_{03} = 0).$$

This is how we can show you general solutions with other constants C_1, C_2, and $C_3(= 0)$ in the top chart of Figure 1.

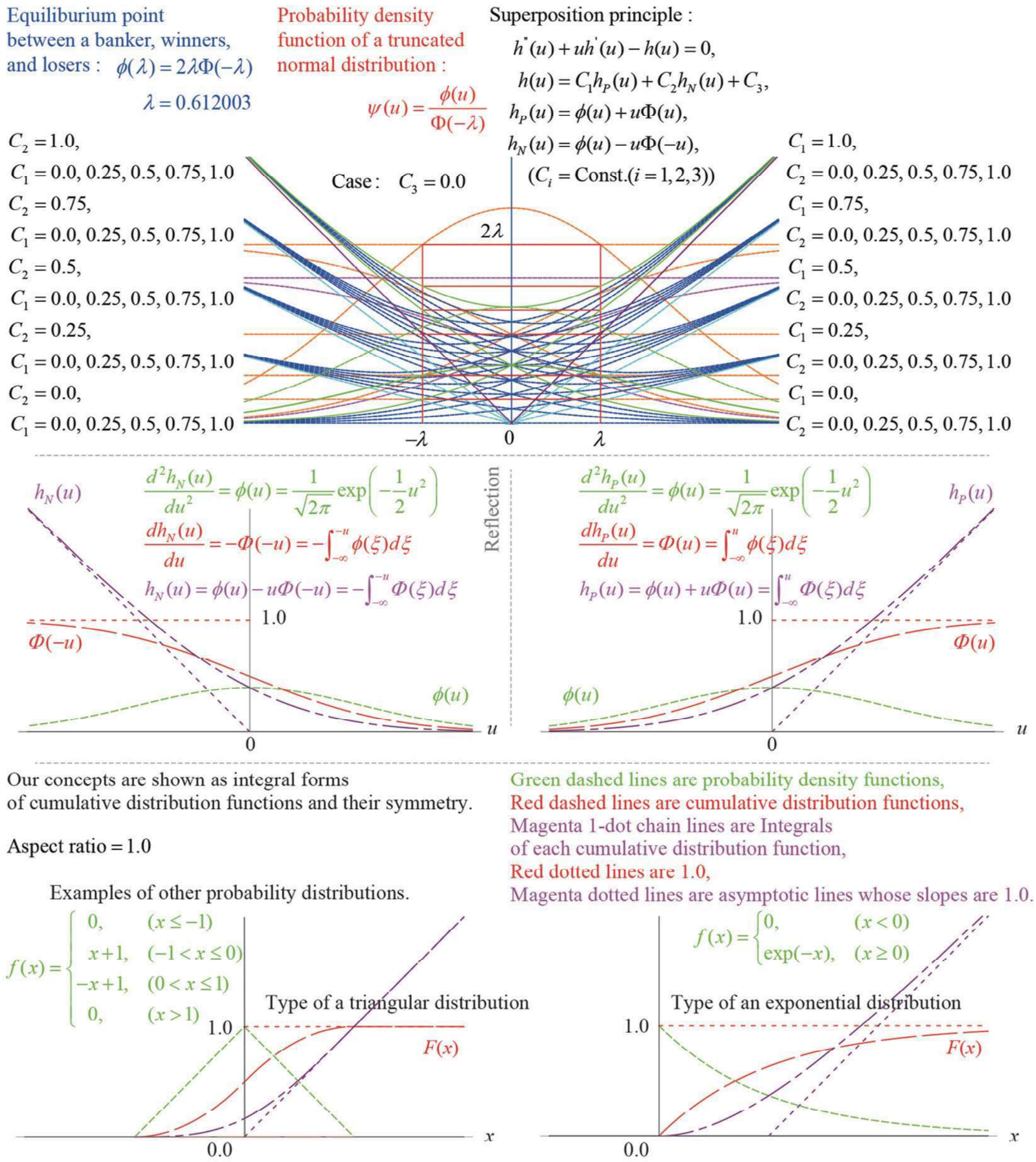

Figure 1 First Concepts of superposition principle and integral forms of cumulative distribution functions about these differential equations (Original References [1-4]).

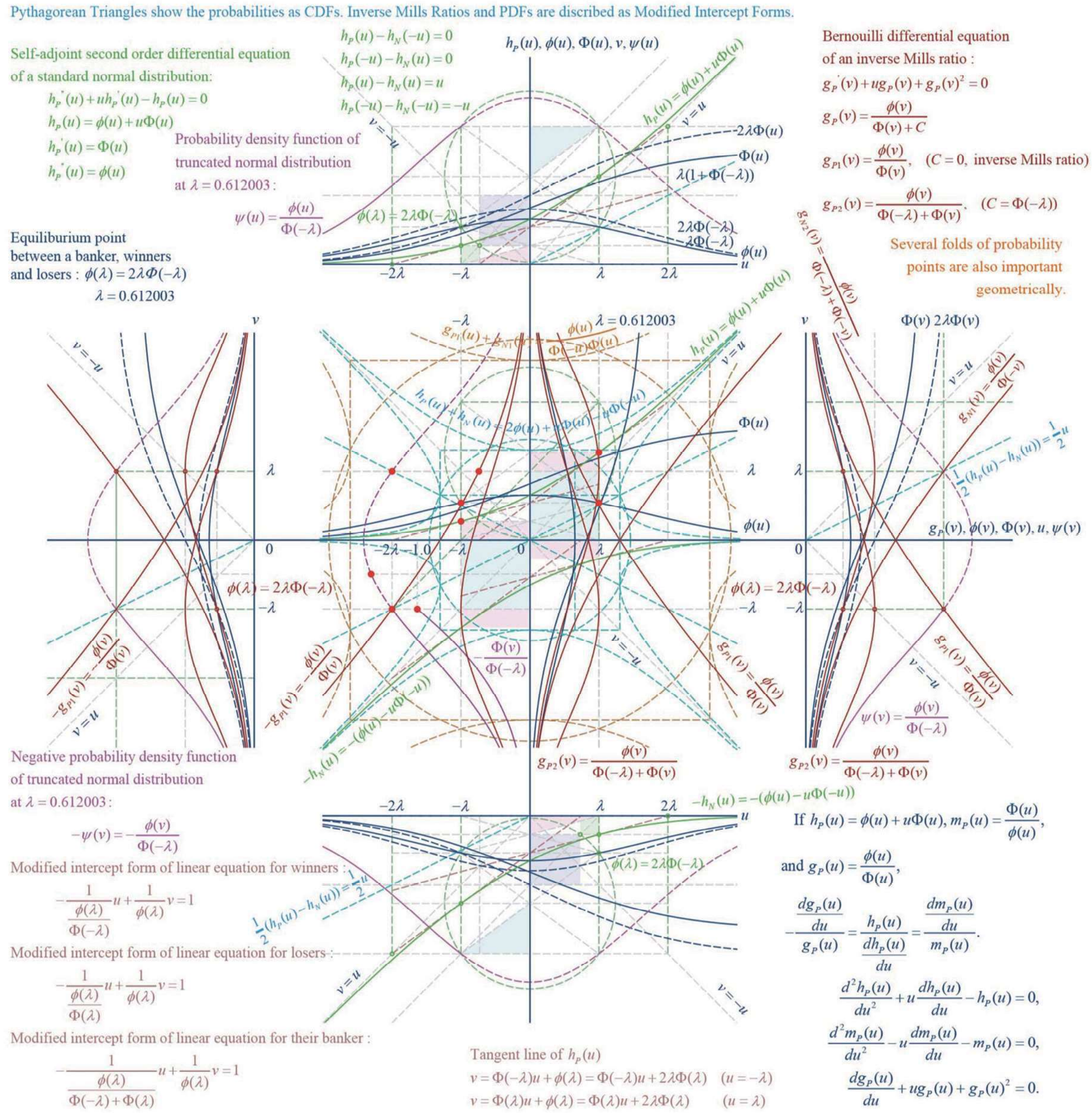

Figure 2 Second concepts using Egyptian drawing styles and Pythagorean theorem for solving the geometric relations between standard normal distribution and inverse Mills ratio based on the probability point $k = \lambda(= 0.612003)$ (Original References [6-9]).

Equations (2.1) and (2.2) bring us the visual and physical meanings coincidentally. If we consider u as time and $h(u)$ as position, Equation (2.1) shows the movements according to the second derivative $h''(u)$ as its acceleration and the first derivative $h'(u)$ as its velocity. Nevertheless, we must also admit that we should consider the meaning of time from $-\infty$ to ∞ based on the principle of physics concretely. Therefore, we think that normal distributions have the parameters both μ and σ^2 to apply various models in practice. However, it is beyond the authors' thoughts because we are not physicists. We can simply understand the only meaning of that a probability and probability density play an important role about the velocity and acceleration without considering uncertainties and shapes of probability theory. That is why we can illustrate that several other probability distributions in the bottom charts of Figure 1 to understand the thoughts.

By the way, we noticed misspelled differential equations about Equation (2.1) in the paper of RIMS2078-10 [1]. We would like to modify its equation to inform many researchers and readers of our correct message [2-4]. The misspelled equation was described as

$$m_P''(u) - u m_P'(u) - m_P(u) = 0, \qquad m_P(0) = \frac{\sqrt{2\pi}}{2}, \qquad m_P'(0) = 1.0. \tag{2.3}$$

From Equation (2.3), we can get the following solution as Mills ratio [2,4,6,35].

$$m_P(u) = \frac{\Phi(u)}{\phi(u)}. \tag{2.4}$$

About two types of the differential equations, we have also found the relations mathematically as follows [2-4,6]. First, an inverse of $\phi(u)$ multiplied by $h_P(u)$ in Equation (2.2) is rewritten as following formulations.

$$\frac{1}{\phi(u)}\left(\frac{d^2h_P(u)}{du^2} + u\frac{dh_P(u)}{du} - h_P(u)\right) = 0 \quad \text{or} \quad \left(\frac{h_P'(u)}{\phi(u)}\right)' - \left(\frac{h_P(u)}{\phi(u)}\right) = 0. \tag{2.5}$$

Second, we can also reconsider $\phi(u)$ multiplied by Equation (2.3) as

$$\phi(u)\left(\frac{d^2m_P(u)}{du^2} - u\frac{dm_P(u)}{du} - m_P(u)\right) = 0 \quad \text{or} \quad \left(\phi(u)m_P'(u)\right)' - \left(\phi(u)m_P(u)\right) = 0. \tag{2.6}$$

Therefore, we can clarify Equations (2.5) and (2.6) are self-adjoint differential equations [58] about standard normal distribution symmetrically. We can find the relations [6]

$$h_P(u) = \phi(u)m_P'(u) \quad \text{and} \quad h_P'(u) = \phi(u)m_P(u). \tag{2.7}$$

Moreover, we would like to consider inverse Mills ratio at the same time with ancient Egyptian drawing styles in Figure 2 geometrically [1,6,7]. The two types of Bernoulli differential equations of inverse Mills ratios along to the vertical axis v in Figure 2 are shown as

$$\frac{dg_P(v)}{dv} + vg_P(v) + g_P(v)^2 = 0 \text{ and } \frac{dg_N(v)}{dv} + vg_N(v) - g_N(v)^2 = 0. \tag{2.8}$$

These general solutions [1] are described as

$$g_P(u) = \frac{\phi(v)}{\Phi(v) + C_{gP}} \quad \text{or} \quad g_N(u) = \frac{\phi(v)}{\Phi(-v) + C_{gN}}. \tag{2.9}$$

In the same way, we can understand the general solutions of three types of differential equations are connected as a unique equation if the constants $C_{gP} = C_{gN} = 0$ in Equation (2.9). That is, from these characterizations, we can show that the mathematical relations about $h_P(u), m_P(u), and\ g_P(u)$ or $h_N(u), m_N(u), and\ g_N(u)$ are the following important formulations [6]

$$-\frac{g_P'(u)}{g_P(u)} = \frac{h_P(u)}{h_P'(u)} = \frac{m_P'(u)}{m_P(u)}, \quad \text{and} \quad -\frac{g_N'(u)}{g_N(u)} = \frac{h_N(u)}{h_N'(u)} = \frac{m_N'(u)}{m_N(u)}. \tag{2.10}$$

By the way, even though we understand the mathematically connected relations in Equations (2.10), we would like to investigate the other characterizations about that geometrically shown in Figure 2 [6,7]. one is a modified intercept form of linear equation for a negative probability point $-k(=-\lambda)$ in Figure 2 as

$$-\frac{1}{\frac{\phi(k)}{\Phi(-k)}}u + \frac{1}{\phi(k)}v = 1, \qquad \because v = \Phi(-k)u + \phi(k). \tag{2.11}$$

Another modified intercept form of linear equation for a positive probability point $k(=\lambda)$ in Figure 2 is also shown as

$$-\frac{1}{\frac{\phi(k)}{\Phi(k)}}u + \frac{1}{\phi(k)}v = 1, \qquad \because v = \Phi(k)u + \phi(k). \tag{2.12}$$

The other modified intercept form of linear equation for probabilities both above negative probability point $-k(=-\lambda)$ and positive probability point $k(=\lambda)$ in Figure 2 is shown as

$$-\frac{1}{\frac{\phi(k)}{\Phi(k) + \Phi(-k)}}u + \frac{1}{\phi(k)}v = 1 \text{ or } -\frac{1}{\phi(k)}u + \frac{1}{\phi(k)}v = 1. \tag{2.13}$$

We can clarify that these intercept forms from Equations (2.11) to (2.13) are composed of probability density function $\phi(k)$ at k and its inverse Mills ratio $\phi(k)/\Phi(k)$ geometrically.

3. Rotationally Symmetric Relations, Superpositions and Shears using Parametric Equations

We understand there should be the relations shown in Equations (2.11), (2.12) and (2.13) between $h_P(u)$ in Equation (2.2) and $g_P(v)$ in Equations (2.9). We can also find the superpositions about the combinations both $h_P(u)$ and $h_N(u)$ in Equations (2.2). If we think of the case of constants in Equation (2.2) such as $C_1 + C_2 = 1$ and $C_3 = 0$, we can estimate that $C_1 = \Phi(k)$, $C_2 = \Phi(-k)$ at the probability point k [7-9]. Therefore, we can show that weighted averages both $h_P(u)$ and

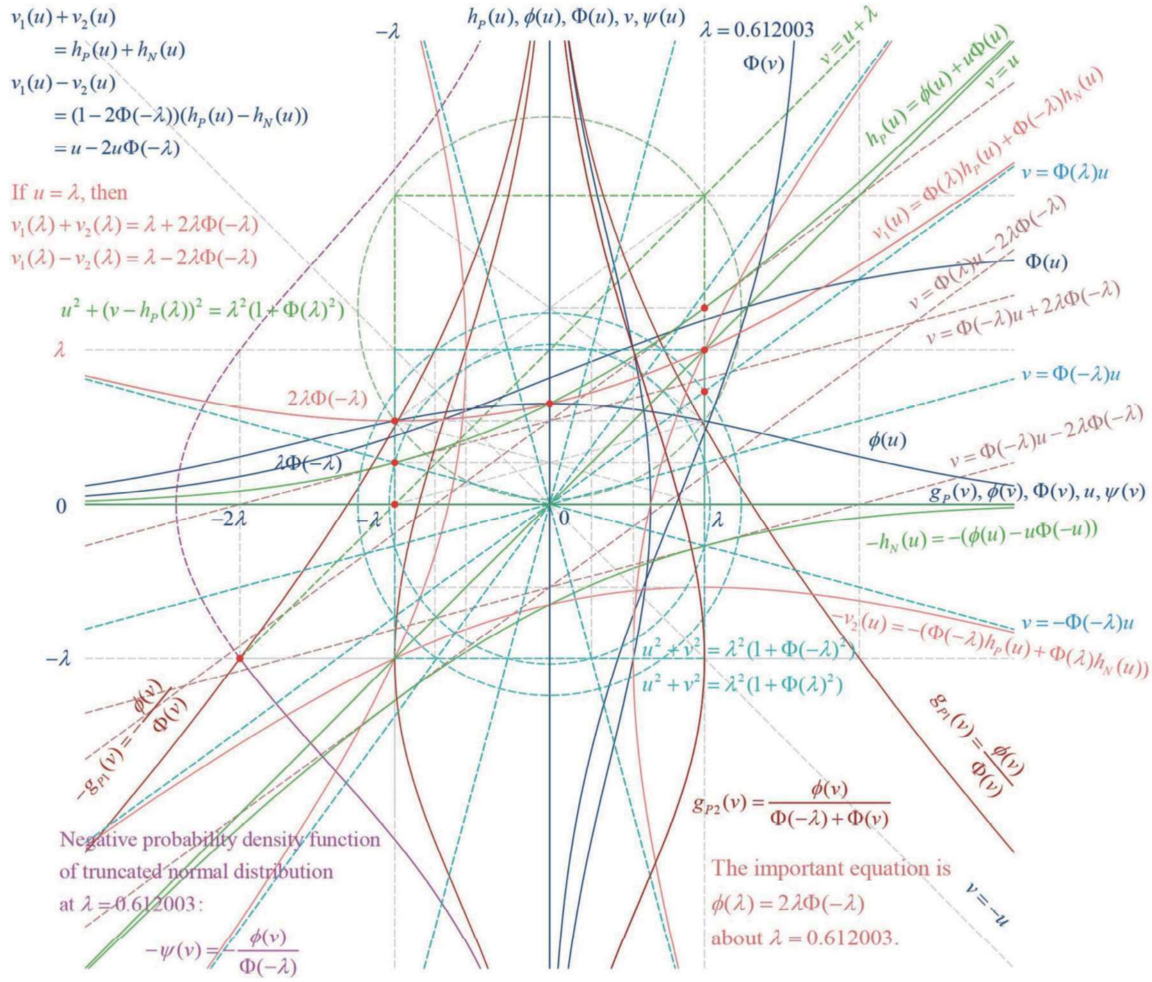

Figure 3 Rotationally symmetric relations of differential equations of standard normal distribution based on the probability point $k=\lambda(=0.612003)$ (Original references [7,9]).

$h_N(u)$ based on $\Phi(k)+\Phi(-k)=1$ and principle of superpositions [59] are illustrated as the special pink solid curves in Figure 3 [7-9]. These equations are shown as

$$v_1(u)=\Phi(k)h_P(u)+\Phi(-k)h_N(u), \tag{3.1}$$

$$v_2(u)=\Phi(-k)h_P(u)+\Phi(k)h_N(u).$$

If we search for the optimal values on the probability point $k=-\lambda$ or λ in Figure 3, we can solve the solutions [7] such as

$$\min v_1(u)=v_1(-\lambda)=\phi(\lambda)\big(=2\lambda\Phi(-\lambda)\big),\quad \because k=-\lambda, \tag{3.2}$$

$$\min v_2(u)=v_2(\lambda)=\phi(\lambda)\big(=2\lambda\Phi(-\lambda)\big),\quad \because k=\lambda.$$

Moreover, we understand that $\min v_1(k)=\phi(k)$ and $\min v_2(k)=\phi(k)$ about any real probability points symmetrically. Since we can get the geometrically special meanings in Equation (3.1), we would like to reconsider that such as following parametric equations. First, the green solid curves in the right column of Figure 4 are described by any probability points x [9] as

$$\begin{cases}u(x)=x,\\ v(x)=h_P(x),\end{cases}\text{ and }\begin{cases}u(x)=h_P(x)+h_N(x),\\ v(x)=h_P(x).\end{cases} \tag{3.3}$$

Second, the cyan solid curves by using parametric equations with shears [60] in the center column of Figures 4 are shown as

$$\begin{cases}u(x)=x,\\ v(x)=\Phi(k)h_P(x)+\Phi(-k)h_N(x),\end{cases}\text{ and }\begin{cases}u(x)=h_P(x)+h_N(x),\\ v(x)=\Phi(k)h_P(x)-\Phi(-k)h_N(x).\end{cases} \tag{3.4}$$

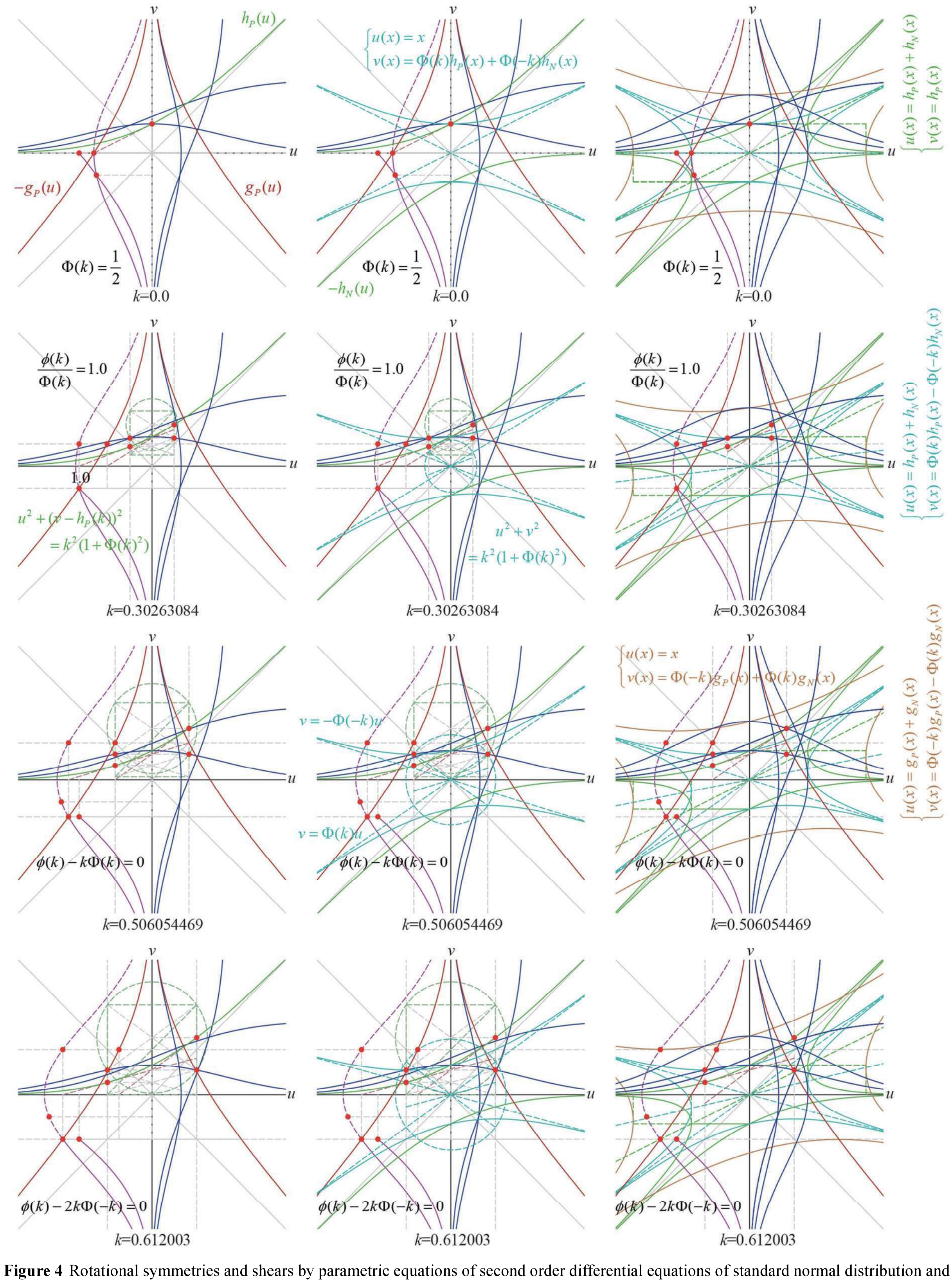

Figure 4 Rotational symmetries and shears by parametric equations of second order differential equations of standard normal distribution and Bernoulli differential equations of inverse Mills ratios (Original References [7-9]).

Third, the orange solid curves in the right columns of Figure 4 are also illustrated as

$$\begin{cases} u(x) = x, \\ v(x) = \Phi(-k)g_P(x) + \Phi(k)g_N(x), \end{cases} \text{ and } \begin{cases} u(x) = g_P(x) + g_N(x), \\ v(x) = \Phi(-k)g_P(x) - \Phi(k)g_N(x). \end{cases} \tag{3.5}$$

Especially, we can confirm the following relations

$$\Phi(-k)g_P(k) + \Phi(k)g_N(k) = 2\big(\Phi(k)h_P(k) + \Phi(-k)h_N(k)\big) \tag{3.6}$$

under the conditions of the two important circles in Figure 4 as

$$u^2 + \big(v - h_P(k)\big)^2 = k^2(1 + \Phi(k)^2) \text{ and } u^2 + v^2 = k^2(1 + \Phi(k)^2). \tag{3.7}$$

Therefore, Equation (3.7) of the two circles are connected to both Equations (3.3) and (3.4) if we think of the concept of Pythagorean theorem and without height such as ancient Egyptian drawing styles. That is

$$\begin{cases} u(x) = x, \\ v(x) = h_P(x)\big(= \Phi(\infty)h_P(x) + \Phi(-\infty)h_N(x)\big). \end{cases} \tag{3.8}$$

By the way, we would like to consider the constant $C_3(\neq 0)$ in Equation (2.2) shown in Figure 5 as two meaningful cases. One is $C_3 = -\phi(k)$ in the top parts of Figure 5. The parametric equation is shown as

$$\begin{cases} u(x) = x, \\ v(x) = \Phi(k)h_P(x) + \Phi(-k)h_N(x) - \phi(k). \end{cases} \tag{3.9}$$

The other case is $C_3 = -\phi(0)$ in the bottom parts of Figure 5. That is

$$\begin{cases} u(x) = x, \\ v(x) = \Phi(k)h_P(x) + \Phi(-k)h_N(x) - \phi(0). \end{cases} \tag{3.10}$$

Since the intersections between the upper green solid curve and the upper cyan solid curve in Equation (3.9) and in the top parts of Figure 5 are described at the probability point $x = -\phi(k)/\Phi(-k)$, we can show you the equivalent relation such as

$$h_P\left(-\frac{\phi(k)}{\Phi(k)}\right) = \Phi(k)h_P\left(-\frac{\phi(k)}{\Phi(k)}\right) + \Phi(-k)h_N\left(-\frac{\phi(k)}{\Phi(k)}\right) - \phi(k). \tag{3.11}$$

That is,

$$\begin{aligned} &\phi\left(-\frac{\phi(k)}{\Phi(k)}\right) - \frac{\phi(k)}{\Phi(k)}\Phi\left(\frac{\phi(k)}{\Phi(k)}\right) \\ &\quad = \Phi(k)\left(\phi\left(-\frac{\phi(k)}{\Phi(k)}\right) - \frac{\phi(k)}{\Phi(k)}\Phi\left(-\frac{\phi(k)}{\Phi(k)}\right)\right) \\ &\quad +\Phi(-k)\left(\phi\left(-\frac{\phi(k)}{\Phi(k)}\right) + \frac{\phi(k)}{\Phi(k)}\left(1 - \Phi\left(-\frac{\phi(k)}{\Phi(k)}\right)\right)\right) - \phi(k). \end{aligned} \tag{3.12}$$

The boundary condition [61] on $u = x$ in Equation (3.9) is shown as

$$\begin{cases} \Phi(k)h_P(x) + \Phi(-k)h_N(x) - \phi(k) = 0, \\ \Phi(k)h'_P(x) + \Phi(-k)h'_N(x) = 0. \end{cases} \tag{3.13}$$

About Equation (3.10), since intersections between the upper green solid curve and the upper cyan solid curve in the bottom parts of Figure 5 are described at the probability point $x = -\phi(0)/\Phi(-k)$, we can show you the equivalent relation such as

$$h_P\left(-\frac{\phi(0)}{\Phi(k)}\right) = \Phi(k)h_P\left(-\frac{\phi(0)}{\Phi(k)}\right) + \Phi(-k)h_N\left(-\frac{\phi(0)}{\Phi(k)}\right) - \phi(0). \tag{3.14}$$

That is,

$$\begin{aligned} &\phi\left(-\frac{\phi(0)}{\Phi(k)}\right) - \frac{\phi(0)}{\Phi(k)}\Phi\left(-\frac{\phi(0)}{\Phi(k)}\right) \\ &\quad = \Phi(k)\left(\phi\left(-\frac{\phi(0)}{\Phi(k)}\right) - \frac{\phi(0)}{\Phi(k)}\Phi\left(-\frac{\phi(0)}{\Phi(k)}\right)\right) \\ &\quad +\Phi(-k)\left(\phi\left(-\frac{\phi(0)}{\Phi(k)}\right) + \frac{\phi(0)}{\Phi(k)}\left(1 - \Phi\left(-\frac{\phi(0)}{\Phi(k)}\right)\right)\right) - \phi(0). \end{aligned} \tag{3.15}$$

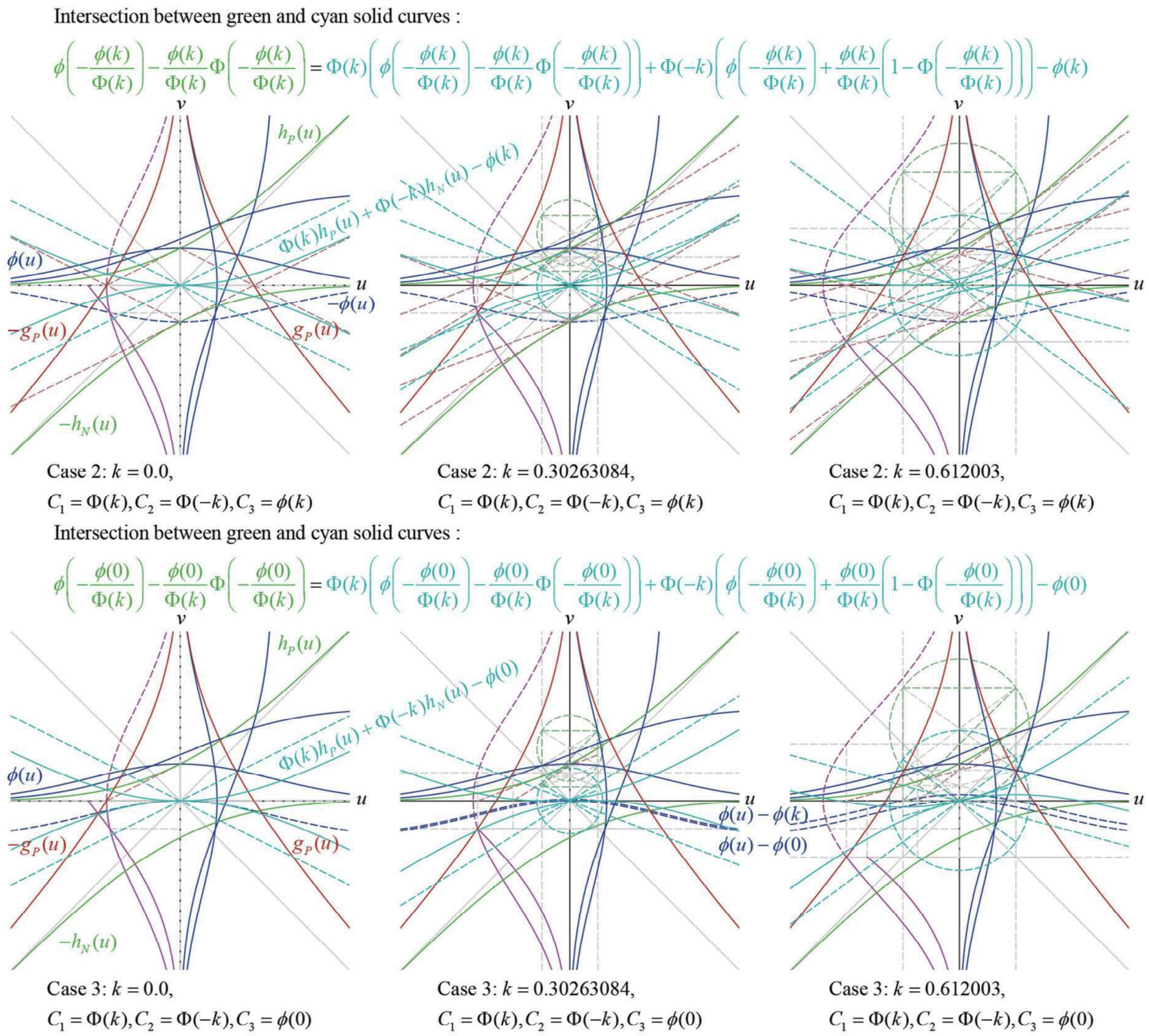

Figure 5 Rotational symmetries and some boundaries by parametric equations of second order differential equations of standard normal distribution and Bernoulli differential equations of inverse Mills ratios.

The boundary condition [61] on $u = 0$ and $u = x$ in Equation (3.10) is shown as

$$\begin{cases}\Phi(k)h_P(0) + \Phi(-k)h_N(0) - \phi(0) = 0, \\ \Phi(k)h_P(x) + \Phi(-k)h_N(x) - \phi(k) = 0.\end{cases} \tag{3.16}$$

4. Relations about Equilateral Triangles or Golden Ratio based on Right Triangles of Geometric Pythagorean Theorem and Considering Beauty with These Small Differences

We have considered a green dashed circle in Figure 4 in Section 3. In this section, we would like to deal with four parts of circles and squares based on several times of the right angle $n\pi/2$ $(n = 0,1,2,3)$ as these rotations. From Figure 6, we can notice that the shapes might be a cross as squaring the circle [62,63], a flower with four-leaves, and a four-leaf clover based on the probability points k. If we do not mind the small differences of the shapes at the probability point k, there might be the possibilities of geometrically meaningful shapes.

If we add the other cyan dashed circle in the center of Figure 6, these shapes might be also illustrated as a five-rims or an Olympic track in Figures 6 and 7B. There are not strictly correct meanings of mathematical principles about these shapes. However, we would like to explain the two geometrically emphasized characterizations to the readers. One is the equilateral triangles which are related to three circles such as Reuleaux triangle [56] and the shape of history of mathematics [57]. Its probability point and the characterization are described as

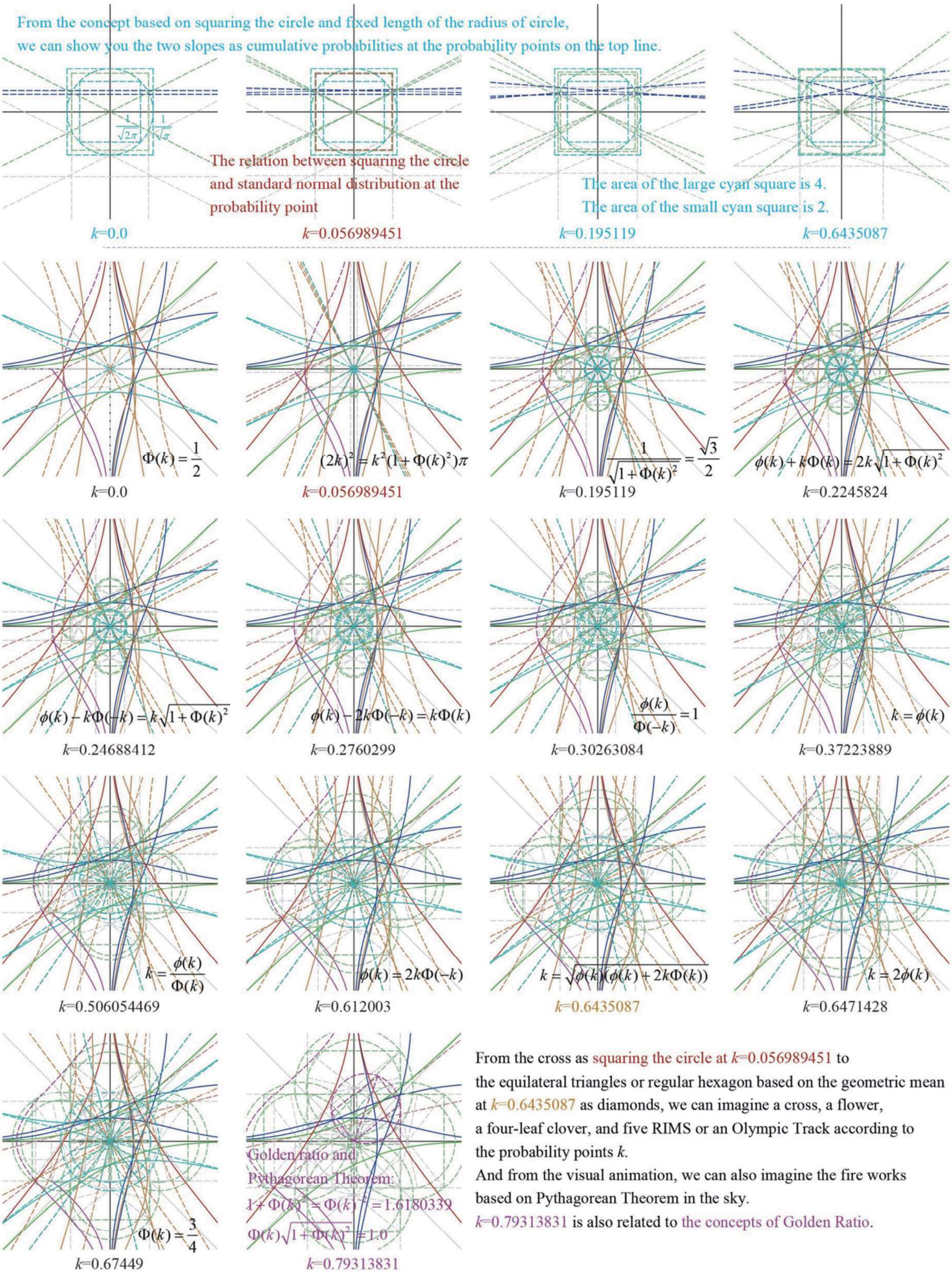

Figure 6 Characterizations about squaring the circle and our four directionally expanding concepts with Egyptian drawing styles.

$$\phi(k)+k\Phi(k)=k\sqrt{1+\Phi(k)^2} \quad \text{or} \quad k=\sqrt{\phi(k)\big(\phi(k)+2k\Phi(k)\big)} \because k=0.6435087. \tag{4.1}$$

We can understand that the relations are illustrated as the exactly right triangle between $\phi(k)+2k\Phi(k)$ and $\phi(k)$

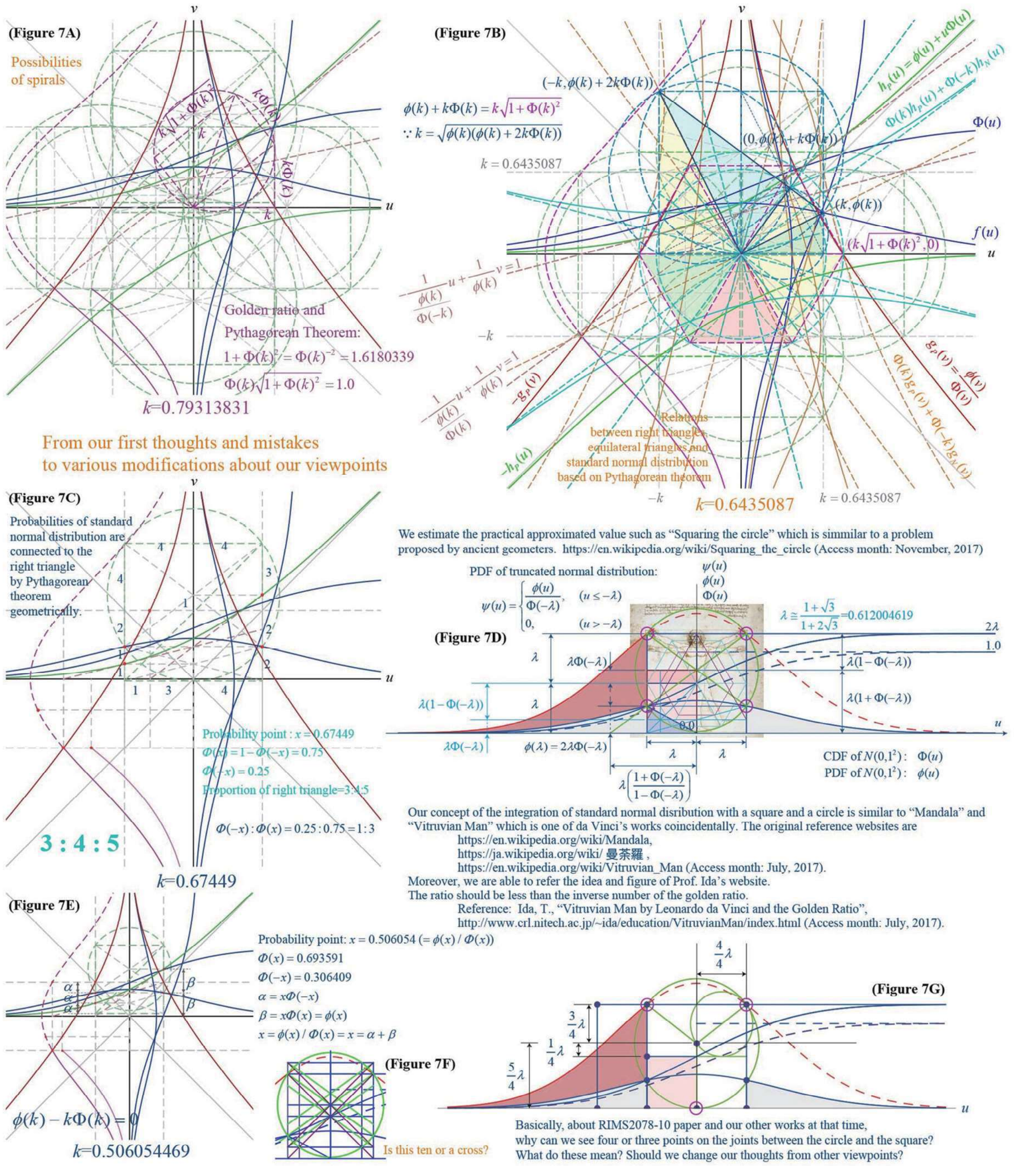

Figure 7 From our first mistakes to various modified relations about Equilateral triangles, golden ratio based on the geometric meanings and Pythagorean theorem.

geometrically. The other is that about the square root of golden ratio [64]. It is shown as

$$\Phi(k)\sqrt{1+\Phi(k)^2}=1 \because k=0.7931383, \qquad \Phi(k)^{-2}=1+\Phi(k)^2=1.6180339. \tag{4.2}$$

We guess that Equation (4.2) bring us the possibilities of some mathematical spirals shown in Figure 7A about a square root of golden ratio or several cumulative probabilities. These ideas make us not only four squares, circles, and standard normal distributions but also the above mentioned various tendencies such as a cross based on the equivalent areas both the circle and square, a flower with four-leaves, a four-leaf clover, five-RIMS, an Olympic track, and a square root of golden ratio. We can also illustrate Figure 7 such as fireworks in the sky because Equations (4.1) and (4.2) can be shown in Figure 7A and 7B. We have reconsidered what these tendencies mean since we found that of three or four points in Figure 7D, 7F, and 7G about Da Vinci's work, "Vitruvian Man" [65-67]. Our ideas and concepts have been widely created and modified to connect to Figures

7C and 7E by using Pythagorean theorem and ancient Egyptian drawing styles since we noticed misspelled equations in section 2. Moreover, it also makes us extended new thoughts about rotationally symmetric relations of standard normal distribution. After that, we can find mathematically correct probability points about equilateral triangles or golden ratio of standard normal distribution. We have been interested in the concepts about the textbooks by Tanioka [68], by Bernstein [69], by Salsburg [70], and the website by Watanabe [71] since our studies about $\lambda = 0.612003$ started. We are very happy since these philosophies bring us various above-mentioned results.

5. Illustrations of the others

In this section, we deal with two-dimensional probability densities of standard normal distribution [72-77] using regression analysis and principal component analysis in Figure 8. If we consider the condition that the two-dimensional maximal probability points on ellipses in Figure 8 are equal to the correlation coefficient both them, we can understand there should be a relation as parabola in Figure 8 between regression analysis and principal component analysis simply [4,7,72]. That is

$$u_1^2 + u_2^2 - 2\rho u_1 u_2 = \rho^2(1-\rho^2). \tag{5.1}$$

From Equation (5.1) and another rotationally symmetric viewpoint, we can imagine an attractively approximated number 0.777 if we geometrically and symmetrically consider the probability point as $\lambda = 0.612003$ and the correlation coefficient is also shown as $\rho = \lambda$. However, strictly speaking, it has a small error. Our mathematical trials and interests will have been continuing to seek new ideas and developments of sciences about our human beings and happiness.

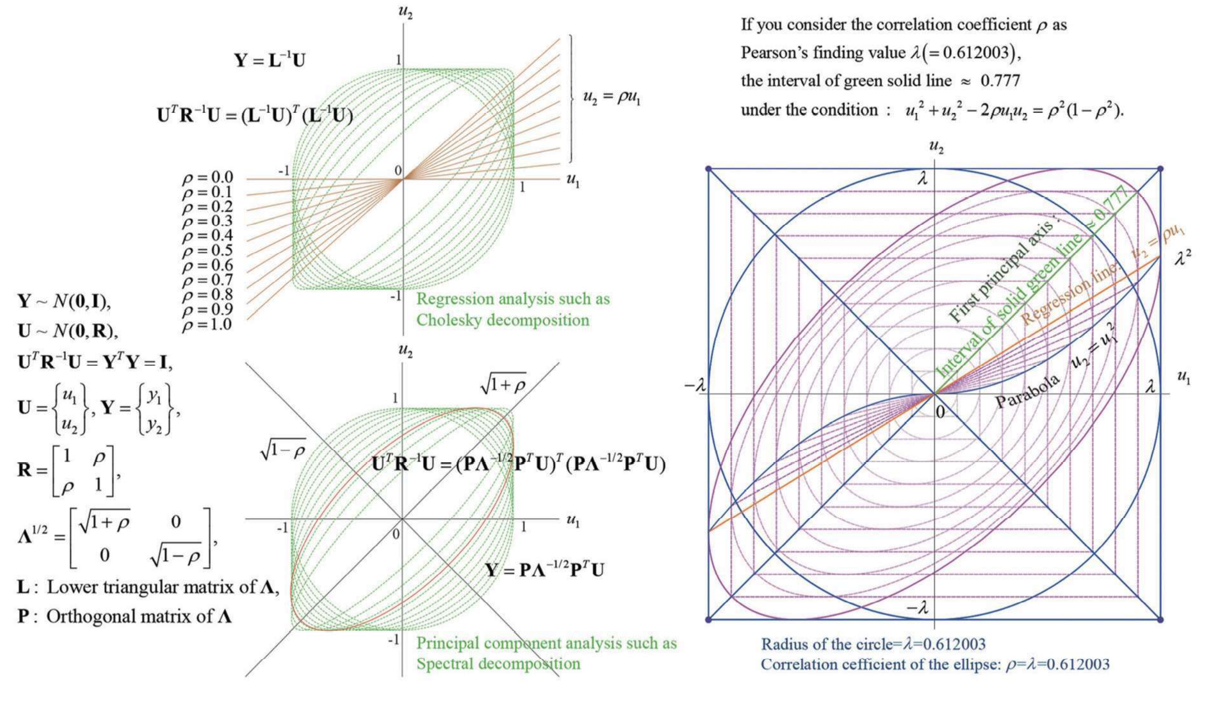

Figure 8 Two-dimensional correlated standard normal distribution under the conditions: Radius of Circles = Correlation Coefficients (Original References [7,75-77]).

6. Conclusions

In this paper, we deal with the geometric characterizations about a standard normal distribution. First, we can reconsider that the relations both inverse Mills ratio and standard normal distribution are formulated as several differential equations. Second, we can confirm that the weighted average proportions of integral forms of a cumulative distribution function of standard normal distribution are expressed as various geometric characterizations. Third, we can draw equilateral triangles correctly when we think of the meaning of Pythagorean right triangles, and the probability point, 0.6435087, and illustrate the other point 0.7931383 as the concept about golden ratio. Although we cannot find any new relations about da Vinci's art "Vitruvian Man", we can imagine "Mandalas" in Section 2 as our extended concepts. Moreover, we can realize that the true height of densities of a standard normal distribution is much more important than we thought of that shown in many textbooks under the aspect ratio is 1.0 throughout our studies.

Finally, we can guess that many readers have been waiting for several modifications and various new messages to develop our sciences and societies correctly and to inform next generations of that precisely. Authors would not also like to expect we have insisted on the first researchers about modified results from misspelled ideas. Therefore, we are happy if readers find what are the correct facts related to our ideas for mathematical sciences about truth and beauty of that.

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