Some properties of interpolation function of truncated multiple zeta value

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It is known that the polygamma function for a positive integer k,

$$\psi^{(k)}(s) = \frac{d^{k+1}}{ds^{k+1}} \log \Gamma(s),$$

interpolates the generalized harmonic sum

$$H_N(k) = \sum_{m=1}^{N} \frac{1}{m^k} \quad (N \in \mathbb{Z}_{\geq 1})$$

by the formula

$$H_N(k) = \frac{1}{(k-1)!} \psi^{(k-1)}(N).$$

Rieman-zeta value $\zeta(k)$ can be represented as a special value of the polygamma function, i.e.,

$$\zeta(k) = \frac{(-1)^{k-1}}{(k-1)!} \psi^{(k-1)}(1).$$

Rieman-zeta value, the generalized harmonic sum and the polygamma function are deeply related and there are several relation formulas, e.g.,

$$(-1)^k (k-1)! \{-\zeta(k) + \psi^{(k-1)}(N+1)\} = H_N(k).$$

For an index $\mathbf{k} = (k_1, \ldots, k_d) \in (\mathbb{Z}_{\geq 1})^d$ with $k_1 \geq 2$, the multiple zeta value and the multiple zeta-star value are defined by the series

$$\zeta(\mathbf{k}) = \sum_{m_1 > m_2 > \cdots > m_d > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_d^{k_d}}$$

and

$$\zeta^\star(\mathbf{k}) = \sum_{m_1 \geq m_2 \geq \cdots \geq m_d > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_d^{k_d}},$$

respectively. The index $\mathbf{k}$ with $k_1 \geq 2$ is called admissible. We call d the depth of $\mathbf{k}$ and $k = |\mathbf{k}| = k_1 + \cdots + k_d$ the weight of $\mathbf{k}$. For a positive integer $N \in \mathbb{Z}_{\geq 1}$ and $\mathbf{k} = (k_1, \ldots, k_d) \in (\mathbb{Z}_{\geq 1})^d$, the truncated multiple zeta value and the truncated multiple zeta-star value are defined by

$$H_N(\mathbf{k}) = \sum_{N \geq m_1 > m_2 > \cdots > m_d > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_d^{k_d}} = \sum_{m=d}^{N} \frac{1}{m^{k_1}} H_{m-1}(\mathbf{k}^1)$$

and

$$H_N^\star(\mathbf{k}) = \sum_{N \geq m_1 \geq m_2 \geq \cdots \geq m_d > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_d^{k_d}},$$

respectively with $\mathbf{k}^1 = (k_2, \ldots, k_d) \in (\mathbb{Z}_{\geq 1})^{d-1}$. Also in studies of multiple zeta values, an interpolation function of the truncated multiple zeta value plays important roles. For example, Kawashima function, defined as Newton series in [3], was introduced for deriving Kawashima relation, that is a class of linear relations of multiple zeta values including several basic classes of relations such as the duality relation and Ohno relation. Besides, S. Zlobin had defined the same function as Kawashima's in a different form in [11] and derived several linear relation formulas concerning multiple zeta values. This function interpolates the truncated multiple zeta-star value $H_N^\star(\mathbf{k})$. In [8], we have defined an interpolation function $\Psi_{\mathbf{k}}(s)$ of the truncated multiple zeta value $H_N(\mathbf{k})$ in a similar way to Zlobin's. In this note, we study the interpolation function $\Psi_{\mathbf{k}}(s)$ from two different perspectives; one is on an application and the other on intrinsic properties. We recall in §1 the definition and some basic properties of $\Psi_{\mathbf{k}}(s)$ investigated in [8], and introduce in §2 an application of $\Psi_{\mathbf{k}}(s)$ concerning multiple zeta values (cf. [6, 7]). In §3, we derive a function $G^d(s; \mathbf{q})$ from $\Psi_{\mathbf{k}}(s)$ that can be used for studying intrinsic properties of multiple zeta values (cf. [2]). §1 and §2 are based on a joint work with Y. Kusunoki and Y. Sasaki, and §3 is based on a joint work with K. Ihara.

1 Interpolation function

In this section, we summerize basic properties of $\Psi_{\mathbf{k}}(s)$ investigated in [8].

For an index $\mathbf{k} = (k_1, \ldots, k_d) \in (\mathbb{Z}_{\geq 1})^d$, put $\mathbf{k}_j = (k_1, \ldots, k_j)$, $\mathbf{k}^j = (k_{j+1}, \ldots, k_d)$ $(0 \leq j \leq d)$ with $\mathbf{k}_0 = \mathbf{k}^d = \emptyset$.

Definition 1 (Interpolaton function)**.** *We define the function* $\Psi_{\mathbf{k}}(s)$ *for* $s \in \mathbb{C} \setminus \mathbb{Z}_{<0}$ *recursively by*

$$\Psi_{\mathbf{k}}(s) = \sum_{j=1}^{\infty} \left\{ \frac{\Psi_{\mathbf{k}^1}(j-1)}{j^{k_1}} - \frac{\Psi_{\mathbf{k}^1}(s+j-1)}{(s+j)^{k_1}} \right\}$$

for $d \geq 1$ *and* $\Psi_{\emptyset}(s) = 1$ *for* $d = 0$*, i.e.,* $\mathbf{k} = \emptyset$*.*

It is easy to see that $\Psi_{\mathbf{k}}(0) = \cdots = \Psi_{\mathbf{k}}(d-1) = 0$ hold. For $s = d > 0$,

$$\begin{aligned}\Psi_{\mathbf{k}}(d) =& \frac{\Psi_{\mathbf{k}^1}(0)}{1^{k_1}} - \frac{\Psi_{\mathbf{k}^1}(d)}{(d+1)^{k_1}} \\ &+ \frac{\Psi_{\mathbf{k}^1}(1)}{2^{k_1}} - \frac{\Psi_{\mathbf{k}^1}(d+1)}{(d+2)^{k_1}} \\ &+ \frac{\Psi_{\mathbf{k}^1}(2)}{3^{k_1}} - \frac{\Psi_{\mathbf{k}^1}(d+2)}{(d+3)^{k_1}} \\ &+ \cdots \\ =& \frac{1}{d^{k_1}} \Psi_{\mathbf{k}^1}(d-1)\end{aligned}$$

holds. Thus, we have the following property:

Proposition 1.1 (Interpolation)**.** *For* $N \in \mathbb{Z}_{\geq 0}$*,*

$$\Psi_{\mathbf{k}}(N) = \begin{cases} 0, & 0 \leq N \leq d-1, \\ H_{\mathbf{k}}(N), & N \geq d \end{cases}$$

holds. Hence, $\Psi_{\mathbf{k}}(s)$ *interpolates the truncated multiple zeta values* $H_N(\mathbf{k})$ *to* $s \in \mathbb{C} \setminus \mathbb{Z}_{<0}$*.*

As a function of a complex variable s, $\Psi_{\mathbf{k}}(s)$ satisfies the followings, see [8] for proofs and details:

Proposition 1.2 (Difference formula)**.** *For* $s \in \mathbb{C} \setminus \mathbb{Z}_{<0}$ *and for any index* $\mathbf{k} = (k_1, \mathbf{k}^1)$*, it holds that*

$$\Psi_{\mathbf{k}}(s+1) - \Psi_{\mathbf{k}}(s) = \frac{1}{(s+1)^{k_1}} \Psi_{\mathbf{k}^1}(s).$$

Proposition 1.3 (Meromorphicity)**.** $\Psi_{\mathbf{k}}(s)$ *is meromorphic with poles at* $s = -l$ $(l \in \mathbb{Z}_{\geq 1})$ *of order* $k = |\mathbf{k}| = k_1 + \cdots + k_d$*.*

Proposition 1.4 (Asymptotic estimates)**.** *Assume that* $|\arg s| < \pi$. *As* $|s| \to \infty$, *the following estimates hold:*

$$\Psi_{\mathbf{k}}(s) = \zeta(\mathbf{k}) + O(|s|^{-k_1+1})$$

for any admissible index $\mathbf{k} = (k_1, \mathbf{k}^1) \in (\mathbb{Z}_{\geq 1})^d$, *and, for an admissible index* $\mathbf{k}'$ *and* $l \in \mathbb{Z}_{\geq 1}$,

$$\Psi_{\underbrace{1,\ldots,1}_{l},\mathbf{k}'}(s) = P_l(\log s) + O(|s|^{-1} \log^J |s|)$$

with some $J \geq 0$ *and a polynomial* $P_l(x)$ *of degree at most* l.

For a tuple $\mathbf{l} = (l_1, \ldots, l_d)$, let

$$\overleftarrow{\square}(l_1, \ldots, l_d) = \sum_{\square} (l_d \square l_{d-1} \square \cdots \square l_1)$$

where $\square$ denotes , or = and the summation is taken over all combinations concerning $\square$. The notation $*$ denotes the harmonic product, i.e.,

$$\mathbf{k} * \mathbf{l} = (k_1, \mathbf{k}^1 * \mathbf{l}) + (l_1, \mathbf{k} * \mathbf{l}^1) + (k_1 + l_1, \mathbf{k}^1 * \mathbf{l}^1)$$

for indices $\mathbf{k} = (k, \mathbf{k}^1)$ and $\mathbf{l} = (l, \mathbf{l}^1)$. Let $(k)_n$ be the rising factorial

$$(k)_n = k(k+1)\cdots(k+n-1)$$

for $k, n \in \mathbb{Z}_{\geq 0}$. Then the derivatives of $\Psi_{\mathbf{k}}(s)$ can be represented as follows:

Proposition 1.5 (Derivatives)**.** *For* $\mathbf{k} \in (\mathbb{Z}_{\geq 1})^d$ *and* $n \in \mathbb{Z}_{\geq 1}$,

$$\begin{aligned} \frac{d^n}{ds^n} \Psi_{\mathbf{k}}(s) = (-1)^n n! \sum_{\tau=1}^{d} \sum_{\substack{\mathbf{l}_\tau \in (\mathbb{Z}_{\geq 0})^\tau \\ l_\tau > 0 \\ |\mathbf{l}_\tau| = n}} \left(\prod_{i=1}^{\tau} \frac{(k_i)_{l_i}}{l_i!} \right) \Big(\Psi_{\mathbf{k}_\tau + \mathbf{l}_\tau, \mathbf{k}^\tau}(s) \\ + \sum_{\kappa=0}^{\tau-1} (-1)^{\tau-\kappa} \sum_{\iota=\kappa}^{\tau-1} \zeta(|(\mathbf{k}_\tau + \mathbf{l}_\tau)^\iota|, \overleftarrow{\square}(\mathbf{k}_\iota + \mathbf{l}_\iota)^\kappa * \mathbf{k}^\tau) \Psi_{\mathbf{k}_\kappa + \mathbf{l}_\kappa}(s) \Big) \end{aligned} \tag{1.1}$$

where $\zeta(\emptyset) = 1$.

We have the power series expansion of $\Psi_{\mathbf{k}}(s)$.

Theorem 1.1 (Taylor expansion). *$\Psi_{\mathbf{k}}(s)$ can be represented around $s = 0$ by*

$$\Psi_{\mathbf{k}}(s) = \sum_{n=1}^{\infty} (-1)^n a_n(\mathbf{k}) s^n$$

with

$$a_n(\mathbf{k}) = \sum_{\nu=1}^{d} (-1)^\nu \sum_{\substack{\mathbf{l}_\nu \in (\mathbb{Z}_{\geq 0})^\nu \\ |\mathbf{l}_\nu| = n \\ l_\nu > 0}} \prod_{i=1}^{\nu} \frac{(k_i)_{l_i}}{l_i!} \sum_{\iota=0}^{\nu-1} \zeta(|(\mathbf{k}_\nu + \mathbf{l}_\nu)^\iota|, \overleftarrow{\Box}(\mathbf{k}_\iota + \mathbf{l}_\iota) * \mathbf{k}^\nu). \quad (1.2)$$

2 Parity result of MZV

In this section, as an application of the interpolation function $\Psi_{\mathbf{k}}(s)$, we introduce a parity result (cf. [1, 10]) of the multiple zeta values with an explicit relation formula. Please see [7] for details.

The multiple polylogarithm

$$\mathrm{Li}_{\mathbf{k}}(z) = \sum_{m_1 > m_2 > \cdots > m_d > 0} \frac{z^{m_1}}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_d}} = \sum_{m=d}^{\infty} \frac{z^m}{m^{k_1}} H_{\mathbf{k}^1}(m-1)$$

is holomorphic in $|z| < 1$, and continuous on $|z| = 1$ except for $z = 1$. If $\mathbf{k}$ is admissible, the value $\mathrm{Li}_{\mathbf{k}}(1)$ exists and $\mathrm{Li}_{\mathbf{k}}(1) = \zeta(\mathbf{k})$ holds.

Assume that $0 < \arg z < 2\pi$. For $\mathbf{k} = (k_1, \mathbf{k}^1) \in (\mathbb{Z}_{\geq 1})^d$, let

$$f_{\mathbf{k}}(z; s) = \frac{2\pi i}{e^{2\pi i s} - 1} \frac{z^s}{s^{k_1}} \Psi_{\mathbf{k}^1}(s-1).$$

The function $f_{\mathbf{k}}(z; s)$ is meromorphic, and has simple poles at $s = m \in \mathbb{Z}_{\geq d}$, a pole of order $k + 1$ at $s = 0$ and poles of order $|\mathbf{k}^1| + 1$ at $s = m \in \mathbb{Z}_{<0}$. For $m \in \mathbb{Z}_{\geq d}$,

$$\begin{aligned} \operatorname*{Res}_{s=m} f_{\mathbf{k}}(z; s) &= \lim_{s \to m} (s-m) f_{\mathbf{k}}(z; s) \\ &= \lim_{s \to m} \frac{2\pi i (s-m)}{e^{2\pi i (s-m)} - 1} \frac{z^s}{s^{k_1}} \Psi_{\mathbf{k}^1}(s-1) \\ &= \frac{z^m}{m^{k_1}} H_{\mathbf{k}^1}(m-1) \end{aligned}$$

holds. Thus, we can represent the multiple polylogarithm by using the interpolation function $\Psi_{\mathbf{k}}(s)$, i.e.,

$$\mathrm{Li}_{\mathbf{k}}(z) = \sum_{m=d}^{\infty} \operatorname*{Res}_{s=m} \frac{2\pi i}{e^{2\pi i s} - 1} \frac{z^s}{s^{k_1}} \Psi_{\mathbf{k}^1}(s-1).$$

Then, by the residue theorem, the summation of residues of $f_{\mathbf{k}}(z;s)$ at all poles gives a formula concerning to multiple polylogarithms. Especially, considering the unit circle case directly, we can prove the parity result of multiple zeta values with an explicit formula. For simplicity, we use the following notations:

$$\underline{(k)_n} = \frac{(-1)^k (k)_n}{n!}$$

with $k \in \mathbb{N}$, $n \in \mathbb{Z}_{\geq 0}$, and

$$\mathcal{B}_n(z) = \frac{(2\pi i)^n}{n!} B_n\left(\frac{\log z}{2\pi i}\right)$$

with Bernouilli polynomials $B_n(x)$. Further, we define

$$\begin{aligned}
\zeta_\sigma(\mathbf{k}, n) &= (-1)^n \frac{d^n}{ds^n} \Psi_{\mathbf{k}}(s)\Big|_{s=0} = (-1)^n a_n(\mathbf{k}) \\
&= (-1)^n \sum_{\nu=1}^{d} (-1)^\nu \sum_{\substack{\mathbf{l}_\nu \in (\mathbb{Z}_{\geq 0})^\nu \\ |\mathbf{l}_\nu| = n \\ l_\nu > 0}} \prod_{i=1}^{\nu} \frac{(k_i)_{l_i}}{l_i!} \sum_{\iota=0}^{\nu-1} \zeta(|(\mathbf{k}_\nu + \mathbf{l}_\nu)^\iota|, \overleftarrow{\Box}(\mathbf{k}_\iota + \mathbf{l}_\iota) * \mathbf{k}^\nu)
\end{aligned}$$

for $n > 0$ and $\zeta_\sigma(\mathbf{k}, n) = 0$ for $n \leq 0$. If $\mathbf{k} = \emptyset$,

$$\zeta_\sigma(\emptyset, n) = \begin{cases} 0, & n > 0, \\ 1, & n \leq 0. \end{cases}$$

Theorem 2.1 (Parity result [7]). *For $\mathbf{k} = (k_1, \ldots, k_d)$, put $k = k_1 + \cdots + k_d$. Assume that $k + d$ is odd. Then $\zeta(\mathbf{k})$ is a $\mathbb{Q}[\zeta(2)]$-linear combination of multiple zeta values of depth at most $d - 1$. The relation formula is given by*

$$\begin{aligned}
&\zeta(\mathbf{k}) + (-1)^{d+k+1} \zeta^\star(\mathbf{k}) \\
&= \sum_{n_1+n_2+n_3=|\mathbf{k}^1|} \frac{(2\pi i)^{n_1} B_{n_1}}{n_1!} \underline{(k_1)_{n_2}} \left\{ \sum_{1 \leq \tau \leq d-1} \sum_{\substack{j=\tau \\ n_3 \geq |\mathbf{k}^j|}}^{d} (-1)^j \right. \\
&\qquad \left. \times \sum_{\substack{\mathbf{t}_\tau \in \mathbb{Z}_{\geq 0}^\tau \\ |\mathbf{t}_\tau| = n_3 - (|(\mathbf{k}_\tau)^1| + |\mathbf{k}^j|)}} \prod_{i=2}^{\tau} \underline{(k_i)_{t_i}} \zeta_\sigma(\mathbf{k}^j, t_1) \zeta^\star(k_1 + n_2, (\mathbf{k}_\tau + \mathbf{t}_\tau)^1) \right\} \\
&\quad + \sum_{j=1}^{d} (-1)^j \sum_{n=0}^{|\mathbf{k}_j|} \frac{(2\pi i)^n B_n}{n!} \zeta_\sigma(\mathbf{k}^j, |\mathbf{k}_j| - n)
\end{aligned}$$

where B_n are Bernouilli numbers with $B_1 = -\frac{1}{2}$.

3 Primitive function

In this section, based on [2], we define a function $G^d(s;\mathbf{q})$ from an integral representation of $\Psi_{\mathbf{k}}(s)$ and study intrinsic properties of $\Psi_{\mathbf{k}}(s)$.

Assume that $|q| < 1$ and $|q_j| < 1$ $(j = 1, \dots, d)$. For a fixed complex number s, we define meromorphic functions in q by

$$G^1(s;q) = \frac{1-q^s}{1-q}$$

and, in $\mathbf{q} = (q_1, \dots, q_d)$ by

$$G^d(s;\mathbf{q}) = \frac{1}{1-q_d}\{G^{d-1}(s;\mathbf{q}_{d-1}) - G^{d-1}(s;(\mathbf{q}_{d-2}, q_{d-1}q_d))\}$$

for $d \geq 2$, where $\arg q$ is treated as zero for q^s.

Example 1. *Let us consider the case $d = 3$. By the definition, we have*

$$\begin{aligned}
&G^3(s;q_1,q_2,q_3)\\
&= \frac{1}{1-q_3}\left\{G^2(s;q_1,q_2) - G^2(s;q_1,q_2q_3)\right\}\\
&= \frac{1}{1-q_3}\left\{\frac{1}{1-q_2}(G^1(s;q_1) - G^1(s;q_1q_2))\right\}\\
&= \frac{1}{1-q_3}\left(\frac{1}{1-q_2}\left(\frac{1-q_1^s}{1-q_1} - \frac{1-q_1^sq_2^s}{1-q_1q_2}\right) - \frac{1}{1-q_2q_3}\left(\frac{1-q_1^s}{1-q_1} - \frac{1-q_1^sq_2^sq_3^s}{1-q_1q_2q_3}\right)\right).
\end{aligned}$$

Remark that Y. Komori defines essentially the same function as $G^d(s;\mathbf{q})$ in his recent research. He expresses multiple polygamma functions in terms of certain contour integral of $G^d(s;\mathbf{q})$. See [4, 5] for details. On the other hand, we derive this function as an integrand of an integral representation of the interpolation function $\Psi_{\mathbf{k}}(s)$.

Theorem 3.1 ([2]). [1]

For $\Re s > 0$, we have

$$\begin{aligned}
\Psi_{\mathbf{k}}(s) &= \frac{(-1)^{k-d}}{\prod_{l=1}^d \Gamma(k_l)}\int_0^1 \cdots \int_0^1 G^d(s;\mathbf{q})\prod_{l=1}^d (\log q_l)^{k_l-1} dq_1\cdots dq_d\\
&= \frac{(-1)^{k-d}}{\prod_{l=1}^d \Gamma(k_l)}\int_0^\infty \cdots \int_0^\infty G^d(s;e^{-t_1},e^{-t_2},\dots,e^{-t_d})\prod_{l=1}^d t_l^{k_l-1} dt_1\cdots dt_d.
\end{aligned}$$

[1] Let me express my gratitude for Prof. S. Yamamoto concerning his suggestion about a method for proving the convergence of the integration.

3.1 Values of $G^d(s;\mathbf{q})$ at positive integers

If $d=1$ and $N\in\mathbb{Z}_{\geq 1}$,

$$G^1(N;q)=1+q+\cdots+q^{N-1}$$

holds. Because of the property $\lim_{q\to 1}G^1(N;q)=N-1$, $G^1(N;q)$ is called q-integer. Let us consider the case $d>1$.

Example 2. *For $d=3$ and $s=4$, by the definition, we have*

$$\begin{aligned}
&G^3(4;q_1,q_2,q_3)\\
=\;&\frac{1}{1-q_3}\left\{\frac{1}{1-q_2}\left((1+q_1+q_1^2+q_1^3)-(1+q_1q_2+q_1^22_2^2+q_1^3q_2^3)\right)\right.\\
&\quad\left.-\frac{1}{1-q_2q_3}\left((1+q_1+q_1^2+q_1^3)-(1+q_1q_2q_3+q_1^22_2^2q_3^2+q_1^3q_2^3q_3^3)\right)\right\}\\
=\;&\frac{1}{1-q_3}\left\{\frac{1}{1-q_2}\left(q_1(1-q_2)+q_1^2(1-q_2^2)+q_1^3(1-q_2^3))\right)\right.\\
&\quad\left.-\frac{1}{1-q_2q_3}\left(q_1(1-q_2q_3)+q_1^2(1-q_2^2q_3^2)+q_1^3(1-q_2^3q_3^3)\right)\right\}\\
=\;&\frac{1}{1-q_3}\left\{\left(q_1+q_1^2(1+q_2)+q_1^3(1+q_2+q_2^2)\right)\right.\\
&\quad\left.-\left(q_1+q_1^2(1+q_2q_3)+q_1^3(1+q_2q_3+q_2^2q_3^2)\right)\right\}\\
=\;&\frac{1}{1-q_3}\left(q_1^2q_2(1-q_3)+q_1^3q_2(1-q_3)+q_1^3q_2^2(1-q_3^2)\right)\\
=\;&q_1^2q_2+q_1^3q_2+q_1^3q_2^2(1+q_3)\\
=\;&q_1^2q_2\quad+q_1^3q_2\quad+q_1^3q_2^2\quad+q_1^3q_2^2q_3
\end{aligned}$$

Thus, $G^3(4;q_1,q_2,q_3)$ is a polynomial of degree $(4-1)!=6$.

In general, we have the following:

Proposition 3.1 ([2])**.** *For a positive integer N, we have*

$$G^d(N;\mathbf{q})=\sum_{N>l_1>l_2>\cdots>l_n\geq 0}q_1^{l_1}q_2^{l_2}\cdots q_d^{l_d}. \tag{3.1}$$

Putting Λ_N the set of exponents in the polynomial expression (3.1) of $G^d(N;\mathbf{q})$, one can express the truncated multiple zeta values as

$$H_{\mathbf{k}}(N-1)=\sum_{(l_1,\ldots,l_d)\in\Lambda_N}\frac{1}{(l_1+1)^{k_1}\cdots(l_d+1)^{k_d}} \tag{3.2}$$

where $\lambda+\mathbf{1}=(l_1+1,l_2+1,\ldots,l_n+1)$.

3.2 Addition and Product

For complex variables s and t, we have the following properties:

Proposition 3.2 ([2]). *For $s, t \in \mathbb{C}$,*

$$G^d(s+t;\mathbf{q}) = G^d(s;\mathbf{q})+q_1^s G^d(t;\mathbf{q})+q_1^s \sum_{j=1}^{d-1} G^{d-j}\left(s; \prod_{l=1}^{j+1} q_l, \mathbf{q}^{j+1}\right) G^j(t; q_1 q_2, (\mathbf{q}_{j+1})^2)$$

and

$$G^d(st;\mathbf{q}) = \sum_{j=1}^{d} \left(\sum_{\substack{i_1+\cdots+i_j=d \\ \forall i_\bullet \geq 0}} \left(\prod_{k=1}^{j} G^{i_k} \left(s; \prod_{l=k}^{d-\sum_{u=1}^{k}(i_u-1)} q_l, (\mathbf{q}_{d-\sum_{u=1}^{k-1}(i_u-1)})^{d-\sum_{u=1}^{k}(i_u-1)} \right)\right)\right)$$
$$\times G^j(t; q_1^s, \ldots, q_j^s)$$

holds.

3.3 Harmonic product

For two tuples of parameters $\mathbf{p} = (p, \mathbf{p}^1)$ and $\mathbf{q} = (q, \mathbf{q}^1)$ where depths of $\mathbf{p}$ and $\mathbf{q}$ need not be the same, let $\underline{*}$ be an operation defined by

$$\mathbf{p}\underline{*}\mathbf{q} = (p_1, \mathbf{p}^1\underline{*}\mathbf{q}) + (q_1, \mathbf{p}\underline{*}\mathbf{q}^1) + (p_1 q_1, \mathbf{p}^1\underline{*}\mathbf{q}^1).$$

We call the operation (multiplicative) harmonic product.

Theorem 3.2 ([2]).

$$G(s;\mathbf{p})G(s;\mathbf{q}) = G(s;\mathbf{p}\underline{*}\mathbf{q}) \tag{3.3}$$

where ommiting upper subscripts are taken suitably depending on the depth of each tuple of parameters.

If $s = N$ is a positive integer, since $G^d(N;\mathbf{q})$ is a polynomial, the assertion of Theorem 3.2 is obvious. Let us see an example.

Example 3. *For the case $s = 4$, the product of $G^2(4; p_1, p_2)$ and $G^3(4; q_1, q_2, q_3)$ is*

$$\begin{aligned} &G^2(4; p_1, p_2)G^3(4; q_1, q_2, q_3) \\ &= (p_1q_1)^2 q_2 + (p_1q_1)^3 q_2 + (p_1q_1)^3 q_2^2 + (p_1q_1)^3 q_2^2 (q_3 p_2) \\ &\quad + (p_1q_1)^2 (p_2q_2) + (p_1q_1)^3 (p_2q_2) + (p_1q_1)^3 (p_2q_2)^2 + (p_1q_1)^3 (p_2q_2)^2 q_3 \end{aligned}$$

$$
\begin{aligned}
&+ q_1^2 p_1 q_2 + q_1^3 p_1 q_2 + q_1^3 (p_1 q_2)^2 + q_1^3 (p_1 q_2)^2 (p_2 q_3) \\
&+ q_1^3 q_2^2 p_1 \\
&+ q_1^3 q_2^2 (p_1 q_3) \\
&+ q_1^3 p_1^2 q_2 \\
&+ q_1^3 p_1^2 (p_2 q_2) \\
&+ q_1^3 (p_1 q_2)^2 q_3 \\
&+ q_1^3 (p_1 q_2)^2 p_2 \\
&+ p_1^3 q_1^2 q_2 \\
&+ (p_1 q_1)^3 q_2^2 q_3 \\
&+ p_1^3 q_1^2 (p_2 q_2) \\
&+ (p_1 q_1)^3 q_2^2 p_2 \\
&+ p_1^3 (p_2 q_1)^2 q_2 \\
&+ (p_1 q_1)^3 p_2^2 q_2.
\end{aligned}
$$

That is,

$$
\begin{aligned}
& G^2(4; p_1, p_2) G^3(4; q_1, q_2, q_3) \\
= \; & G^3(4; p_1q_1, q_2, p_2 + q_3) + G^3(4; p_2q_1, p_2q_2, q_3) + G^3(4; q_1, p_1q_2, q_3p_2) \\
& + G^4(4; q_1, q_2, p_1, q_3p_2) + G^4(4; q_1, q_2, p_1q_3, p_2) + G^4(4; q_1, p_1, q_2, p_2q_3) \\
& + G^4(4; q_1, p_1, p_2q_2, q_3) + G^4(4; q_1, p_1q_2, q_3, p_2) + G^4(4; q_1, p_1q_2, p_2, q_3) \\
& + G^4(4; p_1, q_1, q_2, p_2q_3) + G^4(4; p_1q_1, q_2, q_3, p_2) + G^4(4; p_1, q_1, p_2q_2, q_3) \\
& + G^4(4; p_1q_1, q_2, p_2, q_3) + G^4(4; p_1, p_2q_1, q_2, q_3) + G^4(4; p_1q_1, p_2, q_2, q_3) \\
= \; & G(4; (p_1, p_2) \underline{*} (q_1, q_2, q_3))
\end{aligned}
$$

holds.

Combining Theorem 3.1 and Theorem 3.2, we have the following:

Theorem 3.3 ([2]). $\Psi_{\mathbf{k}}(s)\Psi_{\mathbf{l}}(s) = \Psi_{\mathbf{k}*\mathbf{l}}(s)$ *holds.*

3.4 Behavior around 0 and 1

By the definition, it is easy to see that

Proposition 3.3 ([2]).

$$
G^d(s; \underbrace{0, \ldots, 0}_{d}) := \lim_{(q_1, \ldots, q_d) \to (0, \ldots, 0)} G^d(s; q_1, \ldots, q_d) = \begin{cases} 1 & d = 1, \\ 0 & d \geq 2. \end{cases}
$$

On the other hand, $\mathbf{q} = (1, \ldots, 1)$ is a singularity of $G^d(s; \mathbf{q})$ for a variable $\mathbf{q}$. However, we have the following properties:

Theorem 3.4 ([2])**.** *For* $\mathbf{q} = (q_1, \ldots, q_d)$,

$$
\begin{aligned}
G^d(s; \underbrace{1, \ldots, 1}_{d}) &:= \lim_{(q_1,\ldots,q_d)\to(1,\ldots,1)} G^d(s; \mathbf{q}) \\
&= \binom{s}{d}, \\
G^d(s; \underbrace{1, \ldots, 1}_{d-1}, q_d) &:= \lim_{q_{d-1}\to 1} \lim_{(q_1,\ldots,q_{d-2})\to(1,\ldots,1)} G^d(s; \mathbf{q}) \\
&= \frac{1}{(q_d - 1)^d} \left(q_d^s - \sum_{l=0}^{d-1} \binom{s}{l} (q_d - 1)^l \right), \\
G^d(\mathbf{q}_{d-1}, 1) &:= \lim_{q_d \to 1} G^d(s; \mathbf{q}) \\
&= \frac{(-1)^{d-1} s \prod_{j=1}^{d-1} q_j^s}{\prod_{p=1}^{d-1}(1 - \prod_{j=p}^{d-1} q_j)} \\
&\quad + \sum_{l=1}^{d-1} \frac{(-1)^{d-1-l} \left(\prod_{j=l}^{d-1} q_j\right) G^l(\mathbf{q}_{l-1}, \prod_{j=l}^{d-1} q_j)}{\prod_{p=l}^{d-1}(1 - \prod_{j=p}^{d-1} q_j)}.
\end{aligned}
$$

Combining Proposition 3.2 and the first result of Theorem 3.4, we have formulas for binomial coefficients.

Corollary 3.1. *For* $s, t \in \mathbb{C}$ *and* $d \in \mathbb{Z}_{\geq 0}$,

$$
\binom{s+t}{d} = \sum_{j=0}^{d} \binom{t}{j} \binom{s}{d-j}
$$

and

$$
\binom{st}{d} = \sum_{j=1}^{d} \left(\sum_{\substack{i_1+\cdots+i_j=d \\ \forall i_\bullet \geq 0}} \binom{s}{i_1} \binom{s}{i_2} \cdots \binom{s}{i_j} \right) \binom{t}{j}
$$

holds.

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