Classification of ribbon 2-knots with ribbon crossing number up to four

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1 Introduction

A ribbon 2-knot is a knotted 2-sphere in $\mathbb{R}^4$ that bounds a ribbon 3-disk, which is an immersed 3-disk with only ribbon singularities. The ribbon crossing number of a ribbon 2-knot is the minimal number of the ribbon singularities of any ribbon 3-disk bounding the knot [14]. Yasuda has classified ribbon 2-knots with ribbon crossing number up to three in [13] and has enumerated those with ribbon crossing number four in [15]. In this paper we classify these ribbon 2-knots.

Theorem 1. *The number of mutually non-isotopic ribbon* 2*-knots with ribbon crossing number four is either* 111 *or* 112*. Amongst them* 9 *or* 10 *knots are positive-amphicheiral. So, if each chiral pair is counted as one knot, the number of ribbon* 2*-knots with ribbon crossing number four is either* 60 *or* 61*; see Table* 1*.*

Table 1: Numbers of the ribbon 2-knots with ribbon crossing number up to four.

Ribbon crossing number	0	1	2	3	4
(i) Number of ribbon 2-knots, each chiral pair is counted separetely	1	0	3	13	111/112
(ii) Number of ribbon 2-knots, each chiral pair is counted as one knot	1	0	2	7	60/61

The ribbon 2-knots with ribbon crossing number with up to three are completely classified by the Alexander polynomial. However, those with ribbon crossing number four listed in [15] have not been classified. Theorem 1 means that there is an indistinguishable pair of ribbon 2-knots, Y43 and Y46 in Table 3, which are positive-amphicheiral; they have isomorphic knot group. Also, there is one knot, Y112 (the ribbon handlebody is shown in Fig. 1), which had been missed in [15].

Satoh [8] introduced a virtual arc presentation for a ribbon 2-knot. If a ribbon 2-knot K is presented by a virtual arc with n classical crossings, then the ribbon crossing number of K is at most n. In [2] ribbon 2-knots presented by a virtual arc with up to four crossings are enumerated, and in [6] those ribbon 2-knots are classified. There are 24 ribbon 2-knots with ribbon crossing number up to four, which are not presented by a virtual arc with up to four crossings. So, we have only to consider these knots. We have 27 sets of ribbon 2-knots $\mathcal{A}_i$ ($i = 1, 2, \ldots, 17$) and $\mathcal{A}_j!$ ($j = 2, 3, 4, 7, 8, 10, 11, 12, 14, 16$), which consist of knots sharing the same Alexander polynomial; $\mathcal{A}_j!$ is the set consisting of the mirror images of the knots in $\mathcal{A}_j$. The knots in the sets $\mathcal{A}_i$ with $i \leq 13$ (and so $\mathcal{A}_j!$ with $j \leq 12$) have been classified in [6]. Thus, we classify the knots in $\mathcal{A}_i$ with $i = 14$, 15, 16, 17 (Sec. 5). The knots in these sets are ribbon 2-knots of 1-fusion. In order to classify the knots in these sets we use the trace set, or the twisted Alexander polynomial associated to the representations to $\mathrm{SL}(2, \mathbb{C})$. The *trace set* is an invariant defined for a ribbon 2-knot of 1-fusion from the representations of the knot group to $\mathrm{SL}(2, \mathbb{C})$; see Sec. 4 in [7]. For the twisted Alexander polynomial of a ribbon 2-knot, see [4].

This paper is organized as follows: In Secs. 2 and 3, we review a ribbon handlebody presentation of a ribbon 2-knot and the stable transformations for a ribbon handlebody presentation, which were introduced in [3]. In Sec. 4 we give Yasuda's table of the ribbon 2-knots with ribbon crossing number up to four (Tables 2 and 3), which contain the 1-fusion notation of the knots. In Sec. 5 we classify the knots in $\mathcal{A}_i$, $i = 14$, 15, 16, 17, which completes the proof of Theorem 1.

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2 Ribbon handlebody presentation of a ribbon 2-knot

In this section we review a ribbon handlebody presentation of a ribbon 2-knot introduced in [3]. A *ribbon handlebody* $\mathcal{H}$ is a ribbon 2-disk, which is a 2-dimensional handlebody in $\mathbb{R}^3$ consisting of $(m+1)$ 0-handles $D_0, D_1, \ldots, D_m$ and m 1-handles $B_1, B_2, \ldots, B_m$ such that the preimage of each ribbon singularity consists of an arc in the interior of a 0-handle and a cocore of a 1-handle. We set $\mathcal{H} = \mathcal{D} \cup \mathcal{B}$, where $\mathcal{D} = D_0 \cup D_1 \cup \cdots \cup D_m$ and $\mathcal{B} = B_1 \cup B_2 \cup \cdots \cup B_m$. We associate to a ribbon handlebody $\mathcal{H}$ an immersed 3-disk $V_{\mathcal{H}}$ in $\mathbb{R}^4$ defined by

$$V_{\mathcal{H}} = \mathcal{D} \times [-2, 2] \cup \mathcal{B} \times [-1, 1]. \tag{1}$$

Then $V_{\mathcal{H}}$ is a ribbon 3-disk for the ribbon 2-knot $K_{\mathcal{H}} = \partial V_{\mathcal{H}}$ in $\mathbb{R}^4$. Conversely, for any ribbon 2-knot K in $\mathbb{R}^4$, there exists a ribbon handlebody $\mathcal{H}$ such that K is ambient isotopic to the associated 2-knot $K_{\mathcal{H}}$; see [1, 10, 12].

We suppose that each 1-handle B_q is the image of an embedding $b_q : I \times I \to \mathbb{R}^3$, $q = 1, 2, \ldots, m$. Let $\beta_q : I \to \mathbb{R}^3$ be the center line of the 1-handle B_p defined by $\beta_q(t) = b_q(1/2, t)$, which is an oriented path such that

$$\begin{aligned} \beta_q(I) \cap \mathcal{D} = \left\{\beta_q(0), \beta_q(t_{q,1}), \beta_q(t_{q,2}), \ldots, \beta_q(t_{q,\ell_q}), \beta_q(1)\right\}, \\ 0 < t_{q,1} < t_{q,2} < \cdots < t_{q,\ell_q} < 1. \end{aligned} \tag{2}$$

Let ι_q, τ_q, $\lambda(q, j)$ $(j = 1, 2, \ldots, \ell_q)$ be integers in $\{0, 1, \ldots, m\}$ determined by

$$\beta_q(0) \in \partial D_{\iota_q}, \quad \beta_q(1) \in \partial D_{\tau_q}, \quad \beta_q(t_{q,j}) \in \mathrm{Int} D_{\lambda(q,j)}, \quad j = 1, 2, \ldots, \ell_q. \tag{3}$$

Thus, β is an oriented path joining D_{ι_q} and D_{τ_q}. At the intersection $\beta_q(t_{q,j})$ if β_q passes from the negative side of $D_{\lambda(q,j)}$ through to the positive side we define $\epsilon(q,j)=+1$, and if it passes in the opposite direction we define $\epsilon(q,j)=-1$.

Then for a ribbon handlebody $\mathcal{H}$ we define a *ribbon handlebody presentation* $[\,X \mid R\,]$ consisting of:

- $X=\{x_0,x_1,\ldots,x_m\}$, where each letter x_q corresponds to the 0-handle D_q,
- $R=\{\rho_1,\rho_2,\ldots,\rho_m\}$, where each relation $\rho_q : x_{\iota_q}^{w_q}=x_{\tau_q}$ (or $x_{\tau_q}=x_{\iota_q}^{w_q}$) corresponds to the 1-handle B_q that joins D_{ι_q} to D_{τ_q} passing through 0-handles according to the word w_q:
$$w_q = x_{\lambda(q,1)}^{\epsilon(q,1)} x_{\lambda(q,2)}^{\epsilon(q,2)} \cdots x_{\lambda(q,\ell_q)}^{\epsilon(q,\ell_q)}. \tag{4}$$
In particular, if $\beta_q(I)\cap\mathcal{D}=\{\beta_q(0),\beta_q(1)\}$, then $\rho_q : x_{\iota_q}=x_{\tau_q}$.

For a ribbon handlebody presentation $P=[\,X \,|\, R]$, $X=\{x_0,x_1,\ldots,x_m\}$ and $R=\{\rho_1,\rho_2,\ldots,\rho_m\}$ with $\rho_q : x_{\iota_q}^{w_q}=x_{\tau_q}$, $w_q\in F[X]$, we can associate an oriented labelled tree $\tilde{P}=(X,E,\lambda)$, where X is a set of vertices, E is a set of oriented edges:

$$E=\left\{\overrightarrow{x_{\iota_q}x_{\tau_q}} \mid q=1,\ldots,m\right\}, \tag{5}$$

and $\lambda : E\to F[X]$ is a labeling function defined by $\lambda(\overrightarrow{x_{\iota_q}x_{\tau_q}})=w_q$.

Conversely, for an oriented labeled tree (X,E,λ) as above, we obtain a unique ribbon handlebody presentation $P=[\,X \,|\, R\,]$ and also the associated ribbon 2-knot of m-fusion, which we denote by K_P; cf. Proposition 3.3 in [3]. Note that the knot group of K_P, $\pi_1(\mathbb{R}^4-K_P)$, is presented by $\langle\, X \,|\, \tilde{R}\,\rangle$, where $\tilde{R}$ is a set of relations $\{\tilde{\rho}_1,\tilde{\rho}_2,\ldots,\tilde{\rho}_m\}$ with $\tilde{\rho}_q : w_q^{-1}x_{\iota_q}w_q=x_{\tau_q}$; see [11].

Therefore, any ribbon 2-knot with ribbon crossing number r is obtained from an oriented labeled tree (X,E,λ) as above such that $\sum_{q=1}^{m}\ell_q=r$, where ℓ_q is the word length of the word w_q as in Eq. (4).

3 Stable transformations of a ribbon handlebody presentation

Let $P=[\,X \mid R\,]$ be a ribbon handlebody presentation, where $X=\{x_0,x_1,\ldots,x_m\}$ and $R=\{\rho_1,\ldots,\rho_m\}$ with

$$\rho_q : x_{\iota_q}^{w_q}=x_{\tau_q}, \quad w_q = x_{\lambda(q,1)}^{\epsilon(q,1)} x_{\lambda(q,2)}^{\epsilon(q,2)} \cdots x_{\lambda(q,\ell_q)}^{\epsilon(q,\ell_q)}, \quad \epsilon(q,s)=\pm 1. \tag{6}$$

We call the following transformations of a ribbon handlebody presentation *stable transformations*:

S1. Replace $\rho_q : x_{\iota_q}^{w_q}=x_{\tau_q}$ by $x_{\iota_q}=x_{\tau_q}^{(w_q^{-1})}$.

S2. Replace $\rho_q : x_{\iota_q}^{w_q}=x_{\tau_q}$ by either $x_{\iota_q}^{x_{\iota_q}^{\epsilon}w_q}=x_{\tau_q}$ or $x_{\iota_q}^{w_q x_{\tau_q}^{\epsilon}}=x_{\tau_q}$, $\epsilon=\pm1$.

S3. Add a generator y and a relation $y=x_p^w$ or $x_p=y^w$, where w is a word in x_0, $x_1,\ldots,x_m$.

S3′. Inverse transformation of S3.

S4. (i) Suppose $\tau_p = \iota_q$. Replace either $\rho_p : x_{\iota_p}^{w_p} = x_{\tau_p}$ or $\rho_q : x_{\iota_q}^{w_q} = x_{\tau_q}$ by $x_{\iota_p}^{w_p w_q} = x_{\tau_q}$.

(ii) Suppose $\iota_p = \iota_q$. Replace $\rho_p : x_{\iota_p}^{w_p} = x_{\tau_p}$ by $x_{\tau_q}^{w_q{}^{-1} w_p} = x_{\tau_p}$.

(iii) Suppose $\tau_p = \tau_q$. Replace $\rho_p : x_{\iota_p}^{w_p} = x_{\tau_p}$ by $x_{\iota_p}^{w_p w_q{}^{-1}} = x_{\iota_q}$.

S5. (i) Suppose $\lambda(p, s) = \tau_q$. Replace $x_{\lambda(p,s)}(= x_{\tau_q})$ in w_p in ρ_p by $w_q^{-1} x_{\iota_q} w_q$.

(ii) Suppose $\lambda(p, s) = \iota_q$. Replace $x_{\lambda(p,s)}(= x_{\iota_q})$ in w_p in ρ_p by $w_q x_{\tau_q} w_q^{-1}$.

Then we have the following (Proposition 4.1 in [3]):

Proposition 2. *Suppose that ribbon handlebody presentations P and P' are related by a finite sequence of stable transformations* S1–S5. *Then, the associated ribbon 2-knots K_P and $K_{P'}$ are ambient isotopic.*

We denote by

$$R(p_1, q_1, \ldots, p_n, q_n), \quad p_1, q_1, \ldots, p_n, q_n, \in \mathbb{Z}, \tag{7}$$

a ribbon 2-knot of 1-fusion, which is presented by the ribbon handlebody presentation

$$[\, x, y \mid x = y^w \ (w = x^{p_1} y^{q_1} \cdots x^{p_n} y^{q_n})\,]\,. \tag{8}$$

cf. [5, Sect. 2]. Then, by the transformation S1 we have:

$$R(p_1, q_1, \ldots, p_n, q_n) \approx R(-q_n, -p_n, \ldots, -q_1, -p_1) \tag{9}$$

$$R(p_1, q_1, \ldots, p_n, q_n)! \approx R(-p_1, -q_1, \ldots, -p_n, -q_n) \approx R(q_n, p_n, \ldots, q_1, p_1), \tag{10}$$

where $K \approx K'$ denotes that the two 2-knots K and K' are ambient isotopic and $K!$ the mirror image of K.

Example 3. The ribbon 2-knot Y43 presented by

$$P(\text{Y43}) = [\, x_1, x_2, x_3 \,|\, \rho_1 : x_1^{x_2 x_1} = x_2,\ \rho_2 : x_1^{x_3 x_2} = x_3\,]\,. \tag{11}$$

is isotopic to the ribbon 2-knot of 1-fusion $R(1, 1, -1, -1, -1, -1, 1, 1)$. Thus, by Eqs. (9) and (10) Y43 is positive-amphicheiral.

Proof By the transformation S5(ii), we replace x_1 in the power of ρ_1 with $x_3 x_2 x_3 x_2^{-1} x_3^{-1}$ coming from ρ_2. Then $P(\text{Y43})$ is deformed into

$$P(\text{Y43})_1 = \left[\, x_1, x_2, x_3 \,\middle|\, \rho_1' : x_1^{x_2 x_3 x_2 x_3 x_2^{-1} x_3^{-1}} = x_2,\ \rho_2 : x_1^{x_3 x_2} = x_3 \,\right]. \tag{12}$$

By the transformation S4(ii), we replace ρ_1' by $\rho_1'' : x_3^{(x_3 x_2)^{-1} x_2 x_3 x_2 x_3 x_2^{-1} x_3^{-1}} = x_2$. Then $P(\text{Y43})_1$ is deformed into

$$P(\text{Y43})_2 = \left[\, x_1, x_2, x_3 \,\middle|\, \rho_1'' : x_3^{x_2^{-1} x_3^{-1} x_2 x_3 x_2 x_3 x_2^{-1} x_3^{-1}} = x_2,\ \rho_2 : x_1^{x_3 x_2} = x_3 \,\right]. \tag{13}$$

By the transformation S3′, $P(\text{Y43})_2$ is deformed into

$$P(\text{Y43})_3 = \left[\, x_2, x_3 \,\middle|\, x_3^{x_2^{-1} x_3^{-1} x_2 x_3 x_2 x_3 x_2^{-1} x_3^{-1}} = x_2 \,\right], \tag{14}$$

which presents $R(-1, -1, 1, 1, 1, 1, -1, -1)(\approx R(1, 1, -1, -1, -1, -1, 1, 1))$. □

4 Yasuda's Table

Yasuda enumerated ribbon 2-knots with ribbon crossing number up to three in [13] and ribbon 2-knots with ribbon crossing four in [15]. He claims that any ribbon 2-knots with ribbon crossing number up to four is presented by one of the following ribbon handlebody presentations:

$$P_1(w) = [\, x_1, x_2 \,|\, x_1^{w} = x_2 \,]\,; \tag{15}$$
$$P_2(w_1, w_2) = [\, x_1, x_2, x_3 \,|\, x_1^{w_1} = x_2,\ x_1^{w_2} = x_3 \,]\,; \tag{16}$$
$$P_3(w_1, w_2, w_3) = [\, x_1, x_2, x_3, x_4 \,|\, x_1^{w_1} = x_2,\ x_2^{w_2} = x_3,\ x_3^{w_3} = x_4 \,]\,; \tag{17}$$
$$P_4(w_1, w_2, w_3) = [\, x_1, x_2, x_3, x_4 \,|\, x_1^{w_1} = x_2,\ x_1^{w_2} = x_3,\ x_1^{w_3} = x_4 \,]\,; \tag{18}$$
$$P_5(w_1, w_2, w_3, w_4) = [\, x_1, x_2, x_3, x_4, x_5 \,|\, x_1^{w_1} = x_2,\ x_1^{w_2} = x_3,\ x_1^{w_3} = x_4,\ x_2^{w_4} = x_5 \,]\,; \tag{19}$$
$$P_6(w_1, w_2, w_3, w_4) = [\, x_1, x_2, x_3, x_4, x_5 \,|\, x_1^{w_1} = x_2,\ x_1^{w_2} = x_3,\ x_1^{w_3} = x_4,\ x_1^{w_4} = x_5 \,]\,. \tag{20}$$

Remark 4. A ribbon 2-knot with ribbon crossing number up to four presented by the ribbon handlebody presentation

$$[\, x_1, x_2, x_3, x_4, x_5 \,|\, x_1^{w_1} = x_2,\ x_2^{w_2} = x_3,\ x_3^{w_3} = x_4,\ x_4^{w_4} = x_5 \,]\,, \tag{21}$$

$w_i \in F[x_1, x_2, x_3, x_4, x_5]$, is transformed into a ribbon 2-knot presented by one of the ribbon handlebody presentations (15)–(18).

In a similar way to Example 3, we can deform a ribbon 2-knot with up to four ribbon crossing by a finite sequence of stable transformations S1–S5 (Proposition 2) into one of the following two types:

- Type 1: a ribbon 2-knot of 1-fusion.
- Type 2: a composition of two ribbon 2-knots of 1-fusion.

In order to determine the type of a ribbon 2-knot we use the following proposition (Proposition 3.1 in [6]). Indeed, the fundamental group of a Type 2 ribbon 2-knot with ribbon crossing number up to four is isomorphic to the free product $\mathbb{Z}_3 * \mathbb{Z}_3$ (Proposition 3.2 in [6]).

Proposition 5. *The fundamental group of the* 2*-fold cover of* S^4 *branched over a ribbon* 2*-knot of* 1*-fusion* K *is the finite cyclic group whose order is the determinant of* K*,* $|\Delta_K(-1)|$.

Table 2 lists the ribbon 2-knots with ribbon crossing number up to three given by [13], and Table 3 lists the ribbon 2-knots with ribbon crossing four given by Yasuda [15]. Each column in Tables 2 and 3 shows as follows:

- The first column, Name, shows the names of the ribbon 2-knots:
  - (i) The names Ym_n, Ym_n^* ($m = 2, 3$) in Table 2 denote the knots m_n, m_n^* with ribbon crossing number m in [13]; Ym_n^* is the mirror image of Ym_n.
  - (ii) The name Yn ($1 \le n \le 111$) in Table 3 denotes the ribbon 2-knot K_n^2 with ribbon crossing number four in [15].

- The column, C, shows the chirality of the ribbon 2-knots:
  (i) The symbol "a" means that the ribbon 2-knot is positive-amphicheiral.
  (ii) In Table 3 the mirror image knot is listed.
- The column, Presentation, shows a ribbon handlebody presentation of the ribbon 2-knot: P_i is one of the ribbon handlebody presentations (15)–(20), and the symbols j and $\bar{j}$ ($j = 1, 2, 3, 4, 5$) denote the letters x_j and x_j^{-1}, respectively. For example, $P_2(21, 32)$ for the knot Y43 in Table 3 means the presentation Eq. (11) in Example 3.
- The column, Type, shows the type of the ribbon 2-knot:
  (i) A Type 1 ribbon 2-knot is presented by a 1-fusion notation $R(p_1, q_1, \ldots, p_m, q_m)$.
  (ii) A Type 2 ribbon 2-knot is presented by a composition $R(\epsilon_1, \epsilon_2)\#R(\epsilon_3, \epsilon_4)$, $\epsilon_i = \pm 1$.
- The column, $\Delta(t)$, shows the normalized Alexander polynomial of the ribbon 2-knot in the abbreviated form: $(c_{-m}\, c_{-m+1}\, \ldots\, c_{-1}\, [c_0]\, c_1\, \ldots\, c_{n-1}\, c_n) = \sum_{i=-m}^{n} c_i t^i$, $c_i \in \mathbb{Z}$. We normalize the Alexander polynomial of a ribbon 2-knot $\Delta(t) \in \mathbb{Z}[t^{\pm 1}]$, so that $\Delta(1) = 1$ and $(d/dt)\Delta(1) = 0$; cf. [1].
- The column, Det, shows the determinant of the ribbon 2-knot, which is given by $|\Delta(-1)|$.
- The column, Set, shows the name of the set of the ribbon 2-knots sharing the same Alexander polynomial; $\mathcal{A}_i!$ denotes the set of the mirror images of the knots in $\mathcal{A}_i$. For example, $\mathcal{A}_2 = \{$Y3_1*, Y27$\}$, $\mathcal{A}_2! = \{$Y3_1, Y34$\}$, and the knots in the sets $\mathcal{A}_i$, $i = 1, 5, 6, 9, 13, 15, 17$, have reciprocal Alexander polynomials, and so we do not consider the set of mirror images. The sets $\mathcal{A}_i$ with $i \leq 13$ are the same sets as in [6].

The knot Y112 is missed in [15], which has the same Alexander polynomial as Y109; the ribbon handlebodies are shown as in Fig. 1.

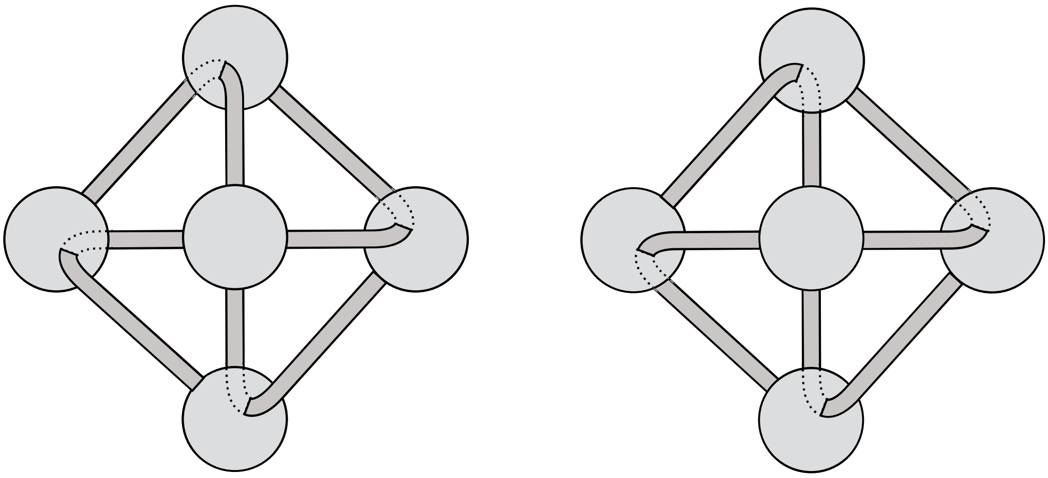

Figure 1: Ribbon handlebodies presenting Y109 and Y112.

Table 2: Ribbon 2-knots with up to three crossings.

Name	C	Presentation	Type	$\Delta(t)$	Det	Set
Y0	a		Trivial knot	([1])	1	$\mathcal{A}_1$
Y2_1	a	$P_1(21)$	$R(1,1)$	(1 [−1] 1)	3	
Y2_2		$P_1(2\bar{1})$	$R(1,-1)$	([0] 2 −1)	3	$\mathcal{A}_3$!
Y2_2*		$P_1(\bar{2}1)$	$R(-1,1)$	(−1 2 [0])	3	$\mathcal{A}_3$
Y3_1		$P_1(211)$	$R(1,2)$	(1 −1 [0] 1)	1	$\mathcal{A}_2$!
Y3_1*		$P_1(221)$	$R(-1,-2)$	(1 [0] −1 1)	1	$\mathcal{A}_2$
Y3_2		$P_1(2\bar{1}\bar{1})$	$R(1,-2)$	([0] 1 1 −1)	1	
Y3_2*		$P_1(\bar{2}11)$	$R(-1,2)$	(−1 1 1 [0])	1	
Y3_3		$P_2(31,2)$	$R(-1,1,1,1)$	(−1 2 −1 [1])	5	
Y3_3*		$P_2(\bar{3}\bar{1},\bar{2})$	$R(1,-1,-1,-1)$	([1] −1 2 −1)	5	
Y3_4		$P_2(31,\bar{2})$	$R(1,1,-1,1)$	(1 −2 [2])	5	
Y3_4*		$P_2(3\bar{1},\bar{2})$	$R(-1,-1,1,-1)$	([2] −2 1)	5	
Y3_5	a	$P_2(3\bar{1},\bar{2})$	$R(1,-1,-1,1)$	(−1 [3] −1)	5	
Y3_6		$P_3(3,4,2)$	$R(-1,1,-1,-1,1,1)$	(1 −3 3 [0])	5	
Y3_6*		$P_3(\bar{3},\bar{4},\bar{2})$	$R(1,-1,1,1,-1,-1)$	([0] 3 −3 1)	5	
Y3_7		$P_3(3,4,\bar{2})$	$R(-1,1,1,-1,1,1)$	(−1 3 [−2] 1)	5	
Y3_7*		$P_3(3,\bar{4},\bar{2})$	$R(1,-1,-1,1,-1,-1)$	(1 [−2] 3 −1)	5	

Table 3: Ribbon 2-knots with four crossings.

Name	C	Presentation	Type	$\Delta(t)$	Det	Set
Y1	Y7	$P_1(2111)$	$R(1,3)$	(1 −1 0 [0] 1)	3	
Y2	a	$P_1(2121)$	$R(1,1,1,1)$	(1 −1 [1] −1 1)	5	
Y3	Y8	$P_1(2\bar{1}\bar{1}\bar{1})$	$R(3,-1)$	([0] 1 0 1 −1)	3	$\mathcal{A}_4$!
Y4	Y9	$P_1(2\bar{1}2\bar{1})$	$R(1,-1,1,-1)$	([0] 0 3 −2)	5	
Y5	a	$P_1(2211)$	$R(2,2)$	(1 0 [−1] 0 1)	1	
Y6	Y10	$P_1(22\bar{1}\bar{1})$	$R(2,-2)$	([0] 0 2 0 -1)	1	
Y7	Y1	$P_1(2221)$	$R(3,1)$	(1 [0] 0 −1 1)	3	
Y8	Y3	$P_1(\bar{2}111)$	$R(-1,3)$	(−1 1 0 1 [0])	3	$\mathcal{A}_4$
Y9	Y4	$P_1(\bar{2}1\bar{2}1)$	$R(-1,1,-1,1)$	(−2 3 0 [0])	5	
Y10	Y6	$P_1(\bar{2}\bar{2}11)$	$R(-2,2)$	(−1 0 2 0 [0])	1	
Y11	Y18	$P_2(231,\bar{2})$	$R(2,1,-1,1)$	(1 0 [−2] 2)	3	
Y12	Y17	$P_2(23\bar{1},\bar{2})$	$R(2,1,-1,-1)$	([1] 1 −2 1)	3	
Y13	Y16	$P_2(2\bar{3}1,\bar{2})$	$R(2,-1,-1,1)$	([0] 2 −1)	3	$\mathcal{A}_3$!
Y14	Y15	$P_2(2\bar{3}\bar{1},\bar{2})$	$R(2,-1,-1,-1)$	([0] 1 0 1 −1)	3	$\mathcal{A}_4$!
Y15	Y14	$P_2(\bar{2}31,2)$	$R(-2,1,1,1)$	(−1 1 0 1 [0])	3	$\mathcal{A}_4$
Y16	Y13	$P_2(\bar{2}3\bar{1},2)$	$R(1,-1,-1,2)$	(−1 2 [0])	3	$\mathcal{A}_3$
Y17	Y12	$P_2(\bar{2}\bar{3}1,2)$	$R(-2,-1,1,1)$	(1 −2 1 [1])	3	
Y18	Y11	$P_2(\bar{2}\bar{3}\bar{1},2)$	$R(-2,-1,1,-1)$	(2 [−2] 0 1)	3	
Y19	Y31	$P_2(311,2)$	$R(-1,1,1,2)$	(−1 2 −1 0 [1])	3	
Y20	Y33	$P_2(313,2)$	$R(-1,1,1,1,-1,1)$	(−1 2 −2 2 [0])	7	
Y21	Y32	$P_2(313,\bar{2})$	$R(1,1,-1,1,1,1)$	(1 −2 2 [−1] 1)	7	
Y22	Y30	$P_2(3\bar{1}3,2)$	$R(1,-1,-1,1,-1,1)$	(−2 4 [−1])	7	
Y23	a	$P_2(3\bar{1}3,\bar{2})$	$R(1,1,-1,-1,1,1)$	(2 [−3] 2)	7	$\mathcal{A}_5$
Y24	Y37	$P_2(321,2)$	$R(-1,1,2,1)$	(−1 2 0 [−1] 1)	1	
Y25	Y36	$P_2(32\bar{1},2)$	$R(1,-2,-1,1)$	(−1 [2] 1 −1)	1	
Y26	Y35	$P_2(3\bar{2}1,\bar{2})$	$R(1,1,-2,1)$	([1])	1	$\mathcal{A}_1$
Y27	Y34	$P_2(3\bar{2}\bar{1},\bar{2})$	$R(1,2,-1,-1)$	(1 [0] −1 1)	1	$\mathcal{A}_2$
Y28	Y39	$P_2(331,2)$	$R(-1,2,1,1)$	(−1 1 1 −1 [1])	1	
Y29	Y38	$P_2(331,\bar{2})$	$R(1,2,-1,1)$	(1 −1 −1 [2])	1	
Y30	Y22	$P_2(\bar{3}1\bar{3},\bar{2})$	$R(1,-1,-1,1,1,-1)$	([−1] 4 −2)	7	
Y31	Y19	$P_2(\bar{3}\bar{1}\bar{1},\bar{2})$	$R(2,1,1,-1)$	([1] 0 −1 2 −1)	3	
Y32	Y21	$P_2(\bar{3}\bar{1}\bar{3},2)$	$R(1,1,1,-1,1,1)$	(1 [−1] 2 −2 1)	7	
Y33	Y20	$P_2(\bar{3}\bar{1}\bar{3},\bar{2})$	$R(1,-1,1,1,1,-1)$	([0] 2 −2 2 −1)	7	
Y34	Y27	$P_2(\bar{3}21,2)$	$R(-1,-1,2,1)$	(1 −1 [0] 1)	1	$\mathcal{A}_2$!
Y35	Y26	$P_2(\bar{3}2\bar{1},2)$	$R(1,-2,1,1)$	([1])	1	$\mathcal{A}_1$
Y36	Y25	$P_2(\bar{3}\bar{2}1,\bar{2})$	$R(1,-1,-2,1)$	(−1 1 [2] −1)	1	
Y37	Y24	$P_2(\bar{3}\bar{2}\bar{1},\bar{2})$	$R(1,2,1,-1)$	(1 [−1] 0 2 −1)	1	
Y38	Y29	$P_2(\bar{3}\bar{3}\bar{1},2)$	$R(-1,-2,1,-1)$	([2] −1 −1 1)	1	
Y39	Y28	$P_2(\bar{3}\bar{3}\bar{1},\bar{2})$	$R(1,-2,-1,-1)$	([1] −1 1 1 −1)	1	

Table 3: Ribbon 2-knots with four crossings (cont'd).

Name	C	Presentation	Type	$\Delta(t)$	Det	Set
Y40	a	$P_2(21, 31)$	$R(1,1)\#R(1,1)$	(1 −2 [3] −2 1)	9	$\mathcal{A}_6$
Y41	Y42	$P_2(21, 3\bar{1})$	$R(1,1)\#R(1,-1)$	([2] −3 3 −1)	9	$\mathcal{A}_7$
Y42	Y41	$P_2(21, \bar{3}1)$	$R(1,1)\#R(-1,1)$	(−1 3 −3 [2])	9	$\mathcal{A}_7$!
Y43	a	$P_2(21, 32)$	$R(-1,-1,1,1,1,1,-1,-1)$	(1 −2 [3] −2 1)	9	$\mathcal{A}_6$
Y44	Y45	$P_2(21, 3\bar{2})$	$R(1,-1,1,1,-1,1,1,-1)$	([2] −3 3 −1)	9	$\mathcal{A}_7$
Y45	Y44	$P_2(21, \bar{3}2)$	$R(-1,1,1,-1,1,1,-1,1)$	(−1 3 −3 [2])	9	$\mathcal{A}_7$!
Y46	a	$P_2(21, \bar{3}\bar{2})$	$R(1,1,1,-1,-1,1,1,1)$	(1 −2 [3] −2 1)	9	$\mathcal{A}_6$
Y47	Y52	$P_2(2\bar{1}, 3\bar{1})$	$R(1,-1)\#R(1,-1)$	([0] 0 4 −4 1)	9	$\mathcal{A}_8$
Y48	a	$P_2(2\bar{1}, \bar{3}1)$	$R(1,-1)\#R(-1,1)$	(−2 [5] −2)	9	$\mathcal{A}_9$
Y49	Y53	$P_2(2\bar{1}, 32)$	$R(-1,-1,1,1,1,-1,-1,-1)$	([2] −3 3 −1)	9	$\mathcal{A}_7$
Y50	Y55	$P_2(2\bar{1}, 3\bar{2})$	$R(1,-1,1,1,-1,-1,1,-1)$	([0] 0 4 −4 1)	9	$\mathcal{A}_8$
Y51	Y54	$P_2(2\bar{1}, \bar{3}2)$	$R(-1,1,1,-1,1,-1,-1,1)$	(−2 [5] −2)	9	$\mathcal{A}_9$
Y52	Y47	$P_2(\bar{2}1, \bar{3}1)$	$R(-1,1)\#R(-1,1)$	(1 −4 4 0 [0])	9	$\mathcal{A}_8$!
Y53	Y49	$P_2(\bar{2}1, 32)$	$R(-1,-1,-1,1,1,1,-1,-1)$	(−1 3 −3 [2])	9	$\mathcal{A}_7$!
Y54	Y51	$P_2(\bar{2}1, 3\bar{2})$	$R(1,-1,-1,1,-1,1,1,-1)$	(−2 [5] −2)	9	$\mathcal{A}_9$
Y55	Y50	$P_2(\bar{2}1, \bar{3}2)$	$R(-1,1,-1,-1,1,1,-1,1)$	(1 −4 4 0 [0])	9	$\mathcal{A}_8$!
Y56	Y58	$P_3(3, 1\bar{4}, \bar{2})$	$R(1,1,-1,1,-1,-1)$	(2 [−3] 2)	7	$\mathcal{A}_5$
Y57	a	$P_3(3, \bar{1}\bar{4}, 2)$	$R(1,-1,-1,-1,-1,1)$	(−1 2 [−1] 2 −1)	7	
Y58	Y56	$P_3(\bar{3}, \bar{1}4, 2)$	$R(-1,-1,1,-1,1,1)$	(2 [−3] 2)	7	$\mathcal{A}_5$
Y59	Y66	$P_4(31, 4, 2)$	$R(-1,-1,1,1,-1,1,1,1)$	(1 −3 3 −1 [1])	9	
Y60	Y64	$P_4(31, 4, \bar{2})$	$R(1,-1,-1,1,1,1,-1,1)$	(−1 3 −3 [2])	9	$\mathcal{A}_7$!
Y61	Y63	$P_4(31, \bar{4}, \bar{2})$	$R(1,1,-1,1,1,-1,-1,1)$	(1 −3 [4] −1)	9	$\mathcal{A}_{10}$!
Y62	Y65	$P_4(3\bar{1}, 4, 2)$	$R(-1,-1,1,1,-1,1,1,-1)$	(1 −3 [4] −1)	9	$\mathcal{A}_{10}$!
Y63	Y61	$P_4(3\bar{1}, 4, \bar{2})$	$R(1,-1,-1,1,1,1,-1,-1)$	(−1 [4] −3 1)	9	$\mathcal{A}_{10}$
Y64	Y60	$P_4(3\bar{1}, \bar{4}, \bar{2})$	$R(1,1,-1,1,1,-1,-1,-1)$	([2] −3 3 −1)	9	$\mathcal{A}_7$
Y65	Y62	$P_4(\bar{3}1, \bar{4}, \bar{2})$	$R(1,1,-1,-1,1,-1,-1,1)$	(−1 [4] −3 1)	9	$\mathcal{A}_{10}$
Y66	Y59	$P_4(\bar{3}1, \bar{4}, 2)$	$R(1,1,-1,-1,1,-1,-1,-1)$	([1] −1 3 −3 1)	9	
Y67	Y82	$P_3(23, 4, 1)$	$R(1,-1,-1,-1,1,1,1,1,-1,-1)$	(1 [−2] 4 −3 1)	11	$\mathcal{A}_{11}$!
Y68	Y81	$P_3(23, 4, \bar{1})$	$R(1,1,-1,-1,1,1,1,-1,-1,-1)$	([0] 3 −4 3 −1)	11	
Y69	Y80	$P_3(23, \bar{4}, 1)$	$R(1,-1,-1,-1,-1,1,1,1,-1,-1)$	(−1 3 [−3] 3 −1)	11	$\mathcal{A}_{13}$
Y70	Y79	$P_3(23, \bar{4}, \bar{1})$	$R(1,1,-1,-1,-1,1,1,-1,-1,-1)$	(1 [−2] 4 −3 1)	11	$\mathcal{A}_{11}$!
Y71	Y78	$P_3(2\bar{3}, 4, 1)$	$R(-1,-1,1,-1,1,1,-1,1,1,-1)$	(2 [−4] 4 -1)	11	$\mathcal{A}_{12}$
Y72	Y77	$P_3(2\bar{3}, 4, \bar{1})$	$R(-1,1,1,-1,1,1,-1,-1,1,-1)$	([-1] 5 −4 1)	11	
Y73	Y76	$P_3(2\bar{3}, \bar{4}, 1)$	$R(-1,-1,1,-1,-1,1,-1,1,1,-1)$	(−2 5 [−3] 1)	11	
Y74	Y75	$P_3(2\bar{3}, \bar{4}, \bar{1})$	$R(-1,1,1,-1,-1,1,-1,-1,1,-1)$	(2 [−4] 4 −1)	11	$\mathcal{A}_{12}$
Y75	Y74	$P_3(\bar{2}3, 4, 1)$	$R(1,-1,-1,1,1,-1,1,1,-1,1)$	(−1 4 [−4] 2)	11	$\mathcal{A}_{12}$!
Y76	Y73	$P_3(\bar{2}3, 4, \bar{1})$	$R(1,1,-1,1,1,-1,1,-1,-1,1)$	(1 [−3] 5 −2)	11	
Y77	Y72	$P_3(\bar{2}3, \bar{4}, 1)$	$R(1,-1,-1,1,-1,-1,1,1,-1,1)$	(1 −4 5 [-1])	11	
Y78	Y71	$P_3(\bar{2}3, \bar{4}, \bar{1})$	$R(1,1,-1,1,-1,-1,1,-1,-1,1)$	(−1 4 [−4] 2)	11	$\mathcal{A}_{12}$!
Y79	Y70	$P_3(\bar{2}\bar{3}, 4, 1)$	$R(-1,-1,1,1,1,-1,-1,1,1,1)$	(1 −3 4 [−2] 1)	11	$\mathcal{A}_{11}$
Y80	Y69	$P_3(\bar{2}\bar{3}, 4, \bar{1})$	$R(-1,1,1,1,1,-1,-1,-1,1,1)$	(−1 3 [−3] 3 −1)	11	$\mathcal{A}_{13}$
Y81	Y68	$P_3(\bar{2}\bar{3}, \bar{4}, 1)$	$R(-1,-1,1,1,-1,-1,-1,1,1,1)$	(−1 3 −4 3 [0])	11	
Y82	Y67	$P_3(\bar{2}\bar{3}, \bar{4}, \bar{1})$	$R(-1,1,1,1,-1,-1,-1,-1,1,1)$	(1 −3 4 [−2] 1)	11	$\mathcal{A}_{11}$

Table 3: Ribbon 2-knots with four crossings (cont'd).

Name	C	Presentation	Type	$\Delta(t)$	Det	Set
Y83	Y90	$P_3(43,1,2)$	$R(1,1,1,1,-1,-1,-1,1)$	(1 −2 [3] −2 1)	9	$\mathcal{A}_6$
Y84	Y89	$P_3(43,1,\bar{2})$	$R(1,-1,1,1,-1,1,-1,-1)$	([3] −4 2)	9	
Y85	Y88	$P_3(43,\bar{1},2)$	$R(1,1,1,-1,-1,-1,-1,1)$	(−1 [3] −2 2 −1)	9	
Y86	Y87	$P_3(43,\bar{1},\bar{2})$	$R(1,-1,1,-1,-1,1,-1,-1)$	([1] −2 4 −2)	9	
Y87	Y86	$P_3(\overline{43},1,2)$	$R(-1,1,-1,1,1,-1,1,1)$	(−2 4 −2 [1])	9	
Y88	Y85	$P_3(\overline{43},1,\bar{2})$	$R(-1,-1,-1,1,1,1,1,-1)$	(−1 2 −2 [3] −1)	9	
Y89	Y84	$P_3(\overline{43},\bar{1},2)$	$R(-1,1,-1,-1,1,-1,1,1)$	(2 −4 [3])	9	
Y90	Y83	$P_3(\overline{43},\bar{1},\bar{2})$	$R(-1,-1,-1,-1,1,1,1,-1)$	(1 −2 [3] −2 1)	9	$\mathcal{A}_6$
Y91	Y106	$P_5(5,2,3,4)$	$R(-1,-1,1,-1,-1,1,1,-1,1,1,-1,1)$	(−1 4 −5 [3])	13	$\mathcal{A}_{14}$
Y92	Y105	$P_5(5,2,3,\bar{4})$	$R(-1,-1,1,1,-1,1,1,-1,1,-1,-1,1)$	(1 −4 [6] −2)	13	
Y93	Y104	$P_5(5,2,\bar{3},4)$	$R(-1,1,1,-1,-1,-1,1,1,1,1,-1,-1)$	(1 −3 [5] −3 1)	13	$\mathcal{A}_{15}$
Y94	Y103	$P_5(5,2,\bar{3},\bar{4})$	$R(-1,1,1,1,-1,-1,1,1,1,-1,-1,-1)$	(−1 [4] −4 3 −1)	13	$\mathcal{A}_{16}$
Y95	Y102	$P_5(5,\bar{2},3,4)$	$R(1,-1,-1,-1,1,1,1,-1,-1,1,1,1)$	(1 −3 [5] −3 1)	13	$\mathcal{A}_{15}$
Y96	Y101	$P_5(5,\bar{2},3,\bar{4})$	$R(1,-1,-1,1,1,1,1,-1,-1,-1,1,1)$	(−1 [4] −4 3 −1)	13	$\mathcal{A}_{16}$
Y97	Y100	$P_5(5,\bar{2},\bar{3},4)$	$R(1,1,-1,-1,1,-1,1,1,-1,1,1,-1)$	([3] −5 4 −1)	13	$\mathcal{A}_{14}$!
Y98	Y99	$P_5(5,\bar{2},\bar{3},\bar{4})$	$R(1,1,-1,1,1,-1,1,1,-1,-1,1,-1)$	([1] −2 5 −4 1)	13	
Y99	Y98	$P_5(\bar{5},2,3,4)$	$R(-1,-1,1,-1,-1,1,-1,-1,1,1,-1,1)$	(1 −4 5 −2 [1])	13	
Y100	Y97	$P_5(\bar{5},2,3,\bar{4})$	$R(-1,-1,1,1,-1,1,-1,-1,1,-1,-1,1)$	(−1 4 −5 [3])	13	$\mathcal{A}_{14}$
Y101	Y96	$P_5(\bar{5},2,\bar{3},4)$	$R(-1,1,1,-1,-1,-1,-1,1,1,1,-1,-1)$	(−1 3 −4 [4] −1)	13	$\mathcal{A}_{16}$!
Y102	Y95	$P_5(\bar{5},2,\bar{3},\bar{4})$	$R(-1,1,1,1,-1,-1,-1,1,1,-1,-1,-1)$	(1 −3 [5] −3 1)	13	$\mathcal{A}_{15}$
Y103	Y94	$P_5(\bar{5},\bar{2},3,4)$	$R(1,-1,-1,-1,1,1,-1,-1,-1,1,1,1)$	(−1 3 −4 [4] −1)	13	$\mathcal{A}_{16}$!
Y104	Y93	$P_5(\bar{5},\bar{2},3,\bar{4})$	$R(1,-1,-1,1,1,1,-1,-1,-1,-1,1,1)$	(1 −3 [5] −3 1)	13	$\mathcal{A}_{15}$
Y105	Y92	$P_5(\bar{5},\bar{2},\bar{3},4)$	$R(1,1,-1,-1,1,-1,-1,1,-1,1,1,-1)$	(−2 [6] −4 1)	13	
Y106	Y91	$P_5(\bar{5},\bar{2},\bar{3},\bar{4})$	$R(1,1,-1,1,1,-1,-1,1,-1,-1,1,-1)$	([3] −5 4 −1)	13	$\mathcal{A}_{14}$!
Y107	Y111	$P_6(3,4,5,2)$	$R(-1,-1,1,-1,-1,1,1,1,-1,-1,1,1,-1,1)$	(−1 4 −6 4 [0])	15	
Y108	Y110	$P_6(3,4,5,\bar{2})$	$R(1,-1,-1,-1,1,1,-1,1,1,-1,-1,1,1,1)$	(1 −4 6 [−3] 1)	15	
Y109	a	$P_6(3,4,\bar{5},\bar{2})$	$R(1,1,-1,-1,1,-1,-1,1,1,1,-1,1,1,-1)$	(−1 4 [−5] 4 −1)	15	$\mathcal{A}_{17}$
Y110	Y108	$P_6(3,\bar{4},\bar{5},\bar{2})$	$R(-1,1,1,1,-1,-1,1,-1,-1,1,1,-1,-1,-1)$	(1 [−3] 6 −4 1)	15	
Y111	Y107	$P_6(\bar{3},\bar{4},\bar{5},\bar{2})$	$R(1,1,-1,1,1,-1,-1,-1,1,1,-1,-1,1,-1)$	([0] 4 −6 4 −1)	15	
Y112	a	$P_6(3,\bar{4},5,\bar{2})$	$R(1,-1,-1,1,1,1,-1,1,1,-1,-1,-1,1,1)$	(−1 4 [−5] 4 −1)	15	$\mathcal{A}_{17}$

5 Classification of the knots

The ribbon 2-knots in the sets $\mathcal{A}_i$, $i = 1, 2, \ldots, 13$, have been classified in [6] except for the pair Y43 and Y46 in $\mathcal{A}_6$, which have isomorphic knot group; see Sect. 7 in [6], where Y43 $= R^8_{8,6}$ and Y46 $= R^8_{8,1}$. In this section we classify the ribbon 2-knots in each of the sets $\mathcal{A}_i$, $i = 14, 15, 16, 17$.

5.1 Classification of the knots in $\mathcal{A}_{14}$

The set $\mathcal{A}_{14}$ consists of the two knots Y91 and Y100, which share the same Alexander polynomial $-t^{-3}+4t^{-2}-5t^{-1}+3$. Since they have different trace sets as shown in Table 4, we obtain Y91 $\not\approx$ Y100.

Table 4: Trace sets of the knots in $\mathcal{A}_{14}$, $\mathcal{A}_{16}$, and $\mathcal{A}_{17}$.

Set	Knot	Trace set
$\mathcal{A}_{14}$	Y91	$\{0,0,0,0,0,0,(\delta+\epsilon\sqrt{5})/2 \mid \delta,\epsilon=\pm1\}$
	Y100	$\left\{0,0,0,0,0,0,\delta\sqrt{5(3+\epsilon\sqrt{3})/6} \,\middle|\, \delta,\epsilon=\pm1\right\}$
$\mathcal{A}_{16}$	Y94	$\{0,0,0,0,0,0\}$
	Y96	$\{0,0,0,0,0,0,\pm\alpha_1,\pm\alpha_2,\pm\alpha_3\}$
$\mathcal{A}_{17}$	Y109	$\{\mathbb{C}-\{\pm\sqrt{3}\},\ \pm\sqrt{2},\ 0,0,0,0,0,0,0,0,\ \mathbb{C}-\{\pm\sqrt{5}\},\ \mathbb{C}-\{\pm\sqrt{5}\},\ \pm1,\ (\delta+\epsilon\sqrt{13})/2,\ (\delta+\epsilon\sqrt{13})/2\ (\delta,\ \epsilon=\pm1),\ \pm\beta_1,\ \pm\beta_2,\ \pm\beta_3,\ \pm\beta_4\}$
	Y112	$\{\mathbb{C}-\{\pm\sqrt{3}\},\ \pm\sqrt{2},\ 0,0,0,0,0,0,0,0,\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4\}$

- The numbers α_k, $k = 1, 2, 3$, are the roots of the cubic equation $1 - x - 2x^2 + x^3 = 0$ with $-1 < \alpha_1 < 0 < \alpha_2 < 1$, $2 < \alpha_3 < 3$.
- The complex numbers β_k, $k = 1, 2, 3, 4$, are the roots of the quartic equation $5 - 2x - 4x^2 + x^3 + x^4 = 0$; $\beta_1, \beta_2 \fallingdotseq 1.25 \pm 0.27i$, $\beta_3, \beta_4 \fallingdotseq -1.75 \pm 0.17i$.
- The complex numbers γ_k, $k = 1, 2, 3, 4$, are the roots of the quartic equation $5 - 4x^2 + x^4 = 0$; $\gamma_k \fallingdotseq \pm 1.46 \pm 0.34i$.

5.2 Classification of the knots in $\mathcal{A}_{15}$

The set $\mathcal{A}_{15}$ consists of the four knots Y93, Y95, Y102(= Y95!), and Y104(= Y93!), which share the same Alexander polynomial $t^{-2} - 3t^{-1} + 5 - 3t + t^2$. Table 5 lists the trace sets of the irreducible representations to $SL(2,\mathbb{C})$ of the knot groups of Y93 and Y95, and the associated twisted Alexander polynomials, which show these four knots are mutually non-isotopic. In fact, since the twisted Alexander polynomials are not reciprocal, the knots Y93 and Y95 are not positive-amhicheiral.

Table 5: Twisted Alexander polynomials of Y93 and Y95 in $\mathcal{A}_{15}$.

Set	Knot	$(s+s^{-1},u)$	Twisted Alexander polynomial
$\mathcal{A}_{15}$	Y93	$\left(\frac{\epsilon}{\sqrt{2}},\frac{3}{2}\right)$ $(\epsilon=\pm 1)$	$1-\epsilon\sqrt{2}t+\frac{5}{2}t^2-\epsilon\frac{3}{\sqrt{2}}t^3+\frac{5}{2}t^4-\epsilon\sqrt{2}t^5+t^6$
		$(0,\alpha_1)$	$1+\beta_1 t^2+\gamma_2 t^4+t^6$
		$(0,\alpha_2)$	$1+\beta_5 t^2+\gamma_1 t^4+t^6$
		$(0,\alpha_3)$	$1+\beta_4 t^2+\gamma_1 t^4+t^6$
		$(0,\alpha_4)$	$1+\beta_2 t^2+\gamma_3 t^4+t^6$
		$(0,\alpha_5)$	$1+\beta_3 t^2+\gamma_2 t^4+t^6$
		$(0,\alpha_6)$	$1+\beta_6 t^2+\gamma_3 t^4+t^6$
	Y95	$(0,\alpha_1)$	$1+\beta_1 t^2+\beta_3 t^4+t^6$
		$(0,\alpha_2)$	$1+\beta_2 t^2+\beta_1 t^4+t^6$
		$(0,\alpha_3)$	$1+\beta_5 t^2+\beta_4 t^4+t^6$
		$(0,\alpha_4)$	$1+\beta_6 t^2+\beta_2 t^4+t^6$
		$(0,\alpha_5)$	$1+\beta_6 t^2+\beta_4 t^4+t^6$
		$(0,\alpha_6)$	$1+\beta_5 t^2+\beta_3 t^4+t^6$

- The numbers α_k, $k=1,\dots,6$, are the roots of the 6th order equation $13-91x+182x^2-156x^3+65x^4-13x^5+x^6=0$ with $0<\alpha_1<0.5<\alpha_2<1<\alpha_3<2<\alpha_4<3<\alpha_5<3.5<\alpha_6<4$.
- The numbers β_k, $k=1,\dots,6$, are the roots of the 6th order equation $-1-81x+201x^2-178x^3+73x^4-14x^5+x^6=0$ with $-1<\beta_1<0<\beta_2<1$, $2<\beta_3<2.4<\beta_4<2.8<\beta_5<3$, $5<\beta_6<6$.
- The numbers γ_k, $k=1,2,3$, are the roots of the cubic equation $-5+12x-7x^2+x^3=0$ with $0<\gamma_1<1<\gamma_2<2$, $4<\gamma_3<5$.

5.3 Classification of the knots in $\mathcal{A}_{16}$

The set $\mathcal{A}_{16}$ consists of the two knots Y94 and Y96, which share the same Alexander polynomial $-t^{-1}+4-4t+3t^2-t^3$. Since they have different trace sets as shown in Table 4, we obtain Y94 $\not\approx$ Y96.

5.4 Classification of the knots in $\mathcal{A}_{17}$

The set $\mathcal{A}_{17}$ consists of the two knots Y109 and Y112, which share the same Alexander polynomial $-t^{-2}+4t^{-1}-5+4t-t^2$. Since they have different trace sets as shown in Table 4, we obtain Y109 $\not\approx$ Y112.

Remark 6. According to Toshio Sumi [9], we can also distinguish the knots Y109 and Y112 in the following ways:

(i) They have distinct twisted Alexander polynomials associated to the nonabelian representations to $\mathrm{SL}(2,2)$ as listed in Table 6.

(ii) They have distinct numbers of the irreducible representations to $\mathrm{SL}(2,7)$.

Table 6: Twisted Alexander polynomials of the knots in $\mathcal{A}_{17}$. .

Set	Knot	$\rho : \pi K \to \mathrm{SL}(2,2)$	$\Delta_{K,\rho}$
$\mathcal{A}_{17}$	Y109	$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, y \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$1+t^6$
	Y112	$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, y \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$1+t^2+t^4+t^6$

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