Cohen real or random real: effect on strong measure zero sets and strongly meager sets

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Abstract

We show that the set of the ground-model reals has strong measure zero (is strongly meager) after adding a single Cohen real (random real). As consequence we prove that the set of the ground-model reals has strong measure zero after adding a single Hechler real.

1 Introduction

Let $\mathcal{N}$ be the σ-ideal of measure zero subsets of 2^ω, and let $\mathcal{M}$ be the σ-ideal of meager sets in 2^ω. More concretely $X \in \mathcal{M}$ if there is some sequence $\langle F_n : n < \omega \rangle$ such that $X = \bigcup_{n<\omega} F_n$ and $\mathrm{int}(\mathrm{cl}(F_n)) = \emptyset$. Let $\mathbb{C}$ and $\mathbb{B}$ be the Cohen algebra and random algebra respectively, let $\mathbb{D}$ be the Hechler forcing, let $\mathbb{L}$ be the Laver forcing, let $\mathbb{M}$ be Mathias forcing, let $\mathbb{V}$ be Silver forcing and let $\mathbb{S}$ be Sacks forcing.

Definition 1.1. For each $\sigma \in (2^{<\omega})^\omega$ define $\mathrm{ht} \in \omega^\omega$ by $\mathrm{ht}_\sigma(n) := |\sigma(n)|$.

Say that $X \subseteq 2^\omega$ *has strong measure zero* ($X \in \mathcal{SN}$) if, for each function $f \in \omega^\omega$ there is some $\sigma \in (2^{<\omega})^\omega$ with $\mathrm{ht}_\sigma = f$ such that $X \subseteq \bigcup_{n<\omega} [\sigma(n)]$.

It is clear that $\mathcal{SN} \subseteq \mathcal{N}$.

Galvin, Mycielski and Solovay [GMS73] gave a very important description of the strong measure zero sets.

Theorem 1.2 ([GMS73]). *The following are equivalent:*

(1) $X \in \mathcal{SN}$,

(2) for every set $F \in \mathcal{M}$, *there is some* $x \in 2^\omega$ *such that* $(x + X) \cap F = \emptyset$.

Using this characterization, we consider the following objects.

Definition 1.3. We say that $X \subseteq 2^\omega$ *is strongly meager* ($X \in \mathcal{SM}$) if, for each $N \in \mathcal{N}$, there is $x \in 2^\omega$ such that $(X + x) \cap N = \emptyset$.

It is clear that $\mathcal{SM} \subseteq \mathcal{M}$.

Kunen [Kun84] proved that after adding a single Cohen real (random real) the set of the ground-model reals becomes null (meager). More presicely,

Theorem 1.4 ([Kun84]). *If c and r are a Cohen real and a random real over V respectively, then*

(i) $V[c] \models 2^\omega \cap V \in \mathcal{N}$ *and* $2^\omega \cap V \notin \mathcal{M}$. *In particular,* $V[c] \models 2^\omega \cap V \notin \mathcal{SM}$.

(ii) $V[r] \models 2^\omega \cap V \in \mathcal{M}$ *and* $2^\omega \cap V \notin \mathcal{N}$. *In particular,* $V[r] \models 2^\omega \cap V \notin \mathcal{SN}$.

Motivated by Theorem 1.4. in this paper we prove that the set of the ground-model reals has strong measure zero after adding a single Cohen real. This was mentioned by Laver [Lav76] (without proof), afterwards, Goldstern sketched this in [Gol11]. We also prove that the set of the ground-model reals is strongly meager after adding a single random real. This was sketched in [Wei13]. The author present a complete proof of these results with some slight variations associated with his perpective.

2 Main result

This section is dedicated to prove the following main result.

Theorem A. *If c and r are a Cohen real and a random real over V respectively, then*

(i) $V[c] \models 2^\omega \cap V \in \mathcal{SN}$.

(ii) $V[r] \models 2^\omega \cap V \in \mathcal{SM}$.

Proof. (i) Enumerate $2^{<\omega} := \{r_n : n < \omega\}$. For each $f \in \omega^\omega$ and $F \in \omega^\omega$ define

$$B^c_{f,F} := \bigcup_{n\in\omega} [r_{c(F(n))}{}^\frown\langle 0,\dots,0\rangle]$$

where for each n, the length of $\langle 0,\dots,0\rangle$ is the greatest between $f(n) - |r_{c_{F(n)}}|$ and 0. Note that $B^c_{f,F}$ is coded in $V[c]$. It is enough to prove that, for any $\mathbb{C}$-name $\dot{f}$ in ω^ω there is a function $F \in \omega^\omega$ such that $\Vdash_{\mathbb{C}} 2^\omega \cap V \subseteq B^{\dot{c}}_{\dot{f},F}$.

In V define a function $F_p \in \omega^\omega$ for each $p \in \mathbb{C}$ by

$$F_p(m) := \min\Big\{k \in \omega : \exists q \in \mathbb{C}(|q| = k \wedge q \leq p \wedge \exists l < \omega(q \Vdash \dot{f}(m) = l))\Big\},$$

Choose $F \in \omega^\omega$ such that $F_p \leq^* F$ for all $p \in \mathbb{C}$. It remains to check that $\Vdash_{\mathbb{C}} 2^\omega \cap V \subseteq B^{\dot{c}}_{\dot{f},F}$. To do this, let p be an arbitrary condition in $\mathbb{C}$. Choose $n < \omega$ such that $F_p(m) \leq F(m)$ for all $m \geq n$. Now choose $q \in \mathbb{C}$ with $|q| = F_p(n)$ and $l < \omega$ such that q extends p and $q \Vdash \dot{f}(n) = l$. Let $x \in 2^\omega \cap V$. Find $i < \omega$ such that $r_i := x \restriction l$. Define a condition $q^* \in \mathbb{C}$ such that $|q^*| = F(n) + 1$, $q^* \Vdash \dot{c}(F(n)) = i$ and $q^* \leq q$.

Then, $q^* \Vdash x \in [r_{\dot{c}(F(n))}] \subseteq B^{\dot{c}}_{\dot{f},F}$ (this contention holds because $|r_{\dot{c}(F(n))}| = l = \dot{f}(n)$).

(ii) For an increasing function $f \in \omega^\omega$ and a function $x \in 2^\omega$ define $x_f \in 2^\omega$ as $x_f(n) := x(f(n))$ for $n \in \omega$. Let A be a Borel set in $V[r] \cap \mathcal{N}$. In V find a Borel null set such that $B \subseteq 2^\omega \times 2^\omega$ and $A = B_r$. Since B has measure zero, choose sequences $s_n, t_n \in 2^{<\omega}$ with $|s_n| = |t_n|$ such that

$$B \subseteq \bigcap_{m<\omega} \bigcup_{n \geq m} [s_n] \times [t_n] \text{ and } \sum_{n=1}^{\infty} 2^{-2|s_n|} < \infty.$$

Find an increasing function $f \in \omega^\omega$ by induction on n such that

(a) $j \leq f(n) \to |s_j| < f(n+1)$.

(b) $\sum_{j \geq f(n)} \mathbf{Lb}([s_j] \times [t_j]) \leq \frac{\mathbf{Lb}([s_n] \times [t_n])}{2^{n+2}}$

From (a) and (b) it follows that

$$(\star) \quad \sum_{f(n) \leq j < f(n+1)} \frac{2^{|f^{-1}[|s_j|]|}}{2^{2|s_j|}} \leq \sum_{f(n) \leq j < f(n+1)} \frac{2^{n+2}}{2^{2|s_j|}} \leq \mathbf{Lb}([s_n] \times [t_n]).$$

We first show that, for each $z \in V \cap 2^\omega$,

$$\left\{ x : \langle x, x_f + z \rangle \in \bigcap_{m<\omega} \bigcup_{n \geq m} [s_n] \times [t_n] \right\}$$

has measure zero. To this end, let

$$H_n^z := \left\{ x : \langle x, x_f \rangle \in [s_n] \times [z \restriction |t_n| + t_n] \right\}$$

Then we have

$$\left\{ x : \langle x, x_f + z \rangle \in \bigcap_{m<\omega} \bigcup_{n \geq m} [s_n] \times [t_n] \right\} = \bigcap_{m<\omega} \bigcup_{n \geq m} H_n^z$$

It remains to prove that $\bigcap_{m<\omega} \bigcup_{n \geq m} H_n^z$ has measure zero.

Claim 2.1.

$$\mathbf{Lb}(H_n^z) \leq \frac{2^{|f^{-1}(|s_n|)|}}{2^{2|s_n|}}$$

Proof. Note that $H_n^z = [s_n] \cap [(z \restriction |t_n| + t_n) \circ f^{-1}]$. Let $t' := (z \restriction |t_n| + t_n) \circ f^{-1}$. Then $H_n^z = [s_n] \cap [(z \restriction |t_n| + t_n) \circ f^{-1}] = \emptyset$ when s_n and t' are incompatible. Otherwise,

$$\begin{aligned} H_n^z &= [s_n \cup ((z \restriction |t_n| + t_n) \circ f^{-1})] \\ &= [s_n \cup t'] \end{aligned}$$

Hence,

$$\begin{aligned}\mathbf{Lb}\Big([s_n \cup t']\Big) &= 2^{-|s_n \cup t'|}\\ &= 2^{-|s_n|-|\{f(n): n<|t_n| \wedge f(n)\geq |s_n|\}|}\\ &\leq 2^{-|s_n|-|t_n|+|f^{-1}[|s_n|]}\\ &= \frac{2^{|f^{-1}(|s_n|)|}}{2^{2|s_n|}}.\end{aligned}$$

This ends the proof of Claim 2.1. □

We continue the proof of (ii). It follows that $\bigcap_{m<\omega}\bigcup_{n\geq m} H^z_n$ has measure zero by the Claim 2.1 and ($\star$). In $V[r]$, since r is a random real over V, $\langle r, r_f + z\rangle \notin B$, which means that $r_f + z \notin A$. Therefore $(2^\omega \cap V) + A \neq 2^\omega$ in $V[r]$. □

As a consequence of Theorem A, we get that the set of the ground-model reals has strong measure zero after adding a single Hechler real.

Corollary 2.2. *If d is a Hechler real, then $V[d] \models 2^\omega \cap V \in \mathcal{SN}$.*

Palumbo [Pal13] proved that $\mathbb{D} * \mathbb{C} \equiv \mathbb{D}$, that is, $V[d'][c] = V[d]$ for some $\mathbb{D}$-generic real d' over V and a Cohen real c over $V[d']$. By Theorem A, $V[d'][c] \models 2^\omega \cap V[d'] \in \mathcal{SN}$, in particular $V[d'][c] \models 2^\omega \cap V \in \mathcal{SN}$. Then $V[d] \models 2^\omega \cap V \in \mathcal{SN}$.

The next result appears implicit in [JMS92].

Theorem 2.3. *If G and G' are a $\mathbb{V}$-generic over V and a $\mathbb{S}$-generic over V respectively, then*

(a) $V[G] \models 2^\omega \cap V \notin \mathcal{N} \cup \mathcal{M}$, in particular, $V[G] \models 2^\omega \cap V \notin \mathcal{SN} \cup \mathcal{SM}$.

(b) $V[G'] \models 2^\omega \cap V \notin \mathcal{N} \cup \mathcal{M}$, in particular, $V[G'] \models 2^\omega \cap V \notin \mathcal{SN} \cup \mathcal{SM}$.

Miller [Mil81] introduced the infinitely often equal real forcing $\mathbb{I}$ to prove that some combinatorial properties of measure and category of the real line are consistent. He also proved that the set of ground-model reals does not become meager (strongly null) after adding a single infinitely often equal real, in particular, the ground-model real does not become strongly meager. To summarize,

Theorem 2.4. *If G is $\mathbb{I}$-generic over V, then*

(i) $V[G] \models 2^\omega \cap V \notin \mathcal{SN}$, and

(ii) $V[G] \models 2^\omega \cap V \notin \mathcal{SM}$.

We finish this section with results related to the Laver property.

Theorem 2.5 ([BJ94],[BJ95, Theorem 8.5.20])**.** *Assume that $\mathbb{P}$ has the Laver property. Then $\Vdash_{\mathbb{P}} 2^\omega \cap V \notin \mathcal{SM}$.*

As a corollary we get

Corollary 2.6. *If G and G' are $\mathbb{M}$-generic over V and $\mathbb{L}$-generic over V respectively, then* *(i)* $V[G] \models 2^\omega \cap V \notin \mathcal{SM}$.

(ii) $V[G'] \models 2^\omega \cap V \notin \mathcal{SM}$.

On the other hand, Laver [Lav76] proved that adding an $\mathbb{M}$-generic over the ground-model V forces all uncountable sets of reals in V to not have strong measure zero in the extension, that is, $V[G] \models 2^\omega \cap V \notin \mathcal{SN}$.

It is known that the set of the ground-model reals does not have measure zero after adding a $\mathbb{L}$-generic over V, that is, $V[G] \models 2^\omega \cap V \notin \mathcal{N}$, in particular $V[G] \models 2^\omega \cap V \notin \mathcal{SN}$.

Open problems

Miller [Mil81] proved that, if c is a Cohen real over V and r is a random real over $V[c]$, then $V[c][r] \models 2^\omega \cap V[r] \notin \mathcal{M}$, in particular $V[c][r] \models 2^\omega \cap V[r] \notin \mathcal{SM}$. Afterwards, Cichoń and Palikowski [CP86] proved that, if r is a random real over V and c is a Cohen over $V[r]$, then $V[r][c] \models 2^\omega \cap V[c] \in \mathcal{N}$. Later Palikowski [Paw86] proved that

(i) If r is a random real over V and c is a Cohen over $V[r]$, then

$$V[r][c] \models 2^\omega \cap V[c] \notin \mathcal{M}.$$

In particular, $V[r][c] \models 2^\omega \cap V[c] \notin \mathcal{SM}$.

(ii) If c is a Cohen real over V and r is a random over $V[c]$, then

$$V[c][r] \models 2^\omega \cap V[r] \in \mathcal{N}.$$

We ask the following problems.

Question 2.7. *If c is a Cohen real over V and r is a random over $V[c]$, does*

$$V[c][r] \models 2^\omega \cap V[r] \in \mathcal{SN}?$$

Question 2.8. *If r is a random real over V and c is a Cohen over $V[r]$, does*

$$V[r][c] \models 2^\omega \cap V[c] \in \mathcal{SN}?$$

In Corollary 2.2 it was proved that the ground-model real become strongly null after adding a single Hechler real, but it is still open the following question.

Question 2.9. *If d is a Hechler real over V, does $V[d] \models 2^\omega \cap V \in \mathcal{SM}$?*

It is known that

(a) $\Vdash 2^\omega \cap V \in \mathcal{N}$.

(b) $\Vdash 2^\omega \cap V \in \mathcal{M}$.

for the following posets:

(1) The eventually different real forcing $\mathbb{E}$.

(2) The localization forcing $\mathbb{LOC}$.

(3) Amoeba forcing $\mathbb{A}$.

It is natural to ask:

Question 2.10. *For the posets in the list above do we have*

(i) $\Vdash 2^\omega \cap V \in \mathcal{SN}$*?*

(ii) $\Vdash 2^\omega \cap V \in \mathcal{SM}$*?*

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