Forcing continuous epsilon-chains with finite side conditions

Tadatoshi Miyamoto

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Abstract

We introduce a poset that generically adds a continuously increasing epsilon-chain of a length the least uncountable cardinal. This poset is similar to a poset that consists of finite conditions and generically adds a closed cofinal subset of the least uncountable cardinal. As an application, we consider a poset for the Strong Reflection Principle of S. Todorcevic along this line.

Introduction

Let us review a poset that generically adds a closed cofinal subset of ω_1 by finite conditions.

Definition. Let $p \in P$, if

- p is a finite partial function from ω_1 to ω_1.
- If $i \in \mathrm{dom}(p)$, then $i \leq p(i)$.
- If $i_1, i_2 \in \mathrm{dom}(p)$ with $i_1 < i_2$, then $p(i_1) < i_2$.

For $p, q \in P$, let $q \leq p$ in P, if $q \supseteq p$.

Hence if

$$p \in P,$$

$$\mathrm{dom}(p) = \{x_0 < x_1 < \cdots < x_{k-1}\},$$

$$p = \big\{\big(x_0, p(x_0)\big), \big(x_1, p(x_1)\big), \cdots, \big(x_{k-1}, p(x_{k-1})\big)\big\},$$

then

$$x_0 \leq p(x_0) < x_1 \leq p(x_1) < \cdots < x_{k-1} \leq p(x_{k-1}) < \omega_1.$$

The following is standard.

Lemma. (1) P is proper.

(2) Let G be P-generic over the ground model V. Then the collection of points in the **domain**s forms a closed cofinal subset of ω_1. More precisely, let

$$\dot{C} = \bigcup\{\mathbf{dom}(p) \mid p \in G\} = \{\mathbf{i} < \omega_1 \mid \exists\, p \in G\ \exists\, j \text{ s.t. } (\mathbf{i}, j) \in p\}.$$

Then $\dot{C}$ is a closed cofinal subset of ω_1.

Let κ be a regular cardinal with $\kappa \geq \omega_2$. In this note, we present a similar proper poset that generically add a sequence $\langle \dot{M}_i \mid i < \omega_1\rangle$ over the ground model V such that

- $\dot{M}_i \in V$ and, in V, $\dot{M}_i$ is a countable elementary substructure of $(H^V_\kappa, \in)$.
- If $i < j < \omega_1$, then $\dot{M}_i \in \dot{M}_j$.
- If j is a limit ordinal, then $\dot{M}_j = \bigcup\{\dot{M}_i \mid i < j\}$.
- $H^V_\kappa = \bigcup\{\dot{M}_i \mid i < \omega_1\}$.

In particular, $\{\omega_1 \cap \dot{M}_i \mid i < \omega_1\}$ forms a closed cofinal subset of ω_1.

As an application of this line of poset, we present a poset for the Strong Reflection Principle (SRP) of S. Todorcevic. (see [B] for a natural construction by the initial segments). There is another application of this method in [MY], where we present a poset for the Mapping Reflection Principle (MRP) of J. Moore. (see [M] for a natural construction by the initial segments.)

Question. Do you see any new application of a plausible reflection principle that combines the two features of SRP and MRP ?

The poset

Definition. Let κ be a regular cardinal with $\kappa \geq \omega_2$. Let us first form a relational structure

$$(H_\kappa, \in).$$

Then we form a club $\mathcal{C}$ in $[H_\kappa]^\omega$ that consists of the countable elementary substructures of $(H_\kappa, \in)$. Hence

$$\mathcal{C} = \{N \in [H_\kappa]^\omega \mid N \prec (H_\kappa, \in)\},$$
$$\mathcal{C} \subset H_\kappa.$$

We next form a relational structure with an additional unary predicate $\mathcal{C}$

$$(H_\kappa, \in, \mathcal{C}).$$

Then we similary form a club $\mathcal{D}$ in $[H_\kappa]^\omega$ that consists of the countable elementary substructures of $(H_\kappa, \in, \mathcal{C})$. Hence

$$\mathcal{D} = \{M \in [H_\kappa]^\omega \mid M \prec (H_\kappa, \in, \mathcal{C})\},$$
$$\mathcal{D} \subset \mathcal{C} \subset H_\kappa.$$

Proposition. Let $M \in \mathcal{D}$. Then for any $x \in M$, there exists $N \in \mathcal{C} \cap M$ with $x \in N$.

Hence M is a union of countable elementary substructures N of $(H_\kappa, \in)$ that belong to M. More precisely,

$$M = \bigcup(\mathcal{C} \cap M).$$

Proof. Let $x \in M$. Then $(H_\kappa, \in, \mathcal{C})$ knows that there exists $N \in \mathcal{C}$ such that $x \in N$. Since $x \in M \prec (H_\kappa, \in, \mathcal{C})$, we can take $N \in M$ as such. Conversely, if $N \in \mathcal{C} \cap M$, then N is countable. Hence $N \in M \in \mathcal{D}$ entails $N = e[\omega] \subset M$, where $e : \omega \longrightarrow N$ onto with $e \in M$.

□

Definition. Let $p \in P$, if

- p is a finite partial function from $\mathcal{D}$ to $\mathcal{C}$ such that $(\mathrm{dom}(p), \in) \models$"linear".
- If $M \in \mathrm{dom}(p)$, then $M \in p(M)$.
- If $M_1, M_2 \in \mathrm{dom}(p)$ with $M_1 \in M_2$, then $p(M_1) \in M_2$.

For $p, q \in P$, let $q \leq p$ in P, if $q \supseteq p$.

Hence if

$$p \in P,$$
$$\mathrm{dom}(p) = \{X_0 \in X_1 \in \cdots \in X_{k-1}\},$$
$$p = \{(X_0, Y_0), (X_1, Y_1), \cdots, (X_{k-1}, Y_{k-1})\},$$

then

$$(\mathrm{dom}(p), \in) \sim \big(\{\omega_1 \cap M \mid M \in \mathrm{dom}(p)\}, < \big) \text{ isomorphic by } M \mapsto \omega_1 \cap M,$$
$$X_0 \in Y_0 \in X_1 \in Y_1 \in \cdots \in X_{k-1} \in Y_{k-1}.$$

Lemma. For any $p \in P$ and $a \in H_\kappa$, there exists $q \in P$ such that $q \leq p$ in P and $a \in \bigcup \mathrm{dom}(q)$.

Proof. Let $p \in P$ and $a \in H_\kappa$. Take (M, N) such that

- $p, a \in M \in N$.
- $M \in \mathcal{D}$.
- $N \in \mathcal{C}$.

Let $q = p \cup \{(M, N)\}$. Then $q \in P$, $q \leq p$ in P, and $a \in M \in \mathrm{dom}(q)$.

□

Lemma. P is proper.

Proof. Let $p \in P$ and $H_\kappa, \mathcal{C}, \mathcal{D}, p, P \in M^*$(countable) $\prec H_\lambda$. Then $M := H_\kappa \cap M^* \in \mathcal{D}$. Let $N \in \mathcal{C}$ with $M \in N$. Let $p_{M^*} = p \cup \{(M, N)\}$. Then $p_{M^*} \in P$ and $p_{M^*} \leq p$ in P.

Claim. p_{M^*} is (P, M^*)-generic.

Proof. Let $D \in M^*$ be predense in P. We show that $D \cap M^*$ is predense below p_{M^*}. To this end, let $\tilde{p} \leq p_{M^*}$ in P. Let $q \leq \tilde{p}$ and $d \in D$ with $q \leq d$ in P. We consider an M^*-copy (q', d', M') of (q, d, M) as follows. Since H_λ knows that there exists $(q', d', M') \in H_\kappa$ such that

- $q' \in P$.
- $d' \in D$.
- $q' \leq d'$ in P.
- $M' \in \mathrm{dom}(q')$.
- $q' \cap M' = (q \cap M)$.

Since $H_\kappa, P, D, (q \cap M) \in M^* \prec H_\lambda$, we can take $(q', d', M') \in H_\kappa \cap M^* = M$ as such. Let $r = q \cup q'$. Then $r \in P$ and $r \leq q, q'$ in P. Hence $D \cap M^*$ is predense below p_{M^*}.

□

□

Lemma. Let G be P-generic over the ground model V. In the generic extension $V[G]$, let

$$\dot{\mathcal{M}} = \bigcup \{\mathrm{dom}(p) \mid p \in G\}.$$

Then

$$\dot{\mathcal{M}} \subset \mathcal{D}$$

$$\bigcup \dot{\mathcal{M}} = H_\kappa^V,$$

$$(\dot{\mathcal{M}}, \in) \models \text{“linear”}.$$

$$\dot{c} : (\dot{\mathcal{M}}, \in) \longrightarrow (\omega_1, <) \text{ by } M \mapsto \dot{c}(M) = \omega_1 \cap M \text{ is order preserving.}$$

Since the range of $\dot{c}$ is cofinal in ω_1, the well-order-type of $(\dot{\mathcal{M}}, \in)$ is exactly ω_1. Hence there exists an isomorphism $\pi : (\omega_1, <) \longrightarrow (\dot{\mathcal{M}}, \in)$. We simply write $\dot{M}_i$ for $\pi(i)$. Hence $\dot{\mathcal{M}}$ gets represented as an $\in$-chain

$$\langle \dot{M}_i \mid i < \omega_1 \rangle.$$

Proof. We show that $(\dot{\mathcal{M}}, \in) \models$“linear”. Let $M_1, M_2 \in \dot{\mathcal{M}}$ s.t. $M_1 \neq M_2$. Take $p \in G$ s.t. $M_1, M_2 \in \mathrm{dom}(p)$. Since $(\mathrm{dom}(p), \in) \models$“linear”, either $M_1 \in M_2$ or $M_2 \in M_1$ holds.

$\square$

Lemma. In $V[G]$, let $\langle \dot{X}_k \mid k < \omega \rangle$ be such that $\dot{X}_k \in \dot{\mathcal{M}}$ and $\dot{X}_k \in \dot{X}_{k+1}$ for all $k < \omega$. Then

$$\bigcup \{\dot{X}_k \mid k < \omega\} \in \dot{\mathcal{M}}.$$

Hence $\langle \dot{M}_i \mid i < \omega_1 \rangle$ is continuously $\subset$-increasing.

Proof. Let $p \Vdash_P$ "$\dot{X}_k \in \dot{\mathcal{M}}$ and $\dot{X}_k \in \dot{X}_{k+1}$ for all $k < \omega$". Since P preserves ω_1, we can assume, by extending p, that there exists $\delta < \omega_1$ such that $p \Vdash_P$ "$\delta = \sup\{\omega_1 \cap \dot{X}_k \mid k < \omega\}$".

Claim 1. There exists $X \in \mathrm{dom}(p)$ s.t. $\delta = \omega_1 \cap X$.

Proof. Suppose not. Then we have (q, M) such that

- $q \in P$.
- $q \leq p$.
- $M \in \mathrm{dom}(q)$.
- $\omega_1 \cap M < \delta$.
- If $Z \in \mathrm{dom}(p)$ with $\omega_1 \cap Z < \delta$, then $p(Z) \in M$.
- $\delta < \omega_1 \cap q(M)$.

Hence $q \Vdash_P$ "there exists no $X \in \dot{\mathcal{M}}$ with $\omega_1 \cap M < \omega_1 \cap X < \delta$". This would be absurd.

$\square$

Claim 2. Let $X \in \mathrm{dom}(p)$ s.t. $\delta = \omega_1 \cap X$. Then $p \Vdash_P$ "$\bigcup\{\dot{X}_k \mid k < \omega\} = X \in \dot{\mathcal{M}}$".

Proof. Let G be P-generic over V with $p \in G$. Argue in $V[G]$. Since $\omega_1 \cap \dot{X}_k < \delta = \omega_1 \cap X$ and $\dot{X}_k, X \in \dot{\mathcal{M}}$, we have $\dot{X}_k \in X$. Hence

$$\bigcup \{\dot{X}_k \mid k < \omega\} \subseteq X.$$

Conversely, let $x \in X$ and $\tilde{p} \leq p$ in P. Since $X \in \mathcal{D}$ and so $X = \bigcup(\mathcal{C} \cap X)$, there exists (M, N) such that

- $M \in N \in X$.
- $M \in \mathcal{D}$.
- $N \in \mathcal{C}$.
- $\tilde{p} \cap X \in M$.
- $\underline{x \in N}$.

Let $q = \tilde{p} \cup \{(M, N)\}$. Then $q \in P$, $q \leq \tilde{p}$, and $q \Vdash_P$ "$\exists\, \dot{X}_k$ s.t. $\underline{x \in N} \in \dot{X}_k$". Hence $p \Vdash_P$ "$X \subseteq \bigcup\{\dot{X}_k \mid k < \omega\}$".

$\square$

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SRP

Definition. ([B]) The Strong Reflection Principle (SRP) holds, if for any set X with $\omega_1 \subseteq X$, any $S \subseteq [X]^\omega$, and any regular cardinal λ s.t. $X, [X]^\omega, S \in H_\lambda$, there exists a sequence $\langle M_i \mid i < \omega_1 \rangle$ such that

- M_i are countable elementray substructures of a first order structure $(H_\lambda, \in, X, S)$, where X and S are constants.
- If $i < j < \omega_1$, then $M_i \in M_j$.
- If $j < \omega_1$ is a limit, then $M_j = \bigcup\{M_i \mid i < j\}$.
- For each $i < \omega_1$, either (yes) or (nono) holds.

  (yes) $X \cap M_i \in S$.

  (nono) For any countable elementary substructure M' of $(H_\lambda, \in, X, S)$ such that $M_i \subseteq_{\omega_1} M'$, we have $X \cap M' \notin S$, where let $M_i \subseteq_{\omega_1} M'$ abbreviate $M_i \subseteq M'$ and $\omega_1 \cap M_i = \omega_1 \cap M'$.

In [B], a natural semi-proper poset for SRP by the initial segments is used under the Semi Proper Forcing Axiom (SPFA). We design a semi-proper poset along the line of previous section.

Definition. Let us form a closed cofinal set $\mathcal{C}$ in $[H_\lambda]^\omega$ by

$$\mathcal{C} = \{N \in [H_\lambda]^\omega \mid N \prec (H_\lambda, \in, X, S)\}.$$

Then we form a closed cofinal set $\mathcal{D}$ in $[H_\lambda]^\omega$ by

$$\mathcal{D} = \{N \in [H_\lambda]^\omega \mid N \prec (H_\lambda, \in, X, S, C)\}, \text{where } \mathcal{C} \text{ is a unary predicate.}$$

Let $p \in P$, if

- p is a finite partial function from $\mathcal{D}$ to $\mathcal{C}$ such that $(\mathrm{dom}(p), \in) \models$"linear".
- If $M \in \mathrm{dom}(p)$, then $M \in p(M)$.
- If $M_1, M_2 \in \mathrm{dom}(p)$ with $M_1 \in M_2$, then $p(M_1) \in M_2$.
- For each $M \in \mathrm{dom}(p)$, either the following (yes) or (nono) holds.

  (yes) $X \cap M \in S$.

  (nono) If $M \subseteq_{\omega_1} \underline{M' \in \mathcal{C}}$, then $X \cap M' \notin S$.

  For $p, q \in P$, let $q \leq p$ in P, if $q \supseteq p$.

Lemma. (Pre-Semi-Generic) Let $p \in P$, M^* be a countable elementary substructure of

$$(H_\theta, \in, H_\lambda, X, S, \mathcal{C}, P),$$

where $H_\lambda, X, S, \mathcal{C}, P$ as constants, and $p \in M^*$. Then there exists $M^\triangle$(countable) $\prec (H_\theta, \in, H_\lambda, X, S, \mathcal{C}, P)$ such that

- $M^* \subseteq_{\omega_1} M^\triangle$.
- $H_\lambda \cap M^\triangle \in \mathcal{D}$.
- $M^\triangle$ satisfies either the following (yes) or (nono).

  (yes) $X \cap (H_\lambda \cap M^\triangle) \in S$.

  (nono) For any M' s.t. $(H_\lambda \cap M^\triangle) \subseteq_{\omega_1} \underline{M' \in \mathcal{C}}$, we have $X \cap M' \notin S$.

  Hence if $N \in \mathcal{C}$ with $H_\lambda \cap M^\triangle \in N$ and we set

$$q = p \cup \{(H_\lambda \cap M^\triangle, N)\},$$

then $q \in P$, $q \leq p$, and $H_\lambda \cap M^\triangle \in \mathrm{dom}(q)$.

Proof. Let p, M^*, H_θ as as above.

Case 1. There exists $M' \in \mathcal{C}$ s.t. $H_\lambda \cap M^* \subseteq_{\omega_1} M'$ and $X \cap M' \in S$. Let

$$M^\triangle := \{f(s) \mid f \in M^* \text{ and } s \in ({}^{<\omega}X) \cap M'\}.$$

Then

Claim. (1) $M^\triangle \prec (H_\theta, \in, H_\lambda, X, S, \mathcal{C}, P)$.

(2) $H_\lambda \cap M^\triangle \in \mathcal{D}$.

(3) $X \cap M^\triangle = X \cap M'$.

(4) $M^* \subseteq_{\omega_1} M^\triangle$.

Proof. (1): We check by the Tarski's criterion. Let

$$(H_\theta, \in, H_\lambda, X, S, \mathcal{C}, P) \models \text{“}\exists\, y\, \phi\big(y, f_1(s_1), \cdots, f_k(s_k)\big)\text{”}.$$

Then there exists $g : {}^{<\omega}X \longrightarrow H_\theta$ s.t. $g \in H_\theta$ and for any $(y, t_1, \cdots, t_k)$ with $y \in H_\theta$, $t_1, \cdots, t_k \in {}^{<\omega}X$, if

$$(H_\theta, \in, H_\lambda, X, S, \mathcal{C}, P) \models \text{“}\phi\big(y, f_1(t_1), \cdots, f_k(t_k)\big)\text{”},$$

then

$$(H_\theta, \in, H_\lambda, X, S, \mathcal{C}, P) \models \text{“}\phi\big(g(\langle t_1, \cdots, t_k\rangle), f_1(t_1), \cdots, f_k(t_k)\big)\text{”},$$

where $\langle t_1, \cdots, t_k\rangle$ is regarded as an element of ${}^{<\omega}X$.

Since $X, \langle f_1, \cdots, f_k\rangle \in M^* \prec (H_\theta, \in, H_\lambda, X, S, \mathcal{C}, P)$, we can take $g \in M^*$. Hence

$$(H_\theta, \in, H_\lambda, X, S, \mathcal{C}, P) \models \text{“}\phi\big(g(\langle s_1, \cdots, s_k\rangle), f_1(s_1), \cdots, f_k(s_k)\big)\text{”},$$

$$g(\langle s_1, \cdots, s_k\rangle) \in M^\triangle.$$

Hence $M^\triangle \prec (H_\theta, \in, H_\lambda, X, S, \mathcal{C}, P)$.

(2): Since $M^\triangle \prec (H_\theta, \in, H_\lambda, X, S, \mathcal{C}, P)$, we have $H_\lambda \cap M^\triangle \prec (H_\lambda, \in, X, S, \mathcal{C})$ by the Tarski's criterion and relativizations. Hence $H_\lambda \cap M^\triangle \in \mathcal{D}$.

(3): Let $x \in X \cap M'$. We want to show $x \in X \cap M^\triangle$. Let us consider a map

$$f : {}^{<\omega}X \longrightarrow H_\theta,$$

$$\langle x_1, \ldots, x_n\rangle \mapsto x_1 \cup \cdots \cup x_n.$$

Then $f \in M^*$, $\langle x\rangle \in ({}^{<\omega}X) \cap M'$, and $f(\langle x\rangle) = x \in X \cap M^\triangle$.

Conversely, let $y = f(s) \in X \cap M^\triangle$. Then there exists $g \in M^*$ s.t. $g : {}^{<\omega}X \longrightarrow X$, if $f(t) \in X$, then $g(t) = f(t)$. We have $g \in H_\lambda \cap M^* \subseteq_{\omega_1} M'$. Hence $y = g(s) \in X \cap M'$.

(4): Let $a \in M^*$. We first show that $a \in M^\triangle$. Let us consider $f : {}^{<\omega}X \longrightarrow \{a\}$ s.t. constantly $f(t) = a$. Then $f \in M^*$ and $a = f(\emptyset) \in M^\triangle$. Next since $X \cap M^\triangle = X \cap M'$, we have

$$\omega_1 \cap M^\triangle = (\omega_1 \cap X) \cap M^\triangle = \omega_1 \cap (X \cap M^\triangle)$$

$$= \omega_1 \cap (X \cap M') = (\omega_1 \cap X) \cap M' = \omega_1 \cap M' = \omega_1 \cap M^*.$$

Case 2. For any $M' \in \mathcal{C}$ s.t. $H_\lambda \cap M^* \subseteq_{\omega_1} M'$, we have $X \cap M' \notin S$. Let $M^\triangle := M^*$. Then this $M^\triangle$ works.

□

Lemma. (Generic) Let $M^{\triangle}$ be a countable elementary substructure of $(H_\theta, \in, H_\lambda, X, S, \mathcal{C}, P)$, $q \in P$, and $H_\lambda \cap M^{\triangle} \in \mathrm{dom}(q)$. Then q is $(P, M^{\triangle})$-**generic**.

Proof. Let $D \in M^{\triangle}$ be predense in P. We want to show that $D \cap M^{\triangle}$ is predense below q. To this end, let $\tilde{q} \leq q$ in P. Let $r \leq \tilde{q}, d$ in P s.t. $d \in D$. We consider $M^{\triangle}$-copy (r', d', M') of $(r, d, H_\lambda \cap M^{\triangle})$ as follows.

$(H_\theta, \in, H_\lambda, X, S, \mathcal{C}, P)$ knows that there exists $(r', d', M') \in H_\lambda$ such that

- $r' \in P$.
- $d' \in D$.
- $r' \leq d'$ in P.
- $M' \in \mathrm{dom}(r')$.
- $r' \cap M' = \big(r \cap (H_\lambda \cap M^{\triangle})\big)$.

Since $H_\lambda, P, D, \big(r \cap (H_\lambda \cap M^{\triangle})\big) \in M^{\triangle} \prec (H_\theta, \in, H_\lambda, X, S, \mathcal{C}, P)$, we can take $(r', d', M') \in H_\lambda \cap M^{\triangle}$ as such. Let $u = r \cup r'$. Then $u \in P$ and $u \leq r, r'$. Hence $D \cap M^{\triangle}$ is predense below q.

□

Lemma. (Semi-Generic) Let $p \in P$, M^* be countable, and $M^* \prec (H_\theta, \in, H_\lambda, X, S, \mathcal{C}, P)$. Then there exists $q \leq p$ in P s.t. q is (P, M^*)-**semi-generic**.

Proof. Let $M^{\triangle}$ be as in the previous lemma. Then we had $q \in P$ such that $q \leq p$ in P and $H_\lambda \cap M^{\triangle} \in \mathrm{dom}(q)$. Hence q is $(P, M^{\triangle})$-generic. Since $M^* \subseteq_{\omega_1} M^{\triangle}$, we conclude that q is (P, M^*)-semi-generic as follows. $q \Vdash_P$"$\theta \cap M^{\triangle}[\dot{G}] = \theta \cap M^{\triangle}$". Hence $q \Vdash_p$"$\omega_1^V \cap M^*[\dot{G}] \subseteq \omega_1^V \cap M^{\triangle}[\dot{G}] = \omega_1^V \cap M^{\triangle} = \omega_1^V \cap M^*$". Hence $q \Vdash_P$"$\omega_1^V \cap M^*[\dot{G}] = \omega_1^V \cap M^*$".

□

Corollary. ([B]) Assume SPFA. Then SRP holds.

Proof. Apply SPFA to P.

□

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miyamoto@nanzan-u.ac.jp

Mathematics

Nanzan University

18 Yamazato-cho, Showa-ku, Nagoya

466-8673 Japan

南山大学　数学　宮元忠敏