

\mathcal{R} -solver and periodic solutions of the Navier-Stokes equations

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1 Introduction

Since Kato-Fujita theory, the Navier-Stokes equations have been studied by a lot of mathematicians based on analytic semigroup properties of Stokes equations. Since the Navier-Stokes equations are a system of semi-linear parabolic equations, and so the analytic semigroup approach has yielded many fruitful results. On the other hand, in many flow problems, for example a falling drop problem, ocean problem, nuclear power, energy conversion technique, environment issues, blood flows... , we meet free boundary problem for the Navier-Stokes equations, which is formulated in an unknown time dependent domain. Under suitable transformation from a time dependent unknown domain with free boundary to a known domain with fixed boundary, the equations become a system of quasilinear parabolic equations with non-homogeneous boundary conditions. The basic tool of proving the local in time existence theorem for such problems is the maximal regularity for the Stokes equations with non-homogeneous boundary or transmission conditions. There are a lot of works have been done by Solonnikov and his colleagues since the early of 1980 in the Hölder spaces, $C^{2+\alpha,1+\alpha/2}$ ($\alpha > 0$), and Sobolev-Slobodetskii spaces $W_2^{2+\ell,1+\ell/2}$ ($1/2 < \ell < 1$), by Jan Pruess and his colleagues in the anisotropic $W_p^{2,1}$ space since the early of 2000, and by Shibata and his colleagues in the anisotropic $W_{q,p}^{2,1}$ space also since the early of 2000. From technical point of view, their approaches are different, and in this note I would like to explain an approach based on \mathcal{R} -solver of the generalized resolvent equations for the Stokes operator with non-homogeneous free boundary conditions, which gives a systematic study of a system of quasilinear parabolic equations with non-homogeneous boundary conditions.

2 Framework based on \mathcal{R} -solvers

I would like to formulate \mathcal{R} -solvers for free boundary problem without surface tension in mind. Let me consider an initial-boundary value problem formulated as follows:

$$\dot{u} - Au = f, \quad Bu = g \quad \text{for } t > 0, \quad u|_{t=0} = u_0, \quad (2.1)$$

where t is the time variable, \dot{u} denotes the time derivative of u , and $Bu = f$ denotes a boundary condition.

- Let X and Y be two UMB Banach spaces and $Y \subset X$.

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- Let $Z = (X, Y)_{[1/2]}$ be a complex interpolation space of order $1/2$.
- Let $A \in \mathcal{L}(Y, X)$, $\mathcal{L}(Y, X)$ denoting the set of all bounded linear operator from Y into X .
- Let $B \in \mathcal{L}(Y, Z) \cap \mathcal{L}(Z, X)$.

Example 1. The following two sets of equations are typical example in the setting of this note:

$$\dot{v} - \Delta v = f \text{ in } \Omega \times (0, \infty), \quad \frac{\partial v}{\partial \nu} = g \text{ on } \partial\Omega \times (0, \infty), \quad v|_{t=0} = v_0,$$

where Ω is a domain, $\partial\Omega$ is its boundary, Δ denotes the Laplace operator and ν denotes the unit outer normal to $\partial\Omega$ (Neumann operator);

$$\begin{aligned} \dot{\mathbf{v}} - \Delta \mathbf{v} + \nabla \mathbf{p} &= \mathbf{f}, \quad \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega \times (0, \infty), \\ (\mathbf{D}(\mathbf{v}) - \mathbf{p}\mathbf{I})\nu &= \mathbf{g} \quad \text{on } \partial\Omega \times (0, \infty), \quad \mathbf{v}|_{t=0} = \mathbf{v}_0 \end{aligned}$$

In these cases, for $1 < p < \infty$, I set $X = L_p$, $Y = H_p^2$, $Z = H_p^1$.

In what follows, for $\epsilon \in (0, \pi/2)$ and $\lambda_0 > 0$ $\Sigma_{\epsilon, \lambda_0}$ denotes a subset of \mathbb{C} defined by setting

$$\Sigma_{\epsilon, \lambda_0} = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon, \quad |\lambda| \geq \lambda_0\}.$$

And, let me consider a generalized resolvent problem corresponding to (2.1) as follows:

$$\lambda u - Au = f, \quad Bu = g \tag{2.2}$$

for $\lambda \in \Sigma_{\epsilon, \lambda_0}$, where "generalized" means the non-homogeneous boundary condition.

A main tool in my approach is Weis's operator valued Fourier multiplier theorem [7], and so I introduce the notion of \mathcal{R} boundedness of operator families.

Definition 2. Let $\mathcal{L}(X, Y)$ denote the set of all bounded linear operators from X into Y . A family of operators, $\mathcal{T} \subset \mathcal{L}(X, Y)$, is called \mathcal{R} -bounded if there exists a constant C and an exponent $p \in [1, \infty)$ such that for all $m \in \mathbb{N}$, $\{T_k\}_{k=1}^m \subset \mathcal{T}$, and $\{x_k\}_{k=1}^m \subset X$, there hold the inequalities:

$$\left\| \sum_{k=1}^m r_k T_k x_k \right\|_{L_p((0,1), Y)} \leq C \left\| \sum_{k=1}^m r_k x_k \right\|_{L_p((0,1), X)}.$$

Here, the Rademacher function r_k , $k \in \mathbb{N}$, are given by $r_k: [0, 1] \rightarrow \{-1, 1\}$, $t \mapsto \operatorname{sign}(\sin(2^k \pi t))$. The smallest such C is called the \mathcal{R} -bound of \mathcal{T} on $\mathcal{L}(X, Y)$ which is written by $\mathcal{R}_{\mathcal{L}(X, Y)}(\mathcal{T})$ in what follows.

In the following, I consider the situation that for any $\lambda \in \Sigma_{\epsilon, \lambda_0}$, $f \in X$ and $g \in X \cap Z$, problem (1) admits a unique solution $u \in Y$ possessing the estimate:

$$\|u\|_Y + \|\lambda u\|_X \leq C(\|f\|_X + \|g\|_Z + \|\lambda^{1/2} g\|_X), \tag{2.3}$$

which has been studied since 1950's as parameter elliptic problems. My concern is to prove the generalized resolvent estimate in terms of \mathcal{R} -norms instead of standard norms, $\|\cdot\|_X$, $\|\cdot\|_Y$, and $\|\cdot\|_Z$.

For any Banach space U , let $\operatorname{Hol}(\Sigma_{\epsilon, \lambda_0}, U)$ denote the set of all U valued holomorphic functions defined on $\Sigma_{\epsilon, \lambda_0}$. Below, I assume the existence of an operator family

$$\mathcal{M}(\lambda) : X \times X \times Z \rightarrow Y; \quad X \times X \times Z \ni (F_1, F_2, F_3) \mapsto \mathcal{M}(\lambda)(F_1, F_2, F_3) \in Y$$

for every $\lambda = \gamma + i\tau \in \Sigma_{\epsilon, \lambda_0}$ with

$$\mathcal{M}(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(X \times X \times Z, Y)), \quad \lambda \mathcal{M}(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(X \times X \times Z, X))$$

such that

- (i) for every $\lambda \in \Sigma_{\epsilon, \lambda_0}$, $f \in X$ and $g \in Z$, $u = \mathcal{M}(\lambda)(f, \lambda^{1/2}g, g)$ is a solution of problem (2.2);
(ii) $\mathcal{M}(\lambda)$ satisfies

$$\begin{aligned} \mathcal{R}_{\mathcal{L}(X \times X \times Z, X)}(\{(\tau \partial_\tau)^\ell(\lambda \mathcal{M}(\lambda)) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq r_b, \\ \mathcal{R}_{\mathcal{L}(X \times X \times Z, Y)}(\{(\tau \partial_\tau)^\ell \mathcal{M}(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq r_b, \end{aligned} \quad (2.4)$$

for $\ell = 0, 1$ with some constant r_b .

Remark 3. Such an operator family $\mathcal{M}(\lambda)$ is called an \mathcal{R} -solver for equations (2.3). Since \mathcal{R} boundedness implies the standard boundedness (in the $m = 1$ case in Definition 2), the estimate (2.3) is derived automatically from (2.4).

I now consider the following time dependent problem:

$$\dot{u} - Au = f, \quad Bu = g \quad (t > 0) \quad u|_{t=0} = u_0. \quad (2.5)$$

The compatibility condition is:

$$g|_{t=0} = Bu_0 \quad (2.6)$$

which is obtained from the boundary condition $Bu = g$ at $t = 0$. Set $u = v + w$, where v and w are solutions of the following equations:

$$\dot{v} - Av = f, \quad Bv = g \quad \text{for } t \in \mathbb{R}, \quad (2.7)$$

$$\dot{w} - Aw = 0, \quad Bw = 0 \quad \text{for } t \in (0, \infty), \quad w|_{t=0} = u_0 - v|_{t=0}. \quad (2.8)$$

I first consider equations (2.7). First of all, let f and g be extended to $t < 0$. Since f is not required to be differentiable in time, and so f is extended by 0, that is $f_0 = f$ for $t > 0$ and $f_0 = 0$ for $t < 0$. On the other hand, g is usually required to be differentiability at least of some fractional order on t and so here it is assumed that g is defined for $t > 0$, and then $g_0(t) = g(t)$ and $g_0(t) = \varphi(t)g(-t)$, where $\varphi(t) \in C^\infty(\mathbb{R})$ which equals one for $t > -1$ and vanishes for $t < -2$. Instead of (2.7), I consider the following equations:

$$\dot{v} - Av = f_0, \quad Bv = g_0 \quad \text{for } t \in \mathbb{R}. \quad (2.9)$$

Applying the Laplace transform to equations (2.9) yields that

$$\lambda \hat{v} - A\hat{v} = \hat{f}_0, \quad B\hat{v} = \hat{g}_0. \quad (2.10)$$

Here, the Laplace transform \hat{v} is defined by setting

$$\hat{v}(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} v(t) dt = \int_{-\infty}^{\infty} e^{-i\tau t} e^{-\gamma t} v(t) dt = \mathcal{F}[e^{-\gamma t} v](\tau)$$

with $\lambda = \gamma + i\tau \in \mathbb{C}$, where \mathcal{F} denotes the Fourier transform. Using the \mathcal{R} -solver $\mathcal{M}(\lambda)$, \hat{v} is represented by $\hat{v} = \mathcal{M}(\lambda)(\hat{f}, \lambda^{1/2}\hat{g}, \hat{g})$. By the Laplace inverse transform,

$$\begin{aligned} v(t) &= \mathcal{L}^{-1}[\mathcal{M}(\lambda)(\hat{f}, \lambda^{1/2}\hat{g}, \hat{g})](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\gamma+i\tau)t} \mathcal{M}(\lambda) \mathcal{F}[e^{-\gamma t}(f_0, \Lambda_\gamma^{1/2}g, g)](\tau) d\tau \\ &= e^{\gamma t} \mathcal{F}^{-1}[\mathcal{M}(\lambda) \mathcal{F}[e^{-\gamma t}(f_0, \Lambda_\gamma^{1/2}g, g)](\tau)](t), \end{aligned}$$

where $\Lambda_\gamma^{1/2}g$ is defined by setting

$$\Lambda_\gamma^{1/2}g = \mathcal{L}^{-1}[\lambda^{1/2}\mathcal{L}[g](\lambda)](t).$$

I now quote the Weis operator valued Fourier multiplier theorem [7].

Theorem 4. Let X and Y be two UMD Banach spaces and let $1 < p < \infty$. Let m be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X, Y))$ such that the following conditions are satisfied:

$$\begin{aligned}\mathcal{R}_{\mathcal{L}(X, Y)}(\{m(\tau), \tau \in \mathbb{R} \setminus \{0\}\}) &= \kappa_0 < \infty, \\ \mathcal{R}_{\mathcal{L}(X, Y)}(\{m(\tau), \tau \in \mathbb{R} \setminus \{0\}\}) &= \kappa_1 < \infty.\end{aligned}$$

Let an operator T_m acting on elements of $\mathcal{F}^{-1}[\mathcal{D}(\mathbb{R}, X)]$ be defined by setting

$$[T_m f](t) = \mathcal{F}^{-1}[m(\tau)\mathcal{F}[f](\tau)](t) \quad \text{for } f \text{ with } \mathcal{F}[f] \in \mathcal{D}(\mathbb{R}, X),$$

where $\mathcal{D}(\mathbb{R}, X)$ denotes the set of all X -valued $C_0^\infty(\mathbb{R})$ functions. Then, the operator T_m is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$ with norm

$$\|T_m\|_{\mathcal{L}(L_p(\mathbb{R}, X), L_p(\mathbb{R}, Y))} \leq C(\kappa_0 + \kappa_1)$$

with some constant depending only on p , X , and Y .

Applying Theorem 4 to the formulas:

$$\begin{aligned}e^{-\gamma t} \dot{v} &= \mathcal{F}^{-1}[\lambda M(\lambda)\mathcal{F}[e^{-\gamma t}(f_0, \Lambda_\gamma^{1/2}g, g)]](t), \\ e^{-\gamma t} v &= \mathcal{F}^{-1}[M(\lambda)\mathcal{F}[e^{-\gamma t}(f_0, \Lambda_\gamma^{1/2}g, g)]](t)\end{aligned}$$

yields that

$$\begin{aligned}&\|e^{-\gamma t} \partial_t v\|_{L_p(\mathbb{R}, X)} + \|e^{-\gamma t} v\|_{L_p(\mathbb{R}, Y)} \\ &\leq C(\|e^{-\gamma t} f\|_{L_p(\mathbb{R}, X)} + \|e^{-\gamma t} \Lambda_\gamma^{1/2} g\|_{L_p(\mathbb{R}, X)} + \|e^{-\gamma t} g\|_{L_p(\mathbb{R}, Z)}),\end{aligned}$$

which is the maximal L_p regularity theorem for problem (2.7).

Problem (2.8) is solved by C^0 analytic semigroup $T(t)$, whose generation is obtained with the help of the \mathcal{R} -solver. In fact, the underlysing space \mathcal{H} , the operator \mathcal{A} and its domain $\mathcal{D}(\mathcal{A})$ are defined as follows:

$$\mathcal{H} = X_0, \quad \mathcal{D}(\mathcal{A}) = \{x \in Y \mid Bx = 0\}, \quad \mathcal{A}x = Ax \text{ for } x \in \mathcal{D}(\mathcal{A}).$$

Problem (2.8) is formulated by

$$\dot{w} - \mathcal{A}w = 0 \quad (t > 0), \quad w|_{t=0} = u_0 - v|_{t=0}. \quad (2.11)$$

The corresponding resolvent problem to (2.11) is

$$\lambda \hat{w} - \mathcal{A} \hat{w} = f, \quad B \hat{w} = 0 \quad (t > 0). \quad (2.12)$$

with $f = u_0 - v|_{t=0}$. Since the \mathcal{R} boundedness implies the boundedness, by the first estimate in (2.4) implies that for any $\lambda \in \Sigma_{\epsilon, \lambda_0}$ problem (2.12) admits a unique solution $\hat{w} \in Y$ possessing the estimate:

$$|\lambda| \|\hat{w}\|_X + \|\hat{w}\|_Y \leq 2r_b \|f\|_X,$$

Thus, there exists a C^0 analytic semigroup $\{T(t)\}_{t \geq 0}$ such that for any $f \in X$, $w = T(t)f$ gives a unique solution of problem (2.12). Moreover, if $f \in (X, \mathcal{D}(\mathcal{A}))_{1-1/p, p}$ which is a real interpolation space between X and $\mathcal{D}(\mathcal{A})$, then w satisfies the estimate:

$$\|e^{-\gamma t} \dot{w}\|_{L_p((0, \infty), X)} + \|e^{-\gamma t} w\|_{L_p((0, \infty), Y)} \leq C \|f\|_{(X, Y)_{1-1/p, p}}$$

for any $\gamma > \lambda_0$, where C is a constant depending solely on λ_0 and ϵ . By the compatibility condition (2.6), $B(u_0 - v|_{t=0}) = Bu_0 - g|_{t=0} = 0$, and so $f = u_0 - v|_{t=0} \in (X, \mathcal{D}(\mathcal{A}))_{1-1/p,p}$ provided that $u_0 \in (X, \mathcal{D}(\mathcal{A}))_{1-1/p,p}$. In fact, a real interpolation theorem yields that $v \in C([0, \infty), (X, Y)_{1-1/p,p})$ and

$$\sup_{t \in [0, \infty)} \|v(t)\|_{(X, \mathcal{D}(\mathcal{A}))_{1-1/p,p}} \leq C(\|e^{-\gamma t} \dot{v}\|_{L_p((0, \infty), X)} + \|e^{-\gamma t} v\|_{L_p((0, \infty), Y)}).$$

Thus, we have

$$\begin{aligned} & \|e^{-\gamma t} \dot{w}\|_{L_p((0, \infty), X)} + \|e^{-\gamma t} w\|_{L_p((0, \infty), Y)} \\ & \leq C(\|u_0\|_{(X, Y)_{1-1/p,p}} + \|e^{-\gamma t} \dot{v}\|_{L_p((0, \infty), X)} + \|e^{-\gamma t} v\|_{L_p((0, \infty), Y)}). \end{aligned}$$

Then, $u = v + w$ is a required solution to problem (2.5).

Summing up, I have proved the following theorem.

Theorem 5. *Let $1 < p < \infty$ and X, Y and Z be three UMD Banach spaces. If \mathcal{R} -solver $\mathcal{M}(\lambda)$ exists for $\lambda \in \Sigma_{\epsilon, \lambda_0}$, then problem (2.5) admits a solution u with*

$$e^{-\gamma t} u \in L_p((0, \infty), Y) \cap H_p^1((0, \infty), X)$$

for any $\gamma > \lambda_0$ possessing the estimate:

$$\begin{aligned} & \|e^{-\gamma t} \dot{u}\|_{L_p((0, \infty), X)} + \|e^{-\gamma t} u\|_{L_p((0, \infty), Y)} \leq C(\|u_0\|_{(X, Y)_{1-1/p,p}} \\ & + \|e^{-\gamma t} f\|_{L_p((0, \infty), X)} + \|e^{-\gamma t} \Lambda_\gamma^{1/2}(\varphi g_0)\|_{L_p(\mathbb{R}, X)} + \|e^{-\gamma t} \varphi g_0\|_{L_p(\mathbb{R}, Z)}). \end{aligned}$$

3 Framework for periodic solutions with the help of \mathcal{R} -solver and transference theorem

Now, let me consider periodic solutions of equations:

$$\dot{v} - Av = f, \quad Bv = g \quad \text{for } t \in (0, 2\pi) = \mathbb{T}, \quad (3.1)$$

where it is assumed that $f(t + 2\pi) = f(t)$ and $g(t + 2\pi) = g(t)$ for $t \in \mathbb{R}$. Let

$$\hat{f}(k) := \mathcal{F}_{\mathbb{T}}[f](k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) dt, \quad \mathcal{F}_{\mathbb{T}}^{-1}[(a_k)_{k \in \mathbb{Z}}](t) := \sum_{k \in \mathbb{Z}} e^{ikt} a_k,$$

$$L_{p, \text{per}}((0, 2\pi), X) = \{f(t) \in L_{p, \text{loc}}(\mathbb{R}, X) \mid f(t + 2\pi) = f(t) \quad (t \in \mathbb{R})\}.$$

Applying Fourier transform gives that

$$ik\hat{v} - A\hat{v} = \hat{f}(k), \quad B\hat{v} = \hat{g}(k). \quad (3.2)$$

Applying \mathcal{R} -solver $\mathcal{M}(\lambda)$ gives that

$$\hat{v}(k) = \mathcal{M}(ik)(\hat{f}(k), (ik)^{1/2} \hat{g}(k), \hat{g}(k))$$

for $|k| \geq \lambda_0$, because $ik \in \Sigma_{\epsilon, \lambda_0}$ for $|k| \geq \lambda_0$. The following theorem was obtained in [2].

Theorem 6 (Eiter-Kyed-Shibata). *Let X, Y be Banach spaces and $p \in (1, \infty)$. Assume that Y is reflexive. If*

$$T_m[f] = \mathcal{F}^{-1}[m(\xi)\mathcal{F}[f](\xi)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi t} m(\xi) \mathcal{F}[f](\xi) d\xi \quad \text{for } f \text{ with } \mathcal{F}[f] \in \mathcal{D}(\mathbb{R}, X)$$

is a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$, that is

$$\|T_m[f]\|_{L_p(\mathbb{R}, Y)} \leq M_p \|f\|_{L_p(\mathbb{R}, X)},$$

then

$$T_{m, \mathbb{T}}[g] := \mathcal{F}_{\mathbb{T}}^{-1}[(m|_{\mathbb{Z}}(ik)\mathcal{F}_{\mathbb{T}}[g](ik))_{k \in \mathbb{Z}}] = \sum_{k \in \mathbb{Z}} e^{ikt} m(k) \mathcal{F}_{\mathbb{T}}[g](k) \quad \text{for } g \in L_{p, \text{per}}((0, 2\pi), X)$$

is also a bounded linear operator from $L_{p, \text{per}}((0, 2\pi), X)$ into $L_{p, \text{per}}((0, 2\pi), Y)$ with essentially the same bound. Namely, we have

$$\|T_{m, \mathbb{T}}[g]\|_{L_p((0, 2\pi), Y)} \leq C_p M_p \|g\|_{L_p((0, 2\pi), X)}$$

for some constant C_p depending solely on p .

Let $\varphi(\tau) \in C^\infty(\mathbb{R})$ which equals 1 for $|\tau| \geq \lambda_0 + 1/2$ and 0 for $|\tau| \leq \lambda_0 + 1/4$. Set

$$v_\varphi = \mathcal{F}_{\mathbb{T}}^{-1}[(\varphi(k)\mathcal{M}(ik)(\hat{f}(ik), (ik)^{1/2}\hat{g}(k), \hat{g}(k)))_{k \in \mathbb{Z}}]$$

where $\hat{h}(ik) = \mathcal{F}_{\mathbb{T}}[h](k)$. And then, v_φ satisfies the equations:

$$\partial_t v_\varphi - Av_\varphi = \mathcal{F}^{-1}[(\varphi(k)\hat{f}(k))_{k \in \mathbb{Z}}], \quad Bv_\varphi = \mathcal{F}^{-1}[(\varphi(k)\hat{g}(k))_{k \in \mathbb{Z}}].$$

By the transference theorem, Theorem 6,

$$\|\partial_t v_\varphi\|_{L_p((0, 2\pi), X)} + \|v_\varphi\|_{L_p((0, 2\pi), Y)} \leq C(\|f\|_{L_p((0, 2\pi), X)} + \|g\|_{H_p^{1/2}((0, 2\pi), X)} + \|g\|_{L_p((0, 2\pi), Z)}). \quad (3.3)$$

A solution v of equations (4.29) is given by

$$v = \sum_{|k| \leq \lambda_0 + 1/2} e^{ikt} v_k + v_\varphi \quad (3.4)$$

where v_k are solutions of the equations:

$$ikv_k - Av_k = \hat{f}(k), \quad Bv_k = \hat{g}(k). \quad (3.5)$$

The part, v_φ , of v in (3.4) is called the high frequency part, and the estimate (3.3) is the maximal L_p regularity of the high frequency part.

4 One phase problem for the Navier-Stokes equations

The material here is taken from my joint paper [1] with Thomas Eiter and Mads Kyed. Free boundary problem for the Navier-Stokes equations is formulated as follows: Let Ω_t be a time dependent domain in the N -dimensional Euclidean space \mathbb{R}^N ($N \geq 2$), which is unknown. Let Γ_t be the boundary of Ω_t and \mathbf{n}_t the unit outer normal to Γ_t . It is assumed that Ω_t is occupied by some incompressible viscous fluid of unit mass density whose viscosity coefficient is a positive constant μ . Let $\mathbf{u} = {}^\top(u_1(x, t), \dots, u_N(x, t))$ be

the velocity field and $\mathbf{p} = \mathbf{p}(x, t)$ the pressure field and then \mathbf{u} and \mathbf{p} satisfies the Navier-Stokes equations in Ω_t with free boundary condition as follows:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \text{Div}(\mu \mathbf{D}(\mathbf{u}) - \mathbf{p} \mathbf{I}) = \mathbf{f} & \text{in } \Omega_t, \\ \text{div } \mathbf{u} = 0 & \text{in } \Omega_t, \\ (\mu \mathbf{D}(\mathbf{u}) - \mathbf{p} \mathbf{I}) \mathbf{n}_t = \sigma H(\Gamma_t) \mathbf{n}_t & \text{on } \Gamma_t, \\ V_{\Gamma_t} = \mathbf{u} \cdot \mathbf{n}_t & \text{on } \Gamma_t \end{cases} \quad (4.1)$$

for $t \in (0, 2\pi)$. Here, $\mathbf{f} = \mathbf{f}(x, t)$ is a prescribed 2π time periodic external force; $H(\Gamma_t)$ denotes the $N - 1$ fold mean curvature of Γ_t which is given by $H(\Gamma_t) \mathbf{n}_t = \Delta_{\Gamma_t} x$ for $x \in \Gamma_t$, where Δ_{Γ_t} is the Laplace-Beltrami operator on Γ_t ; V_{Γ_t} the evolution speed of Γ_t along \mathbf{n}_t ; σ a positive constant representing the surface tension coefficient; $\mathbf{D}(\mathbf{u})$ the doubled deformation tensor given by $\mathbf{D}(\mathbf{u}) = \nabla \mathbf{u} + {}^\top \nabla \mathbf{u}$; and \mathbf{I} the $N \times N$ identity matrix. For any $N \times N$ matrix of functions \mathbf{K} whose $(i, j)^{\text{th}}$ component is K_{ij} , $\text{Div } \mathbf{K}$ denotes an N -vector whose i^{th} component is $\sum_{j=1}^N \partial_j K_{ij}$ and for any N -vector of functions $\mathbf{v} = {}^\top (v_1, \dots, v_N)$, $\mathbf{v} \cdot \nabla \mathbf{v}$ denotes an N vector of functions whose i^{th} component is $\sum_{j=1}^N v_j \partial_j v_i$, where $\partial_j = \partial / \partial x_j$.

Our problem is to find $\Omega_t, \Gamma_t, \mathbf{u}$ and \mathbf{p} satisfying the periodic condition:

$$\Omega_t = \Omega_{t+2\pi}, \quad \Gamma_t = \Gamma_{t+2\pi}, \quad \mathbf{u}(x, t) = \mathbf{u}(x, t + 2\pi), \quad \mathbf{p}(x, t) = \mathbf{p}(x, t + 2\pi) \quad (4.2)$$

for any $t \in \mathbb{R}$.

4.1 Assumptions

Let $\mathbf{p}_i = \mathbf{e}_i = {}^\top (0, \dots, 0, \overset{i\text{-th}}{1}, 0, \dots, 0)$ for $i = 1, \dots, N$ and \mathbf{p}_ℓ ($\ell = N + 1, \dots, M$) be one of $x_i \mathbf{e}_j - x_j \mathbf{e}_i$ ($1 \leq i, j \leq N$). It is known that an N -vector of functions, \mathbf{d} , satisfies $\mathbf{D}(\mathbf{d}) = 0$ if and only if \mathbf{d} is represented as a linear combination of \mathbf{p}_i ($i = 1, \dots, M$). The unknown domain Ω_t will be constructed such that the following three conditions are satisfied:

$$\det \left(\int_0^{2\pi} (\mathbf{p}_\ell, \mathbf{p}_m)_{\Omega_t} dt \right)_{\ell, m=1, \dots, M} \neq 0, \quad (4.3)$$

$$\int_0^{2\pi} \left(\frac{1}{|\Omega_t|} \int_{\Omega_t} x dx \right) dt = 0, \quad (4.4)$$

$$|\Omega_t| = |B_R| \quad \text{for any } t \in (0, 2\pi). \quad (4.5)$$

In what follows, the following symbols will be used:

$$H_{p, \text{per}}^1((0, 2\pi), X) = \{f(\cdot, t) \in L_{p, \text{loc}}(\mathbb{R}, X) \mid \dot{f} \in L_{p, \text{per}}((0, 2\pi), X)\};$$

$$H_{p, \text{per}}^{1/2}((0, 2\pi), X) = \{f(\cdot, t) \in L_{p, \text{loc}}(\mathbb{R}, X) \mid \mathcal{F}_{\mathbb{T}}^{-1}[(1 + k^2)^{1/4} \hat{f}(k)]_{k \in \mathbb{Z}} \in L_{p, \text{per}}((0, 2\pi), X)\};$$

$$\|f\|_{L_p((0, 2\pi), X)} := \left(\int_0^{2\pi} \|f(t)\|_X^p dt \right)^{1/p} < \infty;$$

$$\|f\|_{H_p^{1/2}((0, 2\pi), X)} := \|\mathcal{F}_{\mathbb{T}}^{-1}[(1 + k^2)^{1/4} \hat{f}(k)]_{k \in \mathbb{Z}}\|_{L_p((0, 2\pi), X)};$$

$$(f, g)_G = \int_G f(x) \cdot \overline{g(x)} dx, \quad (f, g)_{\partial G} = \int_{\partial G} f(x) \overline{g(x)} d\sigma.$$

Let Ω_t, \mathbf{u} and \mathbf{p} satisfy equations (4.1) and periodic condition (4.2), and then the divergence theorem of Gauss implies that

$$\begin{aligned} ((\mu \mathbf{D}(\mathbf{u}) - \mathbf{p} \mathbf{I}) \mathbf{n}_t, \mathbf{e}_i)_{\Gamma_t} &= \sigma (\Delta_{\Gamma_t} x, \mathbf{e}_i)_{\Gamma_t} = -\sigma (\nabla_{\Gamma_t} x, \nabla_{\Gamma_t} \mathbf{e}_i)_{\Gamma_t} = 0; \\ ((\mu \mathbf{D}(\mathbf{u}) - \mathbf{p} \mathbf{I}) \mathbf{n}_t, x_i \mathbf{e}_j - x_j \mathbf{e}_i)_{\Gamma_t} &= \sigma (\Delta_{\Gamma_t} x, x_i \mathbf{e}_j - x_j \mathbf{e}_i)_{\Gamma_t} \\ &= -\sigma (\nabla_{\Gamma_t} x_j, \nabla_{\Gamma_t} x_i)_{\Gamma_t} + \sigma (\nabla_{\Gamma_t} x_i, \nabla_{\Gamma_t} x_j)_{\Gamma_t} = 0. \end{aligned} \quad (4.6)$$

Since

$$\frac{d}{dt}(\mathbf{u}, \mathbf{p}_\ell)_{\Omega_t} = (\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{p}_\ell)_{\Omega_t} = (\text{Div}(\mu \mathbf{D}(\mathbf{u}) - \mathbf{p} \mathbf{I}), \mathbf{p}_\ell)_{\Omega_t} + (\mathbf{f}, \mathbf{p}_\ell)_{\Omega_t}$$

as follows from the first equation in (4.1) and $\text{div } \mathbf{u} = 0$, it follows from (4.6) that

$$\frac{d}{dt}(\mathbf{u}, \mathbf{p}_\ell)_{\Omega_t} = (\mathbf{f}, \mathbf{p}_\ell)_{\Omega_t}. \quad (4.7)$$

Assumption on \mathbf{f} . There exists a domain $D \subset \Omega_t$ such that $\text{supp } \mathbf{f}(x, t) \subset D$ for any $t \in \mathbb{R}$. \square

Thus, the periodic condition (4.2) together with (4.7) yields that

$$\int_0^{2\pi} \left(\int_D \mathbf{f}(x, \cdot) \cdot \mathbf{p}_\ell(x) dx \right) dt = 0 \quad \text{for } \ell = 1, \dots, M. \quad (4.8)$$

Instead of problem (4.2), we consider the following equations:

$$\left\{ \begin{array}{ll} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \text{Div}(\mu \mathbf{D}(\mathbf{u}) - \mathbf{p} \mathbf{I}) + \sum_{k=1}^M \int_0^{2\pi} (\mathbf{u}(\cdot, t), \mathbf{p}_k)_{\Omega_t} dt \mathbf{p}_k = \mathbf{f} & \text{in } \Omega_t, \\ \text{div } \mathbf{u} = 0 & \text{in } \Omega_t, \\ (\mu \mathbf{D}(\mathbf{u}) - \mathbf{p} \mathbf{I}) \mathbf{n}_t = \sigma H(\Gamma_t) \mathbf{n}_t & \text{on } \Gamma_t, \\ V_{\Gamma_t} = \mathbf{u} \cdot \mathbf{n}_t & \text{on } \Gamma_t \end{array} \right. \quad (4.9)$$

for $t \in (0, 2\pi)$. In fact, if Ω_t , \mathbf{u} and \mathbf{p} satisfy equations (4.9), the assumption (4.3), and the periodic condition (4.2), then by (4.8) we have

$$(\mathbf{f}, \mathbf{p}_\ell)_{\Omega_t} = \frac{d}{dt}(\mathbf{u}, \mathbf{p}_\ell)_{\Omega_t} + \sum_{k=1}^M \int_0^{2\pi} (\mathbf{u}(\cdot, t), \mathbf{p}_k)_{\Omega_t} dt (\mathbf{p}_k, \mathbf{p}_\ell)_{\Omega_t}.$$

Integrating this formula on $(0, 2\pi)$ and using the periodicity and the assumption on \mathbf{f} , (4.8), gives

$$\sum_{k=1}^M \int_0^{2\pi} (\mathbf{u}(\cdot, t), \mathbf{p}_k)_{\Omega_t} dt \int_0^{2\pi} (\mathbf{p}_k, \mathbf{p}_\ell)_{\Omega_t} dt = 0,$$

which, combined with (4.3), yields that

$$\int_0^{2\pi} (\mathbf{u}(\cdot, t), \mathbf{p}_k)_{\Omega_t} dt = 0. \quad (4.10)$$

Thus, Ω_t , \mathbf{u} and \mathbf{p} satisfy the first equation in (4.1).

4.2 Hanzawa transform

Since Ω_t is unknown, the problem should be formulated in a fixed domain. For this purpose, the Hanzawa transform is used. Let $\xi(t)$ be the barycenter point of Ω_t defined by setting

$$\xi(t) = \frac{1}{|B_R|} \int_{\Omega_t} x dx, \quad (4.11)$$

where the fact that $|\Omega_t| = |B_R|$ has been used, which follows from the assumption (4.5). From the Reynolds transport theorem and $\text{div } \mathbf{u} = 0$ it follows that

$$\frac{d}{dt} \xi(t) = \frac{1}{|B_R|} \int_{\Omega_t} (\partial_t x + \mathbf{u} \cdot \nabla x) dx = \frac{1}{|B_R|} \int_{\Omega_t} \mathbf{u}(x, t) dx, \quad \xi''(t) = \frac{1}{|B_R|} \int_{\Omega_t} \mathbf{f}(x, t) dx. \quad (4.12)$$

Let $\rho(y, t)$ be an unknown periodic function with period 2π such that

$$\Gamma_t = \{x = y + R^{-1}\rho(y, t)y + \xi(t) \mid y \in S_R\},$$

where $S_R = \{x \in \mathbb{R}^N \mid |x| = R\}$ and $R^{-1}y$ is the unit outer normal to S_R for $y \in S_R$. Let H_ρ be a suitable extension of ρ to \mathbb{R}^N such that

$$\begin{aligned} \|H_\rho\|_{H_q^k(B_R)} &\leq C\|\rho\|_{W_q^{k-1/q}(S_R)} \quad \text{for } k = 1, 2, 3, \\ \|\partial_t H_\rho\|_{H_q^k(B_R)} &\leq C\|\partial_t \rho\|_{W_q^{k-1/q}(S_R)} \quad \text{for } k = 1, 2. \end{aligned}$$

Let

$$\begin{aligned} \Omega_t &= \{x = y + R^{-1}H_\rho(y, t)y + \xi(t) \mid y \in B_R\}, \\ \Gamma_t &= \{x = y + R^{-1}\rho(y, t)y + \xi(t) \mid y \in S_R\}. \end{aligned} \tag{4.13}$$

Let $J(t)$ be the Jacobian of the transformation: $x = \Phi(y, t) = y + R^{-1}H_\rho(y, t)y + \xi(t)$. Assume that

$$\sup_{t \in (0, 2\pi)} \|H_\rho(\cdot, t)\|_{H_\infty^1(B_R)} \leq \delta \tag{4.14}$$

with some small constant $\delta > 0$, which is chosen so small that the map $x = \Phi(y, t)$ is injective for any $t \in (0, 2\pi)$, and so the inverse map $y = \Phi^{-1}(x, t)$ exists and has the same regularity property as that Φ has.

4.3 Kinematic equations

Let $\mathbf{u}(x, t)$ and $\mathbf{p}(x, t)$ satisfy equations (4.1), and let $\mathbf{v}(y, t) = \mathbf{u}(\Phi(y, t), t)$ and $\mathbf{q}(y, t) = \mathbf{p}(\Phi(y, t), t)$. An equation for \mathbf{v} and ρ is derived from the kinematic condition: $V_{\Gamma_t} = \mathbf{u} \cdot \mathbf{n}_t$ on Γ_t . From the fact that Γ_t is represented by $x = y + R^{-1}\rho(y, t)y + \xi(t)$ it follows that

$$V_{\Gamma_t} = \frac{\partial x}{\partial t} \cdot \mathbf{n}_t = \left(\frac{\partial \rho}{\partial t} \mathbf{n} + \xi'(t)\right) \cdot \mathbf{n}_t, \tag{4.15}$$

where $\mathbf{n} = R^{-1}y$. To represent $\xi'(t)$, let me represent the Jacobian, $J(t)$, of the map $x = \Phi(y, t)$ as $J(t) = 1 + J_0(t)$ with

$$J_0(t) = \det(\delta_{ij} + R^{-1} \frac{\partial}{\partial y_i} (H_\rho(y, t)y_j))_{i,j=1,\dots,N} - 1.$$

Thus,

$$\xi'(t) = \frac{1}{|B_R|} \int_{\Omega_t} \mathbf{u} dx = \frac{1}{|B_R|} \int_{B_R} \mathbf{v}(y, t) dy + \frac{1}{|B_R|} \int_{B_R} \mathbf{v}(y, t) J_0(t) dy,$$

which, combined with (4.16) and the kinematic condition: $V_{\Gamma_t} = \mathbf{v} \cdot \mathbf{n}_t$, yields that

$$\partial_t \rho - \left(\mathbf{v} - \frac{1}{|B_R|} \int_{B_R} \mathbf{v}(y, t) dy\right) \cdot \mathbf{n}_t = d(\mathbf{v}, \rho) \tag{4.16}$$

with

$$d(\mathbf{v}, \rho) = \frac{1}{|B_R|} \int_{B_R} \mathbf{v}(y, t) J_0(t) dy + \frac{\partial \rho}{\partial t} \mathbf{n} \cdot (\mathbf{n} - \mathbf{n}_t) + \mathbf{v} \cdot (\mathbf{n}_t - \mathbf{n}). \tag{4.17}$$

4.4 Mass conservation and Barycenter

The the case where Ω_t is close to B_R is considered below, and so Δ_{Γ_t} is a small perturbation from Δ_{S_R} , where Δ_{S_R} is the Laplace-Beltrami operator on S_R . Thus,

$$\langle H(\Gamma_t)\mathbf{n}_t, \mathbf{n}_t \rangle = (\Delta_{S_R} + (N-1)/R^2)\rho - (N-1)/R + \text{nonlinear terms.}$$

Here, $-(N-1)/R^2$ is the first eigen-value of the Laplace-Beltrami operator Δ_{S_R} on S_R with eigenfunctions y_j/R for $y = (y_1, \dots, y_N) \in S_R$. To avoid the zero and first eigen-values of Δ_{S_R} in this linear analysis, the following observation is useful: Since $\Gamma_t = \partial\Omega_t = \{x = y + R^{-1}\rho(y, t)y + \xi(t) \mid y \in S_R\}$,

$$\begin{aligned} |B_R| &= |\Omega_t| = \int_{S_R} \left(\int_0^{R+\rho(\omega, t)} r^{N-1} dr \right) d\omega = \frac{1}{N} \int_{S_R} (R + \rho(\omega))^N d\omega \\ &= |B_R| + \int_{S_R} \rho d\omega + \sum_{k=2}^N \frac{N C_k}{N} \int_{S_R} \rho^k d\omega = 0, \end{aligned}$$

and so

$$\int_{S_R} \rho d\omega + \sum_{k=2}^N \frac{N C_k}{N} \int_{S_R} \rho^k d\omega = 0, \quad (4.18)$$

where $d\omega$ denotes the surface element of S_R . The relations $\Gamma_t = \partial\Omega_t = \{x = y + R^{-1}\rho(y, t)y + \xi(t) \mid y \in S_R\}$ and $\xi(t) = \frac{1}{|B_R|} \int_{\Omega_t} x dx$ gives that

$$\begin{aligned} 0 &= \frac{1}{|B_R|} \int_{\Omega_t} (x - \xi(t)) dx = \frac{1}{|B_R|} \int_{S_R} \left(\int_0^{R+\rho} r^N \omega dr \right) d\omega \\ &= \frac{1}{|B_R|} \frac{1}{N+1} \int_{S_R} (R + \rho)^{N+1} \omega d\omega \\ &= \frac{1}{|B_R|} \left(\int_{S_R} \rho \omega d\omega + \sum_{k=2}^{N+1} \frac{N+1 C_k}{N+1} \int_{S_R} \rho^k \omega d\omega \right), \end{aligned}$$

from which it follows that

$$\int_{S_R} \rho \omega_j d\omega + \sum_{k=2}^{N+1} \sum_{k=2}^{N+1} \frac{N+1 C_k}{N+1} \int_{S_R} \rho^k \omega_j d\omega = 0 \quad (4.19)$$

for $j = 1, \dots, N$.

4.5 New kinematic equation

Using these two formulas (4.18) and (4.19), one can see that the kinematic equation is equivalent to equation:

$$\partial_t \rho + \int_{S_R} \rho d\omega + \sum_{k=1}^N \left(\int_{S_R} \rho \omega_k d\omega \right) y_k - \left(\mathbf{v} - \frac{1}{|B_R|} \int_{B_R} \mathbf{v} dy \right) \cdot \mathbf{n} = \tilde{d}(\mathbf{v}, \rho) \quad (4.20)$$

on $S_R \times (0, 2\pi)$ with

$$\tilde{d}(\mathbf{v}, \rho) = d(\mathbf{v}, \rho) - \sum_{k=2}^N \frac{N C_k}{N} \int_{S_R} \rho^k d\omega - \sum_{k=2}^{N+1} \frac{N+1 C_k}{N+1} \left(\int_{S_R} \rho^k \omega d\omega \right) y_k. \quad (4.21)$$

4.6 Linearization Principle

To prove the existence of $(\Omega_t, \mathbf{u}, \mathbf{p})$ satisfying (4.1), it is enough to prove the existence of periodic solutions to the following equations:

$$\begin{cases} \partial_t \mathbf{v} + \mathcal{L}\mathbf{v}_S - \operatorname{Div}(\mu(\mathbf{D}(\mathbf{v}) - \mathbf{q}\mathbf{I}) = \mathbf{G} + \mathbf{F}(\mathbf{v}, \rho) & \text{in } B_R \times (0, 2\pi), \\ \operatorname{div} \mathbf{v} = g(\mathbf{v}, \rho) = \operatorname{div} \mathbf{g}(\mathbf{v}, \rho) & \text{in } B_R \times (0, 2\pi), \\ \partial_t \rho + \mathcal{M}\rho - \mathcal{A}\mathbf{v} \cdot \mathbf{n} = \tilde{d}(\mathbf{v}, \rho) & \text{on } S_R \times (0, 2\pi), \\ (\mu\mathbf{D}(\mathbf{v}) - \mathbf{q})\mathbf{n} - (\mathcal{B}_R\rho)\mathbf{n} = \mathbf{h}(\mathbf{v}, \rho) & \text{on } S_R \times (0, 2\pi), \end{cases} \quad (4.22)$$

where $\mathbf{G}(y, t) = \nabla\Phi(y, t)\mathbf{f}(\Phi(y, t), t)$, $\mathbf{F}(\mathbf{v}, \rho)$, $g(\mathbf{v}, \rho)$, $\mathbf{g}(\mathbf{v}, \rho)$, and $\mathbf{h}(\mathbf{v}, \rho)$ are nonlinear terms, and we have set

$$\begin{aligned} \mathcal{L}\mathbf{v}_S &= 2\pi \sum_{k=1}^M (\mathbf{v}_S, \mathbf{p}_k)_{\mathbb{T}} \mathbf{p}_k, \quad \mathbf{v}_S = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{v}(\cdot, s) ds, \\ \mathcal{A}\mathbf{v} &= \mathbf{v} - \frac{1}{|B_R|} \int_{B_R} \mathbf{v} dy; \quad \mathcal{M}\rho = \int_{S_R} \rho d\omega + \sum_{k=1}^N \left(\int_{S_R} \rho \omega_k d\omega \right) y_k; \\ \mathcal{B}_R\rho &= (\Delta_{S_R} + \frac{N-1}{R^2})\rho = R^{-2}(\Delta_{S_1} + (N-1))\rho, \end{aligned} \quad (4.23)$$

where Δ_{S_1} is the Laplace-Beltrami operator on the unit sphere S_1 .

4.7 Main Result

Theorem 7. *Let $1 < p, q < \infty$ and $2/p + N/q < 1$. Let D be a domain $\subset B_R = \{x \in \mathbb{R}^N \mid |x| < R\}$. Assume that*

- $\mathbf{f} \in L_{p,\text{per}}((0, 2\pi), L_q(D)^N)$ and $\operatorname{supp} \mathbf{f}(\cdot, t) \subset D$ for any $t \in (0, 2\pi)$,

-

$$\int_0^{2\pi} (\mathbf{f}(\cdot, t), \mathbf{p}_\ell)_D dt = 0 \quad \text{for } \ell = 1, \dots, M, \quad (4.24)$$

- there exists a small $\epsilon > 0$ such that $\|\mathbf{f}\|_{L_{p,\text{per}}((0, 2\pi), L_q(D)^N)} \leq \epsilon$.

Then, there exist $\mathbf{v}(y, t)$, $\mathbf{q}(y, t)$, and $\rho(y, t)$ with

$$\begin{aligned} \mathbf{v} &\in L_{p,\text{per}}((0, 2\pi), H_q^2(B_R)^N) \cap H_{p,\text{per}}^1((0, 2\pi), L_q(B_R)^N), \\ \mathbf{q} &\in L_{p,\text{per}}((0, 2\pi), H_q^1(B_R)), \\ \rho &\in L_{p,\text{per}}((0, 2\pi), W_q^{3-1/q}(B_R)^N) \cap H_{p,\text{per}}^1((0, 2\pi), W_q^{2-1/q}(S_R)), \end{aligned} \quad (4.25)$$

such that the correspondence: $x = \Phi(y, t) := y + R^{-1}H_\rho(y, t)y + \xi(t)$ is an injective map defined on B_R and

$$\begin{aligned} \Omega_t &= \{x = \Phi(y, t) \mid y \in B_R\}, \quad \Gamma_t = \{x = y + R^{-1}\rho(y, t)y + \xi(t) \mid y \in S_R\}, \\ \mathbf{u}(x, t) &= \mathbf{v}(\Phi^{-1}(x, t), t), \quad \mathbf{p}(x, t) = \mathbf{q}(\Phi^{-1}(x, t), t), \end{aligned}$$

where $\Phi^{-1}(x, t)$ is the inverse map of the correspondence: $x = \Phi(y, t)$ for any $t \in (0, 2\pi)$, are unique solutions of equations (4.1) satisfying the periodic condition (4.2), where $\xi(t)$ is a 2π periodic function which satisfies the equation:

$$\xi(t) = \frac{1}{|\Omega_t|} \int_{\Omega_t} x dx.$$

Moreover, \mathbf{v} and ρ satisfy the estimate:

$$\begin{aligned} & \|\mathbf{v}\|_{L_p((0,2\pi),H_q^2(B_R))} + \|\partial_t \mathbf{v}\|_{L_p((0,2\pi),L_q(B_R))} \\ & + \|\rho\|_{L_p((0,2\pi),W_q^{3-1/q}(S_R))} + \|\partial_t \rho\|_{L_p((0,2\pi),W_q^{2-1/q}(S_R))} + \|\partial_t \rho\|_{L_\infty((0,2\pi),W_q^{1-1/q}(S_R))} \leq C\epsilon \end{aligned} \quad (4.26)$$

for some constant C independent of ϵ .

4.8 Linearized equations

The linearized equations for problem (4.22) are the set of the following equations:

$$\begin{cases} \partial_t \mathbf{v} + \mathcal{L} \mathbf{v}_S - \text{Div}(\mu(\mathbf{D}(\mathbf{v}) - \mathbf{q}\mathbf{I})) = \mathbf{f} & \text{in } B_R \times (0, 2\pi), \\ \text{div } \mathbf{v} = g = \text{div } \mathbf{g} & \text{in } B_R \times (0, 2\pi), \\ \partial_t \rho + \mathcal{M} \rho - \mathcal{A} \mathbf{v} \cdot \mathbf{n} = d & \text{on } S_R \times (0, 2\pi), \\ (\mu \mathbf{D}(\mathbf{v}) - \mathbf{q}) \mathbf{n} - (\mathcal{B}_R \rho) \mathbf{n} = \mathbf{h} & \text{on } S_R \times (0, 2\pi). \end{cases} \quad (4.27)$$

The following theorem is the unique existence of periodic solutions of problem (4.27).

Theorem 8. *Let $1 < p, q < \infty$. Then, for any*

$$\begin{aligned} & \mathbf{f} \in L_{p,\text{per}}((0, 2\pi), L_q(B_R)^N), \\ & g \in L_{p,\text{per}}((0, 2\pi), H_q^1(B_R)) \cap H_{p,\text{per}}^{1/2}((0, 2\pi), L_q(B_R)), \\ & \mathbf{g} \in H_{p,\text{per}}^1((0, 2\pi), L_q(B_R)^N), \quad d \in L_{p,\text{per}}((0, 2\pi), W_q^{2-1/q}(S_R)), \\ & \mathbf{h} \in L_{p,\text{per}}((0, 2\pi), H_q^1(B_R)^N) \cap H_{p,\text{per}}^{1/2}((0, 2\pi), L_q(B_R)^N), \end{aligned}$$

problem (4.27) admits unique solutions \mathbf{v} , \mathbf{q} , and ρ with

$$\begin{aligned} & \mathbf{v} \in L_{p,\text{per}}((0, 2\pi), H_q^2(B_R)^N) \cap H_{p,\text{per}}^1((0, 2\pi), L_q(B_R)^N), \\ & \nabla \mathbf{q} \in L_{p,\text{per}}((0, 2\pi), L_q(B_R)^N), \\ & \rho \in L_{p,\text{per}}((0, 2\pi), W_q^{3-1/q}(S_R)) \cap H_{p,\text{per}}^1((0, 2\pi), W_q^{2-1/q}(S_R)) \end{aligned}$$

possessing the estimate:

$$\begin{aligned} & \|\mathbf{v}\|_{L_p((0,2\pi),H_q^2(B_R))} + \|\partial_t \mathbf{v}\|_{L_p((0,2\pi),L_q(B_R))} + \|\nabla \mathbf{q}\|_{L_p((0,2\pi),L_q(B_R))} \\ & + \|\rho\|_{L_p((0,2\pi),W_q^{3-1/q}(S_R))} + \|\partial_t \rho\|_{L_p((0,2\pi),W_q^{2-1/q}(S_R))} \\ & \leq C \{ \|\mathbf{f}\|_{L_p((0,2\pi),L_q(B_R))} + \|d\|_{L_p((0,2\pi),W_q^{2-1/q}(S_R))} \\ & + \|\mathbf{g}\|_{H_p^1((0,2\pi),L_q(B_R))} + \|(g, \mathbf{h})\|_{H_p^{1/2}((0,2\pi),L_q(B_R))} + \|(g, \mathbf{h})\|_{L_p((0,2\pi),H_q^1(B_R))} \}. \end{aligned}$$

In what follows, I will give an idea how to prove Theorem 8.

4.9 \mathcal{R} -solver and High frequency part

For any periodic function, f , the stationary part f_S and oscillatory part f_{per} are defined by setting

$$f_S = \frac{1}{2\pi} \int_0^{2\pi} f(\cdot, s) ds, \quad f_{\text{per}}(\cdot, t) = f(\cdot, t) - f_S(\cdot).$$

And then, problem (4.27) is divided as follows:

$$\begin{cases} \mathcal{L}\mathbf{v}_S - \text{Div}(\mu(\mathbf{D}(\mathbf{v}_S) - \mathbf{q}_S\mathbf{I}) = \mathbf{f}_S & \text{in } B_R, \\ \text{div } \mathbf{v}_S = g_S = \text{div } \mathbf{g}_S & \text{in } B_R, \\ \mathcal{M}\rho_S - \mathcal{A}\mathbf{v}_S \cdot \mathbf{n} = d_S & \text{on } S_R \times (0, 2\pi), \\ (\mu\mathbf{D}(\mathbf{v}_S) - \mathbf{q}_S)\mathbf{n} - (\mathcal{B}_R\rho_S)\mathbf{n} = \mathbf{h}(\mathbf{v}, \rho)_S & \text{on } S_R \times (0, 2\pi), \end{cases} \quad (4.28)$$

and

$$\begin{cases} \partial_t \mathbf{v}_{\text{per}} - \text{Div}(\mu(\mathbf{D}(\mathbf{v}_{\text{per}}) - \mathbf{q}_{\text{per}}\mathbf{I}) = \mathbf{f}_{\text{per}} & \text{in } B_R \times (0, 2\pi), \\ \text{div } \mathbf{v}_{\text{per}} = g_{\text{per}} = \text{div } \mathbf{g}_{\text{per}} & \text{in } B_R \times (0, 2\pi), \\ \partial_t \rho_{\text{per}} + \mathcal{M}\rho_{\text{per}} - \mathcal{A}\mathbf{v}_{\text{per}} \cdot \mathbf{n} = d_{\text{per}} & \text{on } S_R \times (0, 2\pi), \\ (\mu\mathbf{D}(\mathbf{v}_{\text{per}}) - \mathbf{q}_{\text{per}})\mathbf{n} - (\mathcal{B}_R\rho_{\text{per}})\mathbf{n} = \mathbf{h}_{\text{per}} & \text{on } S_R \times (0, 2\pi), \end{cases} \quad (4.29)$$

In this subsection, I consider problem (4.29) for the high frequency part.

According to Sect. 3, I consider the generalized resolvent problem:

$$\begin{aligned} \lambda \mathbf{u} - \text{Div}(\mu\mathbf{D}(\mathbf{u}) - \mathbf{p}\mathbf{I}) &= \hat{\mathbf{f}} & \text{in } B_R, \\ \text{div } \mathbf{u} &= \hat{g} = \text{div } \hat{\mathbf{g}} & \text{in } B_R, \\ \lambda \eta + \mathcal{M}\eta - (\mathcal{A}\mathbf{u}) \cdot \mathbf{n} &= \hat{d} & \text{on } S_R, \\ (\mu\mathbf{D}(\mathbf{u}) - \mathbf{p}\mathbf{I})\mathbf{n} - (\mathcal{B}_R\eta)\mathbf{n} &= \hat{\mathbf{h}} & \text{on } S_R \end{aligned} \quad (4.30)$$

for any $\lambda \in \Sigma_{\epsilon, \lambda_0}$ with any $\epsilon \in (0, \pi/2)$ and some large positive number λ_0 depending on ϵ . And then, from the result due to Shibata [4, 5], the following theorem follows.

Theorem 9. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Let*

$$\begin{aligned} X_q(B_R) &= \{(\hat{\mathbf{f}}, \hat{d}, \hat{\mathbf{h}}, \hat{g}, \hat{\mathbf{g}}) \mid \hat{\mathbf{f}} \in L_q(B_R)^N, \hat{d} \in W_q^{2-1/q}, \hat{\mathbf{h}} \in H_q^1(B_R)^N, \\ &\quad \hat{g} \in H_q^1(B_R), \hat{\mathbf{g}} \in L_q(\mathbb{R}^N)^N\}, \\ \mathcal{X}_q(B_R) &= \{F = (F_1, F_2, \dots, F_7) \mid F_1, F_3, F_7 \in L_q(B_R)^N, F_2 \in W_q^{2-1/q}(S_R), \\ &\quad F_4 \in H_q^1(B_R)^N, F_5 \in L_q(B_R), F_6 \in H_q^1(\Omega)\}. \end{aligned}$$

Here, $F_1, F_2, F_3, F_4, F_5, F_6$, and F_7 are corresponding variables to $\hat{\mathbf{f}}, \hat{d}, \lambda^{1/2}\hat{\mathbf{h}}, \hat{\mathbf{h}}, \lambda^{1/2}\hat{g}, \hat{g}$, and $\lambda\hat{\mathbf{g}}$, respectively.

Then, there exist a constant $\lambda_0 > 0$ and operator families $\mathcal{A}(\lambda), \mathcal{P}(\lambda)$, and $\mathcal{H}(\lambda)$ with

$$\begin{aligned} 2\mathcal{A}(\lambda) &\in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(\mathcal{X}_q(B_R), H_q^2(B_R)^N)), \\ \mathcal{P}(\lambda) &\in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(\mathcal{X}_q(B_R), H_q^1(B_R))), \\ \mathcal{H}(\lambda) &\in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(\mathcal{X}_q(B_R), W_q^{3-1/q}(S_R))), \end{aligned}$$

where $\text{Hol}(\Sigma_{\epsilon, \lambda_0}, X)$ denotes the set of all X -valued holomorphic functions defined on $\Sigma_{\epsilon, \lambda_0}$, such that for any $(\hat{\mathbf{f}}, \hat{d}, \hat{\mathbf{h}}, \hat{g}, \hat{\mathbf{g}}) \in X_q(B_R)$ and $\lambda \in \Sigma_{\epsilon, \lambda_0}$, $\mathbf{v} = \mathcal{A}(\lambda)\mathcal{F}_\lambda$, $\mathbf{q} = \mathcal{P}(\lambda)\mathcal{F}_\lambda$ and $\eta = \mathcal{H}(\lambda)\mathcal{F}_\lambda$, where

$$\mathcal{F}_\lambda = (\hat{\mathbf{f}}, \hat{d}, \lambda^{1/2}\hat{\mathbf{h}}, \hat{\mathbf{h}}, \lambda^{1/2}\hat{g}, \hat{g}, \lambda\hat{\mathbf{g}}),$$

are unique solutions of equations (4.30), and

$$\begin{aligned} \mathcal{R}_{\mathcal{L}(\mathcal{X}_q(B_R), H_q^{2-m}(B_R)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{m/2}\mathcal{A}(\lambda)) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq r_b, \\ \mathcal{R}_{\mathcal{L}(\mathcal{X}_q(B_R), L_q(B_R)^N)}(\{(\tau\partial_\tau)^\ell\nabla\mathcal{P}(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq r_b, \\ \mathcal{R}_{\mathcal{L}(\mathcal{X}_q(B_R), W_q^{3-n-1/q}(S_R))}(\{(\tau\partial_\tau)^\ell(\lambda^n\mathcal{H}(\lambda)) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq r_b \end{aligned} \quad (4.31)$$

for $\ell = 0, 1$, $m = 0, 1, 2$ and $n = 0, 1$ with some constant r_b , where $\lambda = \gamma + i\tau \in \Sigma_{\epsilon, \lambda_0} \subset \mathbb{C}$.

Let k_0 be a natural number such that $\lambda_0 < k_0$ and φ a $C^\infty(\mathbb{R})$ function which equals one for $|k| \geq k_0 + 1$ and zero for $|k| \leq k_0 + 1/2$. Let

$$\begin{aligned}\mathbf{F}_\varphi &= \mathcal{F}_\mathbb{T}^{-1}[(\varphi(k)\hat{\mathbf{f}}_{\text{per}}(ik))_{k \in \mathbb{Z}}], & G_\varphi &= \mathcal{F}_\mathbb{T}^{-1}[(\varphi(k)\hat{g}_{\text{per}}(ik))_{k \in \mathbb{Z}}], & \mathbf{G}_\varphi &= \mathcal{F}_\mathbb{T}^{-1}[(\varphi(k)\hat{\mathbf{g}}_{\text{per}}(ik))_{k \in \mathbb{Z}}], \\ D_\varphi &= \mathcal{F}_\mathbb{T}^{-1}[(\varphi(k)\hat{d}_{\text{per}}(ik))_{k \in \mathbb{Z}}], & \mathbf{H}_\varphi &= \mathcal{F}_\mathbb{T}^{-1}[(\varphi(k)\hat{\mathbf{h}}_{\text{per}}(ik))_{k \in \mathbb{Z}}].\end{aligned}$$

Let

$$\mathbf{v}_\varphi = \mathcal{F}_\mathbb{T}^{-1}[(\varphi(k)\mathcal{A}(ik)\mathbf{F}_k)_{k \in \mathbb{Z}}], \quad \mathbf{q}_\varphi = \mathcal{F}_\mathbb{T}^{-1}[(\varphi(k)\mathcal{P}(ik)\mathbf{F}_k)_{k \in \mathbb{Z}}], \quad \rho_\varphi = \mathcal{F}_\mathbb{T}^{-1}[(\varphi(k)\mathcal{H}(ik)\mathbf{F}_k)_{k \in \mathbb{Z}}],$$

where $\mathbf{F}_k = (\hat{\mathbf{f}}_{\text{per}}(ik), \hat{d}_{\text{per}}(ik), (ik)^{1/2}\hat{\mathbf{h}}_{\text{per}}(ik), \hat{\mathbf{h}}_{\text{per}}(ik), (ik)^{1/2}\hat{g}_{\text{per}}(ik), \hat{g}_{\text{per}}(ik), ik\hat{\mathbf{g}}_{\text{per}}(ik))$. Then, \mathbf{v}_φ , \mathbf{q}_φ and ρ_φ are unique solutions of equations:

$$\begin{aligned}\partial_t \mathbf{v}_\varphi - \text{Div}(\mu \mathbf{D}(\mathbf{v}_\varphi) - \mathbf{q}_\varphi \mathbf{I}) &= \mathbf{F}_\varphi & \text{in } B_R \times (0, 2\pi), \\ \text{div } \mathbf{v}_\varphi &= G_\varphi = \text{div } \mathbf{G}_\varphi & \text{in } B_R \times (0, 2\pi), \\ \partial_t \rho_\varphi + \mathcal{M}\rho_\varphi - (\mathcal{A}\mathbf{v}_\varphi) \cdot \mathbf{n} &= D_\varphi & \text{on } S_R \times (0, 2\pi), \\ (\mu \mathbf{D}(\mathbf{v}_\varphi) - \mathbf{q}_\varphi \mathbf{I})\mathbf{n} - (\mathcal{B}_R \rho_\varphi)\mathbf{n} &= \mathbf{H}_\varphi & \text{on } S_R \times (0, 2\pi),\end{aligned}\tag{4.32}$$

with

$$\begin{aligned}\mathbf{v}_\varphi &\in L_{p,\text{per}}((0, 2\pi), H_q^2(B_R)^N) \cap H_{p,\text{per}}^1((0, 2\pi), L_q(\mathbb{R}^N)^N), \quad \nabla \mathbf{q}_\varphi \in L_{p,\text{per}}((0, 2\pi), L_q(B_R)^N), \\ \rho_\varphi &\in L_{p,\text{per}}((0, 2\pi), W_q^{3-1/q}(S_R)) \cap H_{p,\text{per}}^1((0, 2\pi), W_q^{2-1/q}(S_R)).\end{aligned}$$

Moreover, the following estimate holds:

$$\begin{aligned}&\|\mathbf{v}_\varphi\|_{L_p((0,2\pi), H_q^2(B_R))} + \|\partial_t \mathbf{v}_\varphi\|_{L_p((0,2\pi), L_q(B_R))} + \|\nabla \mathbf{q}_\varphi\|_{L_p((0,2\pi), L_q(B_R))} \\ &+ \|\rho_\varphi\|_{L_p((0,2\pi), W_q^{3-1/q}(S_R))} + \|\partial_t \rho_\varphi\|_{H_p^1((0,2\pi), W_q^{2-1/q}(S_R))} \\ &\leq C\{\|\mathbf{F}_\varphi\|_{L_p((0,2\pi), L_q(B_R))} + \|D_\varphi\|_{L_p((0,2\pi), W_q^{2-1/q}(S_R))} + \|(G_\varphi, \mathbf{H}_\varphi)\|_{H_p^{1/2}((0,2\pi), L_q(B_R))} \\ &+ \|(G_\varphi, \mathbf{H}_\varphi)\|_{L_p((0,2\pi), H_q^1(B_R))} + \|\partial_t \mathbf{G}_\varphi\|_{L_p((0,2\pi), L_q(B_R))}\}\end{aligned}\tag{4.33}$$

for some constant $C > 0$. To estimate the right side of (4.33), we use the inequality:

$$\|\mathcal{F}_\mathbb{T}^{-1}[(\varphi(k)\hat{f}(ik))_{k \in \mathbb{Z}}]\|_{L_p((0,2\pi), X)} \leq C_p \|f\|_{L_p((0,2\pi), X)}$$

where X is a UMD Banach space, which follows from Weis' operator valued Fourier multiplier theorem, Theorem 4.

4.10 Low frequency part

I now consider the generalized resolvent problem corresponding to (4.29) for $k \in [-k_0, k_0]$. Namely, I consider the following equations:

$$\begin{aligned}ik\mathbf{v}_k - \text{Div}(\mu \mathbf{D}(\mathbf{v}_k) - \mathbf{p}_k \mathbf{I}) &= \hat{\mathbf{f}}_{\text{per}}(ik) & \text{in } B_R, \\ \text{div } \mathbf{v}_k &= \hat{g}(ik) = \text{div } \hat{\mathbf{g}}_{\text{per}}(ik) & \text{in } B_R, \\ ik\rho_k + \mathcal{M}\rho_k - (\mathcal{A}\mathbf{v}_k) \cdot \mathbf{n} &= \hat{d}_{\text{per}}(ik) & \text{on } S_R, \\ (\mu \mathbf{D}(\mathbf{v}_k) - \mathbf{p}_k \mathbf{I})\mathbf{n} - (\mathcal{B}_R \rho_k)\mathbf{n} &= \hat{\mathbf{h}}_{\text{per}}(ik) & \text{on } S_R\end{aligned}\tag{4.34}$$

for $k \in [-k_0, k_0] \setminus \{0\}$. Then, the following theorem holds.

Theorem 10. *Let $1 < q < \infty$ and $k \in \mathbb{Z}$ with $1 \leq |k| \leq k_0$. Then, for any $\hat{\mathbf{f}}_{\text{per}}(ik) \in L_q(B_R)^N$, $\hat{g}_{\text{per}}(ik) \in H_q^1(B_R)$, $\hat{d}_{\text{per}}(ik) \in W_q^{2-1/q}(S_R)$, $\hat{\mathbf{h}}_{\text{per}}(ik) \in H_q^1(B_R)^N$, and $\hat{\mathbf{g}}_{\text{per}}(ik) \in L_q(B_R)^N$, problem (4.38) admits unique solutions $\mathbf{v}_k \in H_q^2(B_R)^N$, $\mathbf{q}_k \in H_q^1(B_R)$, and $\eta_k \in W_q^{3-1/q}(S_R)$ possessing the estimate:*

$$\begin{aligned} & \|\mathbf{v}_k\|_{H_q^2(B_R)} + \|\nabla \mathbf{q}_k\|_{L_q(B_R)} + \|\eta_k\|_{W_q^{3-1/q}(S_R)} \\ & \leq C(\|\hat{\mathbf{f}}_{\text{per}}(ik)\|_{L_q(B_R)} + \|\hat{d}_{\text{per}}(ik)\|_{W_q^{2-1/q}(S_R)} + \|(\hat{g}_{\text{per}}(ik), \hat{\mathbf{h}}(ik))\|_{H_q^1(B_R)} + \|\hat{\mathbf{g}}_{\text{per}}(ik)\|_{L_q(B_R)}) \end{aligned} \quad (4.35)$$

for some constant $C > 0$ independent of k with $|k| \leq k_0$.

Remark 11. To estimate the right side of (4.39), we use the inequality:

$$\|\hat{f}(ik)\|_X \leq (2\pi)^{-1} \int_0^{2\pi} \|f(s)\|_X ds \leq (2\pi)^{-1/p'} \|f\|_{L_p((0,2\pi),X)}$$

for any $f \in L_p((0,2\pi), X)$, where X is a Banach space and $\|\cdot\|_X$ is its norm.

To prove Theorem 10, in view of the Riesz-Schauder theorem, Fredholm alternative principle, it is sufficient to prove the uniqueness in the L_2 framework. Let $\mathbf{w} \in H_2^2(B_R)^N$, $\mathbf{q} \in H_2^1(B_R)$ and $\zeta \in W_2^{3-1/2}(S_R)$ satisfy the homogeneous equations:

$$\begin{aligned} ik\mathbf{w} - \text{Div}(\mu\mathbf{D}(\mathbf{w}) - \mathbf{q}\mathbf{I}) &= 0, \quad \text{div } \mathbf{w} = 0 && \text{in } B_R, \\ ik\zeta + \mathcal{M}\zeta - (\mathcal{A}\mathbf{w}) \cdot \mathbf{n} &= 0 && \text{on } S_R, \\ (\mu\mathbf{D}(\mathbf{w}) - \mathbf{q}\mathbf{I})\mathbf{n} - \sigma(\mathcal{B}_R\zeta)\mathbf{n} &= 0 && \text{on } S_R. \end{aligned} \quad (4.36)$$

Recall: $\mathcal{M}\zeta = \int_{S_R} \zeta d\omega + \sum_{k=1}^N \left(\int_{S_R} \zeta \omega_k d\omega \right) y_k$ and $\mathcal{A}\mathbf{v} = \mathbf{v} - \frac{1}{|B_R|} \int_{B_R} \mathbf{v} dy$. We first prove that

$$(\zeta, 1)_{S_R} = 0, \quad (\zeta, x_j)_{S_R} = 0 \quad \text{for } j = 1, \dots, N. \quad (4.37)$$

Integrating the second equation of equations (4.40) and applying the divergence theorem of Gauss gives that

$$0 = ik(\zeta, 1)_{S_R} + (\zeta, 1)_{S_R} |S_R| - \int_{B_R} \text{div } \mathcal{A}\mathbf{w} dx = (ik + |S_R|)(\zeta, 1)_{S_R},$$

where we have set $|S_R| = \int_{S_R} d\omega$ and we have used the fact that $\text{div } \mathbf{w} = 0$ in B_R . Thus, we have $(\zeta, 1)_{S_R} = 0$.

Multiplying the second equation of equations (4.40) with x_j , integrating the resultant formula over S_R and using the divergence theorem of Gauss gives that

$$0 = ik(\zeta, x_k)_{S_R} + (\zeta, x_k)_{S_R} (x_k, x_k)_{S_R} - \int_{B_R} \text{div} (x_k \mathcal{A}\mathbf{w}) dx,$$

because $(x_j, x_k)_{S_R} = 0$ for $j \neq k$. Since

$$\int_{B_R} \text{div} (x_k \mathcal{A}\mathbf{w}) dx = \int_{B_R} (\mathbf{w}_k - \frac{1}{|B_R|} \int_{B_R} \mathbf{w}_k dx) dx = 0,$$

we have $(\zeta, x_k)_{S_R} = 0$, because $(x_k, x_k)_{S_R} = (R^2/N)|S_R| > 0$. Thus, we have proved (4.37). In particular, $\mathcal{M}\zeta = 0$ in (4.40).

We now prove that $\mathbf{w} = 0$. Multiplying the first equation of (4.40) with \mathbf{w} and integrating the resultant formula over B_R and using the divergence theorem of Gauss gives that

$$0 = ik\|\mathbf{w}\|_{L_2(B_R)}^2 - \sigma(\mathcal{B}_R\zeta, \mathbf{n} \cdot \mathbf{w})_{S_R} + \frac{\mu}{2}\|\mathbf{D}(\mathbf{w})\|_{L_2(B_R)}^2,$$

because $\operatorname{div} \mathbf{w} = 0$ in B_R . By the second equation of (4.40) with $\mathcal{M}\zeta = 0$, we have

$$\sigma(\mathcal{B}_R\zeta, \mathbf{n} \cdot \mathbf{w})_{S_R} = \sigma(\mathcal{B}_R\zeta, ik\zeta)_{S_R} + \sum_{j=1}^N \frac{1}{|B_R|} \int_{B_R} w_j dt(\mathcal{B}_R\zeta, R^{-1}x_j)_{S_R},$$

where we have used $\mathbf{n} = R^{-1}x = R^{-1}(x_1, \dots, x_N)$ for $x \in S_R$. Thus,

$$(\mathcal{B}_R\zeta, x_j)_{S_R} = (\zeta, (\Delta_{S_R} + \frac{N-1}{R^2})x_j)_{S_R} = 0.$$

Moreover, since ζ satisfies (4.37), we know that $-(\mathcal{B}_R\zeta, \zeta)_{S_R} \geq c\|\zeta\|_{L_2(S_R)}^2$ for some positive constant c , and therefore we have $\mathbf{w} = 0$. And then, $\nabla \mathbf{q} = 0$, which yields that \mathbf{q} is a constant. Since $\mathcal{B}_R\zeta - \mathbf{q} = 0$ on S_R , integrating this formula on S_R , we have $\mathbf{q}|S_R| = 0$, because $(\mathcal{B}_R\zeta, 1)_{S_R} = (N-1)R^{-2}(\zeta, 1)_{S_R} = 0$, and so $\mathbf{q} = 0$.

Finally, combining $\mathcal{B}_R\zeta = 0$ on S_R and $(\zeta, 1)_{S_R} = (\zeta, x_j)_{S_R} = 0$ gives that $\zeta = 0$. This completes the proof of the uniqueness.

4.11 Stationary solution

Let me consider the following stationary problem:

$$\begin{aligned} \mathcal{L}\mathbf{v}_S - \operatorname{Div}(\mu\mathbf{D}(\mathbf{v}_S) - \mathbf{q}_S\mathbf{I}) &= \mathbf{f}_S && \text{in } B_R, \\ \operatorname{div} \mathbf{v}_S &= g_S = \operatorname{div} \mathbf{g}_S && \text{in } B_R, \\ \mathcal{M}\eta_S - (\mathcal{A}\mathbf{v}_S) \cdot \mathbf{n} &= d_S && \text{on } S_R, \\ (\mu\mathbf{D}(\mathbf{v}_S) - \mathbf{q}_S\mathbf{I})\mathbf{n} - (\mathcal{B}_R\eta_S)\mathbf{n} &= \mathbf{h}_S && \text{on } S_R. \end{aligned} \tag{4.38}$$

The following theorem holds.

Theorem 12. *Let $1 < q < \infty$. Then, for any $\mathbf{f}_S \in L_q(B_R)^N$, $g_S \in H_q^1(B_R)$, $d_S \in W_q^{2-1/q}(S_R)$, $\mathbf{h}_S \in H_q^1(B_R)^N$, and $\mathbf{g}_S \in L_q(B_R)^N$, problem (4.38) admits unique solutions $\mathbf{v}_S \in H_q^2(B_R)^N$, $\mathbf{q}_S \in H_q^1(B_R)$, and $\rho_S \in W_q^{3-1/q}(S_R)$ possessing the estimate:*

$$\begin{aligned} \|\mathbf{v}_S\|_{H_q^2(B_R)} + \|\nabla \mathbf{q}_S\|_{L_q(B_R)} + \|\rho_S\|_{W_q^{3-1/q}(S_R)} \\ \leq C(\|\mathbf{f}_S\|_{L_q(B_R)} + \|d_S\|_{W_q^{2-1/q}(S_R)} + \|(g_S, \mathbf{h}_S)\|_{H_q^1(B_R)} + \|\mathbf{g}_S\|_{L_q(B_R)}) \end{aligned} \tag{4.39}$$

for some constant $C > 0$.

To prove Theorem 12, in view of the Riesz-Schauder theorem, Fredholm alternative principle, it is sufficient to prove the uniqueness in the L_2 framework. Let $\mathbf{w} \in H_2^2(B_R)^N$, $\mathbf{q} \in H_2^1(B_R)$ and $\zeta \in W_2^{3-1/2}(S_R)$ satisfy the homogeneous equations:

$$\begin{aligned} \mathcal{L}\mathbf{w} - \operatorname{Div}(\mu\mathbf{D}(\mathbf{w}) - \mathbf{q}\mathbf{I}) &= 0, \quad \operatorname{div} \mathbf{w} = 0 && \text{in } B_R, \\ \mathcal{M}\zeta - (\mathcal{A}\mathbf{w}) \cdot \mathbf{n} &= 0 && \text{on } S_R, \\ (\mu\mathbf{D}(\mathbf{w}) - \mathbf{q}\mathbf{I})\mathbf{n} - \sigma(\mathcal{B}_R\zeta)\mathbf{n} &= 0 && \text{on } S_R. \end{aligned} \tag{4.40}$$

Employing the same argument as in Subsec.4.10, we have

$$(\zeta, 1)_{S_R} = 0, \quad (\zeta, x_j)_{S_R} = 0 \quad \text{for } j = 1, \dots, N. \quad (4.41)$$

We now prove that $\mathbf{w} = 0$. Multiplying the first equation of (4.40) with \mathbf{w} and integrating the resultant formula over B_R and using the divergence theorem of Gauss gives that

$$0 = (\mathcal{L}\mathbf{w}, \mathbf{w})_{B_R} - \sigma(\mathcal{B}_R\zeta, \mathbf{n} \cdot \mathbf{w})_{S_R} + \frac{\mu}{2} \|\mathbf{D}(\mathbf{w})\|_{L_2(B_R)}^2,$$

because $\text{div } \mathbf{w} = 0$ in B_R . Recalling that $\mathcal{L}\mathbf{v}_S = 2\pi \sum_{k=1}^M (\mathbf{v}_S, \mathbf{p}_k)_T \mathbf{p}_k$, we have

$$(\mathcal{L}\mathbf{w}, \mathbf{w})_{B_R} = \sum_{k=1}^M |(\mathbf{w}, \mathbf{p}_k)_{B_R}|^2.$$

Employing the same argument as in Subsec.4.10, we have

$$\sigma(\mathcal{B}_R\zeta, \mathbf{n} \cdot \mathbf{w})_{S_R} = \sum_{k=1}^N \frac{1}{|B_R|} \int_{B_R} w_j dt(\mathcal{B}_R\zeta, R^{-1}x_j)_{S_R} = 0.$$

Thus,

$$0 = \sum_{k=1}^M |(\mathbf{w}, \mathbf{p}_k)_{B_R}|^2 + \frac{\mu}{2} \|\mathbf{D}(\mathbf{w})\|_{L_2(B_R)}^2,$$

which yields that $\mathbf{w} = 0$. And then, $\nabla \mathbf{q} = 0$, which shows that \mathbf{q} is a constant. Thus, $\mathcal{B}_R\zeta - \mathbf{q} = 0$ on S_R . Integrating this formula on S_R , we have $\mathbf{q}|_{S_R} = 0$, and so $\mathbf{q} = 0$.

Finally, combining $\mathcal{B}_R\zeta = 0$ on S_R and $(\zeta, 1)_{S_R} = (\zeta, x_j)_{S_R} = 0$ gives that $\zeta = 0$. This completes the proof of the uniqueness.

Proof of Theorem 8. Since solutions \mathbf{v} , \mathbf{q} and ρ of equations (4.27) are represented as

$$(\mathbf{v}, \mathbf{q}, \rho) = (\mathbf{v}_\varphi, \mathbf{q}_\varphi, \rho_\varphi) + \sum_{1 \leq |k| \leq k_0} (\mathbf{v}_k, \mathbf{q}_k, \rho_k) + (\mathbf{v}_S, \mathbf{q}_S, \rho_S),$$

applying estimate (4.33), Theorem 10, and Theorem 12 yields Theorem 8. \square

5 Proof of Theorem 7

Theorem 7 is proved by the standard Banch fixed point theorem. Let $\epsilon > 0$ be a small number determined later and let \mathcal{I}_ϵ be an underlying space defined by setting

$$\begin{aligned} \mathcal{I}_\epsilon = \{ & (\mathbf{v}, \rho) \mid \mathbf{v} \in L_{p,\text{per}}((0, 2\pi), H_q^2(B_R)^N) \cap H_{p,\text{per}}^1((0, 2\pi), L_q(B_R)^N), \\ & \rho \in L_{p,\text{per}}((0, 2\pi), W_q^{3-1/q}(S_R)) \cap H_{p,\text{per}}^1((0, 2\pi), W_q^{2-1/q}(S_R)) \cap H_{\infty,\text{per}}^1((0, 2\pi), W_q^{1-1/q}(S_R)), \\ & \sup_{t \in (0, 2\pi)} \|H_\rho\|_{H_\infty^1(B_R)} \leq \delta, \quad E(\mathbf{v}, \rho) \leq \epsilon \}, \end{aligned} \quad (5.1)$$

where we have set

$$E(\mathbf{v}, \rho) = \|\mathbf{v}\|_{L_p((0, 2\pi), H_q^2(B_R)^N)} + \|\mathbf{v}\|_{H_p^1((0, 2\pi), L_q(B_R)^N)}$$

$$+ \|\rho\|_{L_p((0,2\pi), W_q^{3-1/q}(S_R))} + \|\rho\|_{H_p^1((0,2\pi), W_q^{2-1/q}(S_R))} + \|\partial_t \rho\|_{L_\infty((0,2\pi), W_q^{1-1/q}(S_R))}.$$

Let $(\mathbf{v}, \rho) \in \mathcal{I}_\epsilon$ and let \mathbf{u} , \mathbf{q} and η be solutions of linear equations:

$$\begin{cases} \partial_t \mathbf{u} + \mathcal{L}\mathbf{u}_S - \text{Div}(\mu(\mathbf{D}(\mathbf{u}) - \mathbf{q}\mathbf{I}) = \mathbf{G} + \mathbf{F}(\mathbf{v}, \rho) & \text{in } B_R \times (0, 2\pi), \\ \text{div } \mathbf{u} = g(\mathbf{u}, \rho) = \text{div } \mathbf{g}(\mathbf{v}, \rho) & \text{in } B_R \times (0, 2\pi), \\ \partial_t \eta + \mathcal{M}\eta - \mathbf{A}\mathbf{u} \cdot \mathbf{n} = \tilde{d}(\mathbf{v}, \rho) & \text{on } S_R \times (0, 2\pi), \\ (\mu\mathbf{D}(\mathbf{u}) - \mathbf{q})\mathbf{n} - (\mathcal{B}_R\eta)\mathbf{n} = \mathbf{h}(\mathbf{v}, \rho) & \text{on } S_R \times (0, 2\pi), \end{cases} \quad (5.2)$$

Applying Theorem 8 to equations (5.2) yields that

$$\begin{aligned} & \|\mathbf{v}\|_{L_p((0,2\pi), H_q^2(B_R)^N)} + \|\mathbf{v}\|_{H_p^1((0,2\pi), L_q(B_R)^N)} \\ & + \|\rho\|_{L_p((0,2\pi), W_q^{3-1/q}(S_R))} + \|\rho\|_{H_p^1((0,2\pi), W_q^{2-1/q}(S_R))} \\ & \leq C(\|\mathbf{G}\|_{L_p((0,2\pi), L_q(B_R))} + \mathcal{E}(\mathbf{v}, \rho)) \end{aligned} \quad (5.3)$$

with

$$\begin{aligned} \mathcal{E}(\mathbf{v}, \rho) &= \|\mathbf{F}(\mathbf{v}, \rho)\|_{L_p((0,2\pi), L_q(B_R))} + \|\tilde{d}(\mathbf{v}, \rho)\|_{L_p((0,2\pi), W_q^{2-1/q}(S_R))} + \|\mathbf{g}(\mathbf{v}, \rho)\|_{H_p^1((0,2\pi), L_q(B_R))} \\ & + \|(g(\mathbf{v}, \rho), \mathbf{h}(\mathbf{v}, \rho))\|_{H_p^{1/2}((0,2\pi), L_q(B_R))} + \|(g(\mathbf{v}, \rho), \mathbf{h}(\mathbf{v}, \rho))\|_{L_p((0,2\pi), H_q^1(B_R))}. \end{aligned}$$

To estimate $\|\partial_t \eta\|_{L_\infty((0,2\pi), W_q^{1-1/q}(S_R))}$, the following estimate is used:

$$\|\partial_t \eta\|_{L_\infty((0,2\pi), W_q^{1-1/q}(S_R))} \leq C(\|\mathcal{M}\rho\|_{L_\infty, W_q^{1-1/q}(S_R)} + \|\mathbf{v}\|_{L_\infty((0,2\pi), H_q^1(B_R))} + \|\tilde{d}\|_{L_\infty((0,2\pi), W_q^{1-1/q}(S_R))}),$$

which follows from the third equation of equations (5.2). The main task is to prove that

$$\mathcal{E}(\mathbf{v}, \rho) + \|\tilde{d}\|_{L_\infty((0,2\pi), W_q^{1-1/q}(S_R))} \leq C\epsilon^2 \quad (5.4)$$

with some constant $C > 0$ independent of ϵ . In the proof, it is assumed that $N < q < \infty$, $2 < p < \infty$ and $2/p + N/q < 1$. In particular, the first assumption is to use Sobolev's imbedding theorem. In fact, the following inequalities are used:

$$\begin{aligned} \|f\|_{L_\infty(B_R)} &\leq C\|f\|_{H_q^1(B_R)}, \\ \|fg\|_{H_q^1(B_R)} &\leq C\|f\|_{H_q^1(B_R)}\|g\|_{H_q^1(B_R)}, \\ \|fg\|_{H_q^2(B_R)} &\leq C(\|f\|_{H_q^2(B_R)}\|g\|_{H_q^1(B_R)} + \|f\|_{H_q^1(B_R)}\|g\|_{H_q^2(B_R)}), \\ \|fg\|_{W_q^{1-1/q}(S_R)} &\leq C\|f\|_{W_q^{1-1/q}(S_R)}\|g\|_{W_q^{1-1/q}(S_R)}, \\ \|fg\|_{W_q^{2-1/q}(S_R)} &\leq C(\|f\|_{W_q^{2-1/q}(S_R)}\|g\|_{W_q^{1-1/q}(S_R)} + \|f\|_{W_q^{1-1/q}(S_R)}\|g\|_{W_q^{2-1/q}(S_R)}), \end{aligned}$$

which follows from the Sobolev inequality and the fact that $\|u|_{S_R}\|_{W_q^{1-1/q}(S_R)} \leq C\|u\|_{H_q^1(B_R)}$ for $u \in H_q^1(B_R)$. To estimate the lower order derivatives of \mathbf{v} and ρ , the following inequalities are used:

$$\begin{aligned} \|\mathbf{v}\|_{L_\infty((0,2\pi), B_{q,p}^{2(1-1/p)}(B_R))} &\leq C(\|\mathbf{v}\|_{L_p((0,2\pi), H_q^2(B_R))} + \|\partial_t \mathbf{v}\|_{L_p((0,2\pi), L_q(B_R))}), \\ \|\rho\|_{L_\infty((0,2\pi), W_{q,p}^{3-1/p-1/q}(B_R))} &\leq C(\|\rho\|_{L_p((0,2\pi), W_q^{3-1/q}(B_R))} + \|\partial_t \rho\|_{L_p((0,2\pi), W_q^{2-1/q}(B_R))}), \end{aligned}$$

which follows from real interpolation theorem. In particular, to obtain $\nabla \mathbf{v} \in L_\infty$, it is used the assumption: $2/p + N/q < 1$.

To estimate $\|(g(\mathbf{v}, \rho), \mathbf{h}(\mathbf{v}, \rho))\|_{H_p^{1/2}((0,2\pi), L_q(B_R))} + \|(g(\mathbf{v}, \rho), \mathbf{h}(\mathbf{v}, \rho))\|_{L_p((0,2\pi), H_q^1(B_R))}$, the following two lemmas are used:

Lemma 13. Let $1 < p < \infty$ and $N < q < \infty$. Let

$$\begin{aligned} a &\in H_{\infty,\text{per}}^1((0, 2\pi), L_q(B_R)) \cap L_{\infty,\text{per}}((0, 2\pi), H_q^1(B_R)), \\ b &\in H_{p,\text{per}}^{1/2}((0, 2\pi), L_q(B_R)) \cap L_{p,\text{per}}((0, 2\pi), H_q^1(B_R)). \end{aligned}$$

Then,

$$\begin{aligned} &\|ab\|_{H_p^{1/2}((0,2\pi),L_q(B_R))} + \|ab\|_{L_p((0,2\pi),H_q^1(B_R))} \\ &\leq C(\|a\|_{H_{\infty}^1((0,2\pi),L_q(B_R))} + \|a\|_{L_{\infty}((0,2\pi),H_q^1(B_R))})^{1/2} \|a\|_{L_{\infty}((0,2\pi),H_q^1(B_R))}^{1/2} \\ &\quad \times (\|b\|_{H_p^{1/2}((0,2\pi),L_q(B_R))} + \|b\|_{L_p((0,2\pi),H_q^1(B_R))}). \end{aligned}$$

Remark 14. This lemma holds for more general domains.

Proof. The lemma follows from the following complex interpolation relation of order $1/2$:

$$\begin{aligned} &H_{p,\text{per}}^{1/2}((0, 2\pi), L_q(B_R)) \cap L_{p,\text{per}}((0, 2\pi), H_q^{1/2}(B_R)) \\ &= (L_{p,\text{per}}((0, 2\pi), L_q(B_R)), H_{p,\text{per}}^1((0, 2\pi), L_q(B_R)) \cap L_{p,\text{per}}((0, 2\pi), H_q^1(B_R)))_{1/2}. \end{aligned}$$

□

Lemma 15. Let $1 < p, q < \infty$. Then, there exists a constant C such that for any u with

$$u \in H_{p,\text{per}}^1((0, 2\pi), L_q(B_R)) \cap L_{p,\text{per}}((0, 2\pi), H_q^2(B_R)),$$

we have

$$\|u\|_{H_p^{1/2}((0,2\pi),H_q^1(B_R))} \leq C(\|u\|_{H_p^1((0,2\pi),L_q(B_R))} + \|u\|_{L_p((0,2\pi),H_q^2(B_R))})$$

for some constant $C > 0$.

Remark 16. This lemma holds for more general domains.

Proof. For a proof, refer to [6].

□

Proof of Theorem 7. Combining (5.3) and (5.4) yields that

$$E(\mathbf{u}, \eta) \leq C\|\mathbf{G}\|_{L_p((0,\infty),L_q(B_R))} + C\epsilon^2$$

for some constant $C > 0$ independent of ϵ . Thus, choosing $\epsilon > 0$ so small that $C\epsilon < 1/2$ yields that

$$E(\mathbf{u}, \eta) \leq C\|\mathbf{G}\|_{L_p((0,\infty),L_q(B_R))} + \epsilon/2.$$

Choosing \mathbf{f} so small that $C\|\mathbf{G}\|_{L_p((0,2\pi),L_q(B_R))} \leq \epsilon/2$ yields that $E(\mathbf{u}, \eta) \leq \epsilon$, and so $(\mathbf{u}, \eta) \in \mathcal{I}_{\epsilon}$. Let Ψ be a map acting on $(\mathbf{u}, \rho) \in \mathcal{I}_{\epsilon}$ defined by $\Psi(\mathbf{u}, \rho) = (\mathbf{v}, \eta)$, and then Ψ is a map from \mathcal{I}_{ϵ} into itself. It also can be proved that

$$E(\Psi(\mathbf{v}_1, \rho_1) - \Psi(\mathbf{v}_2, \rho_2)) \leq C\epsilon E((\mathbf{v}_1, \rho_1) - (\mathbf{v}_2, \rho_2))$$

for any $(\mathbf{v}_i, \rho_i) \in \mathcal{I}_{\epsilon}$ ($i = 1, 2$). Choosing $\epsilon > 0$ smaller if necessary, we may assume that $C\epsilon < 1$, and so Ψ is a contraction map from \mathcal{I}_{ϵ} into itself. Thus, there exists a unique fixed point $(\mathbf{v}, \rho) \in \mathcal{I}_{\epsilon}$, which is a required unique solution of equations (4.22).

Finally, we define $\xi(t)$ by setting

$$\xi(t) = \int_0^t \xi'(s) ds + c = \frac{1}{|B_R|} \int_0^t \int_{B_R} \mathbf{v}(x, s)(1 + J_0(x, s)) dx ds + c$$

where c is a constant for which

$$\int_0^{2\pi} \xi(s) ds = 0, \text{ that is, } c = -\frac{1}{2\pi|B_R|} \int_0^{2\pi} \left(\int_0^t \int_{B_R} (\mathbf{v}(x, s)(1 + J_0(x, s)) dx ds \right) dt.$$

We define Ω_t and Γ_t by the formulas in (4.13). And then, setting $\mathbf{u}(x, t) = \mathbf{v}(\Phi^{-1}(x, t), t)$ and $\mathbf{p}(x, t) = \mathbf{q}(\Phi^{-1}(x, t), t)$, we see that Ω_t , Γ_t , $\mathbf{u}(x, t)$ and $\mathbf{q}(x, t)$ satisfy the equations:

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \text{Div}(\mu \mathbf{D}(\mathbf{u}) - \mathbf{p} \mathbf{I}) + \sum_{k=1}^M \int_0^{2\pi} (\mathbf{u}(\cdot, t), \mathbf{p}_k)_{\Omega_t} dt \mathbf{p}_k &= \mathbf{f}, \quad \text{div } \mathbf{u} = 0 && \text{in } \Omega_t, \\ (\mu(\mathbf{D}(\mathbf{u}) - \mathbf{p} \mathbf{I}) \mathbf{n}_t = \sigma H(\Gamma_t) \mathbf{n}_t &&& \text{on } \Gamma_t, \end{aligned}$$

In particular, $\text{div } \mathbf{u} = 0$ implies that $|\Omega_t|$ is a constant, and so we set $|\Omega| = |B_R|$. And also, we see that

$$\xi(t) = \int_{\Omega_t} x dx,$$

and so by (4.18), (4.20) and (4.21),

$$\partial_t \rho - \mathcal{A} \mathbf{v} \cdot \mathbf{n} = d(\mathbf{v}, \rho).$$

Thus, the kinematic condition: $V_{\Gamma_t} = \mathbf{u} \cdot \mathbf{n}_t$ holds on Γ_t . Finally, the assumption on \mathbf{f} implies (4.10), and therefore, \mathbf{u} and \mathbf{p} satisfy equations (4.1). This completes the proof of Theorem 7. For the detailed proof, see Eiter, Kyed and Shibata [1].

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