On the resolvent estimates of compressible flows with free surfaces

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Abstract

This report concerns some resolvent estimates arising from the study of compressible flows with free surfaces, which were obtain in the recent work [12]. In [12], the resolvent estimates of some compressible model problem were established for general (bounded or unbounded) domains within the framework of the maximal $L_p - L_q$ regularity. For simplicity, we outline the whole strategy of the proof in [12] for exterior domains in this report.

1 Introduction

1.1 Model problem

Consider the free boundary value problem of the compressible Navier-Stokes equations in general domain Ω_t with taking the surface tension into account,

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \boldsymbol{v}) = 0 & \text{in} \quad \bigcup_{t \in]0,T[} \Omega_t \times \{t\}, \\ \partial_t(\rho \boldsymbol{v}) + \operatorname{Div}(\rho \boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{Div} \mathbb{T}\big(\boldsymbol{v}, \mathfrak{p}(\rho)\big) = \rho \boldsymbol{f} & \text{in} \quad \bigcup_{t \in]0,T[} \Omega_t \times \{t\}, \\ \mathbb{T}\big(\boldsymbol{v}, \mathfrak{p}(\rho)\big) \boldsymbol{n}_{\Gamma_t} - \sigma \mathcal{H}_{\Gamma_t} \boldsymbol{n}_{\Gamma_t} = -\mathfrak{p}_0 \boldsymbol{n}_{\Gamma_t}, \ V_{\Gamma_t} = \boldsymbol{v} \cdot \boldsymbol{n}_{\Gamma_t} & \text{on} \quad \bigcup_{t \in]0,T[} \Gamma_t \times \{t\}, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on} \quad S \times]0,T[, \\ (\rho, \boldsymbol{v}, \Omega_t)|_{t=0} = (\rho_\infty + \rho_0, \boldsymbol{v}_0, \Omega). \end{cases} \tag{1.1}$$

where the stress tensor $\mathbb{T}(\boldsymbol{v}, \mathfrak{p}) := \mathbb{S}(\boldsymbol{v}) - \mathfrak{p}\mathbb{I}$, $\mathbb{S}(\boldsymbol{u}) := \mu \mathbb{D}(\boldsymbol{u}) + (\nu - \mu) \operatorname{div} \boldsymbol{u}\, \mathbb{I}$, $\mathbb{D}(\boldsymbol{v})_{jk} := \partial_k v_j + \partial_j v_k$, $j, k = 1, \dots, N$, and the coefficients $\sigma, \mu, \nu > 0$. In addition, we also denote $\rho_\infty := \lim_{|x| \to \infty} \rho(\cdot, x) > 0$. Furthermore, $\mathfrak{p} = \mathfrak{p}(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth pressure law, $\mathcal{H}_{\Gamma_t}$ is the mean curvature of the surface Γ_t for $t \geq 0$, $\boldsymbol{n}_{\Gamma_t}$ is the unit normal vector of Γ_t, $\mathfrak{p}_0$ is the given pressure of the atmosphere, and $\boldsymbol{f}$ is some given external force. Here we call V_{Γ_t} the normal velocity of Γ_t. Moreover, S and Γ_t are disjoint for any $t \geq 0$. In fact, there is one typical case where Ω_t is an infinite layer with the moving top surface Γ_t and the fixed bottom S. Of course, if $S = \emptyset$, i.e. lack of $(1.1)_4$, then Ω_t may stand for some bounded droplet, some exterior region or some half plane without the bottom.

To study (1.1), the key step is the linearization. By the so-called Lagrangian coordinates

$$\boldsymbol{X}_u(\xi,t) := \xi + \int_0^t \boldsymbol{u}(\xi,s)\,ds \in \Omega_t, \ \ \forall \xi \ \in \Omega,$$

one can transfer (1.1) to the following model problem in the fixed initial (or reference) domain Ω,

$$\begin{cases} \partial_t \eta + \gamma_1 \operatorname{div} \boldsymbol{u} = d & \text{in} \quad \Omega \times]0,T[, \\ \gamma_1 \partial_t \boldsymbol{u} - \operatorname{Div}\big(\mathbb{S}(\boldsymbol{u}) - \gamma_2 \eta \mathbb{I}\big) = \boldsymbol{F} & \text{in} \quad \Omega \times]0,T[, \\ \big(\mathbb{S}(\boldsymbol{u}) - \gamma_2 \eta \mathbb{I}\big)\boldsymbol{n}_{\Gamma_0} + \sigma(m - \Delta_{\Gamma_0}) h\, \boldsymbol{n}_{\Gamma_0} = \boldsymbol{g} & \text{on} \quad \Gamma_0 \times]0,T[, \\ \partial_t h - \boldsymbol{u} \cdot \boldsymbol{n}_{\Gamma_0} = k & \text{on} \quad \Gamma_0 \times]0,T[, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on} \quad S \times]0,T[, \\ (\eta, \boldsymbol{u}, h)|_{t=0} = (\rho_0, \boldsymbol{v}_0, 0) & \text{in} \quad \Omega, \end{cases} \tag{1.2}$$

where $\gamma_1 = \gamma_1(x)$ and $\gamma_2 = \gamma_2(x)$ are uniformly continuous functions defined on $\overline{\Omega}$, $\boldsymbol{n}_{\Gamma_0}$ stands for the unit normal vector field of Γ_0 and Δ_{Γ_0} is the standard Laplace-Beltrami operator of Γ_0. Although the equations in (1.2) are in the fixed domain, the role of the new variable h is to handle the variation of the Γ_t in (1.1). The linearization approach is similar to the case of incompressible flows [9, 6], and we omit the details here. As (1.2) is more or less a standard system in Ω, we shall consider the solutions of (1.2) within the classical maximal $L_p - L_q$ regularity framework.

Let us give some comments on the property of the maximal regularity here, while the interested reader may refer to [12, Sec. 1] for more detailed references on the compressible flows. The maximal regularity is an important tool to understand the parabolic type equations. In particular, for the classic viscous fluid dynamics, the momentum conservation law can be linearized to the second order parabolic equations. We ask that whether the second order derivatives of the velocity fields are integrable with L_p in time and L_q in space for $(p,q) \in]1,\infty[^2$, so long as such external forces are given. If so, we say such equations admit the so-called the maximal $L_p - L_q$ regularity. For instance, the maximal regularity property of the Stokes equations with respect to incompressible fluids are studied in e.g. [10, 4] for the non-slip boundary conditions and in e.g. [5, 1, 8] for the slip boundary conditions. For the compressible flow, the maximal regularity property of the Lamé operators corresponding to compressible fluids is verified in e.g. [2, 3]. Let us emphasize that the equations for the compressible flow are usually mixed with the hyperbolic part and the parabolic part as in (1.1). The progress in [12] concerns the maximal regularity property of the Lamé operator with surface tension term involved, namely $\sigma > 0$ in $(1.2)_3$.

1.2 Notations

Let us end up this part with some useful notations. $L_q(G)$ is the standard Lebesgue space in the domain $G \subset \mathbb{R}^N$, and $H^k_p(G)$ with $k \in \mathbb{N}$ and $1 < q < \infty$ stands for the Sobolev space. In addition, the Besov space $B^s_{q,p}(G)$ for some $k-1 < s \leq k$ and for any $(p,q) \in]1,\infty[^2$ is defined by the real interpolation functor

$$B^s_{q,p}(G) := \big(L_q(G), H^k_q(G)\big)_{s/k,p}.$$

In particular, we write $W^s_q(G) = B^s_{q,q}(G)$ for simplicity, and $W^{-\bar{s}}_q(G)$ is the dual space of $W^{\bar{s}}_{q'}(G)$ for $0 < \bar{s} < 1$ and the conjugate index $q' := q/(q-1)$.

For any Banach spaces X, Y, the total of the bounded linear transformations from X to Y is denoted by $\mathcal{L}(X;Y)$. We also write $\mathcal{L}(X)$ for short if $X = Y$. In addition, $\mathrm{Hol}\,(\Lambda; X)$ denotes the set of X valued mappings defined on some domain $\Lambda \subset \mathbb{C}$.

For $\lambda = \gamma + i\tau \in \mathbb{C}$, the Laplace transform and its inverse are formulated by

$$\begin{aligned} \mathcal{L}[f](\lambda) &:= \int_{\mathbb{R}} e^{-\lambda t} f(t)dt = \mathcal{F}_t[e^{-\gamma t} f(t)](\tau), \\ \mathcal{L}^{-1}[g](t) &:= \frac{1}{2\pi}\int_{\mathbb{R}} e^{\lambda t} g(\tau) d\tau = e^{\gamma t}\mathcal{F}^{-1}_\tau[g(\tau)](t), \end{aligned}$$

where $\mathcal{F}_t$ and $\mathcal{F}^{-1}_\tau$ denote the Fourier transformation and its inverse. For any X valued function f, we set $\Lambda^s_\gamma f(t) := \mathcal{L}^{-1}\big[\lambda^s \mathcal{L}[f](\lambda)\big]$ for any $s > 0$. Then the Bessel potential spaces are defined as follows,

$$\begin{aligned} H^s_{p,\gamma}(\mathbb{R}; X) &:= \{f \in L_p(\mathbb{R}; X) : e^{-\gamma t}(\Lambda^s_\gamma f)(t) \in L_p(\mathbb{R}; X)\}, \\ H^s_{p,\gamma,0}(\mathbb{R}; X) &:= \{f \in H^s_{p,\gamma}(\mathbb{R}; X) : f(t) = 0 \text{ for } t < 0\}, \end{aligned}$$

for any $\gamma > 0$ and $1 < p < \infty$.

2 Main results

From now on, we assume that Ω is some exterior domain with the (compact) boundary Γ_0 only, and consider that

$$\begin{cases} \partial_t \eta + \gamma_1 \operatorname{div} \boldsymbol{u} = d & \text{in } \Omega \times \mathbb{R}_+, \\ \gamma_1 \partial_t \boldsymbol{u} - \operatorname{Div}\big(\mathbb{S}(\boldsymbol{u}) - \gamma_2 \eta \mathbb{I}\big) = \boldsymbol{f} & \text{in } \Omega \times \mathbb{R}_+, \\ \big(\mathbb{S}(\boldsymbol{u}) - \gamma_2 \eta \mathbb{I}\big)\boldsymbol{n}_{\Gamma_0} + \sigma(m - \Delta_{\Gamma_0}) h\, \boldsymbol{n}_{\Gamma_0} = \boldsymbol{g} & \text{on } \Gamma_0 \times \mathbb{R}_+, \\ \partial_t h - \boldsymbol{u} \cdot \boldsymbol{n}_{\Gamma_0} = k & \text{on } \Gamma_0 \times \mathbb{R}_+, \\ (\eta, \boldsymbol{u}, h)|_{t=0} = (0, \boldsymbol{0}, 0) & \text{in } \Omega. \end{cases} \tag{2.1}$$

Above there exist constants ρ_1, ρ_2, ρ_3 such that

$$\begin{aligned} &0 < \rho_1 \leq \gamma_1(x) \leq \rho_2, \quad 0 < \gamma_2(x) \leq \rho_2, \ \ \forall\, x \in \overline{\Omega}, \\ &\|(\nabla\gamma_1, \nabla\gamma_2)\|_{L_r(\Omega)} \leq \rho_2, \quad \rho_3 := \max\big\{\rho_2, \|\gamma_1\gamma_2\|_{L_\infty(\Omega)\cap \widehat{H}^1_r(\Omega)}\big\}, \end{aligned} \tag{2.2}$$

with $N < r < \infty$. For the general domain Ω case, one may refer to [12] for more details. Now, let us fix the functional spaces for the data $(d, \boldsymbol{f}, \boldsymbol{g}, k)$ in (2.1). For any $(p, q) \in]1, \infty[^2$, we say $(d, \boldsymbol{f}, \boldsymbol{g}, k) \in \mathscr{F}_{p,q,\gamma}$ $(\gamma > 0)$, if d, $\boldsymbol{f}$, $\boldsymbol{g}$ and k fulfil that

$$\begin{aligned} &d \in L_{p,\gamma,0}\big(\mathbb{R}; H^1_q(\Omega)\big), \quad \boldsymbol{f} \in L_{p,\gamma,0}\big(\mathbb{R}; L_q(\Omega)^N\big), \\ &\boldsymbol{g} \in L_{p,\gamma,0}\big(\mathbb{R}; H^1_q(\Omega)^N\big) \cap H^{1/2}_{p,\gamma,0}\big(\mathbb{R}; L_q(\Omega)^N\big), \quad k \in L_{p,\gamma,0}\big(\mathbb{R}; W^{2-1/q}_q(\Gamma_0)\big), \end{aligned}$$

with the quantity

$$\begin{aligned} \|(d, \boldsymbol{f}, \boldsymbol{g}, k)\|_{\mathscr{F}_{p,q,\gamma}} :=& \|e^{-\gamma t} d\|_{L_p(\mathbb{R}; H^1_q(\Omega))} + \|e^{-\gamma t}(\boldsymbol{f}, \Lambda^{1/2}_\gamma \boldsymbol{g})\|_{L_p(\mathbb{R}; L_q(\Omega))} \\ &+ \|e^{-\gamma t}\boldsymbol{g}\|_{L_p(\mathbb{R}; H^1_q(\Omega))} + \|e^{-\gamma t} k\|_{L_p(\mathbb{R}; W^{2-1/q}_q(\Gamma_0))} < \infty. \end{aligned}$$

Next, we recall the smoothness of the domain and an auxiliary result proved in [7, Theorem 2.1] on the Laplace-Beltrami operator.

Definition 2.1. *We say that a connected open subset Ω in $\mathbb{R}^N$ ($N \geq 2$) is of class uniform $W^{m-1/r}_r$ for some integer $m \geq 2$ and $1 < r < \infty$. if and only if the boundary $\partial\Omega$ is uniformly characterized by local $W^{m-1/r}_r$ graph functions. That is, for any point $x_0 = (x_0', x_{0N}) \in \partial\Omega$, one can choose a Cartesian coordinate system with origin x_0 and coordinates $y = (y', y_N) := (y_1, ..., y_{N-1}, y_N)$, as well as positive constants α, β, K and some $W^{m-1/r}_r$ function h with $\|h\|_{W^{m-1/r}_r(B'_\alpha(x_0'))} \leq K$ such that*

$$\begin{aligned} \{(y', y_N) : h(y') - \beta < y_N < h(y'), |y'| < \alpha\} = \Omega \cap U_{\alpha,\beta,h}(x_0), \\ \{(y', y_N) : y_N = h(y'), |y'| < \alpha\} = \partial\Omega \cap U_{\alpha,\beta,h}(x_0), \end{aligned}$$

where $U_{\alpha,\beta,h}(x_0) := \{(y', y_N) : h(y') - \beta < y_N < h(y') + \beta, |y'| < \alpha\}$ and $B'_\alpha(x_0') := \{y' \in \mathbb{R}^{N-1} : |y' - x_0'| < \alpha\}$. Moreover, the choices of α, β, K are independent of the location of x_0. $\partial\Omega$ is uniform whenever $\partial\Omega$ is compact.

Proposition 2.2. *[7, Theorem 2.1] Let $0 < \varepsilon < \pi/2$, $1 < q, q' := q/(q-1) < \infty$, $N < r < \infty$ and $r \geq \max\{q, q'\}$. For any uniform $W^{2-1/r}_r$ boundary $\Gamma \subset \partial\Omega$, there exists a constant $\lambda_1 = \lambda_1(\varepsilon, \Gamma) > 0$, such that $\Sigma_{\varepsilon,\lambda_1}$ is contained in the resolvent set $\rho(\Delta_\Gamma)$ of Δ_Γ. That is, for any $\lambda \in \Sigma_{\varepsilon,\lambda_1}$ and $f \in W^{-1/q}_q(\Gamma)$, the resolvent problem*

$$(\lambda - \Delta_\Gamma)u = f \ \text{ on } \ \Gamma$$

admits a unique solution $u \in W^{2-1/q}_q(\Gamma)$ possessing the estimates

$$\|u\|_{W^{2-1/q}_q(\Gamma)} \leq C_{\varepsilon,q,r,\Gamma}\|f\|_{W^{-1/q}_q(\Gamma)}.$$

Then we can establish the property of the maximal $L_p - L_q$ regularity for (2.1) as follows.

Theorem 2.3 (Maximal $L_p - L_q$ regularity)**.** *Let $0 < \varepsilon < \pi/2$, $\sigma, \mu, \nu, \rho_1, \rho_2, \rho_3 > 0$, $1 < q, q' := q/(q-1) < \infty$, $N < r < \infty$ and $r \geq \max\{q, q'\}$. Assume that the exterior domain Ω is of type $W^{3-1/r}_r$, $m \geq \lambda_1(\varepsilon, \Gamma_0)$ by Proposition 2.2, and (2.2) is satisfied. Then there exist constants $\gamma_0, C > 0$ such that the following assertions hold true. For any $(d, \boldsymbol{f}, \boldsymbol{g}, k) \in \mathscr{F}_{p,q,\gamma_0}$, (2.1) admits a unique solution*

$$\eta \in H^1_{p,\gamma_0,0}\big(\mathbb{R}; H^1_q(\Omega)\big), \quad \boldsymbol{u} \in L_{p,\gamma_0,0}\big(\mathbb{R}; H^2_q(\Omega)^N\big) \cap H^1_{p,\gamma_0,0}\big(\mathbb{R}; L_q(\Omega)^N\big),$$
$$h \in L_{p,\gamma_0,0}\big(\mathbb{R}; W^{3-1/q}_q(\Gamma_0)\big) \cap H^1_{p,\gamma_0,0}\big(\mathbb{R}; W^{2-1/q}_q(\Gamma_0)\big).$$

Moreover, we have

$$\begin{aligned}\|e^{-\gamma_0 t}(\partial_t \eta, \eta)\|_{L_p(\mathbb{R}; H^1_q(\Omega))} + \|e^{-\gamma_0 t}(\partial_t \boldsymbol{u}, \Lambda^{1/2}_{\gamma_0} \nabla \boldsymbol{u})\|_{L_p(\mathbb{R}; L_q(\Omega))} + \|e^{-\gamma_0 t} \boldsymbol{u}\|_{L_p(\mathbb{R}; H^2_q(\Omega))} \\ + \|e^{-\gamma_0 t} \partial_t h\|_{L_p(\mathbb{R}; W^{2-1/q}_q(\Gamma_0))} + \|e^{-\gamma_0 t} h\|_{L_p(\mathbb{R}; W^{3-1/q}_q(\Gamma_0))} \leq C \|(d, \boldsymbol{f}, \boldsymbol{g}, k)\|_{\mathscr{F}_{p,q,\gamma_0}}.\end{aligned}$$

Similar to [12, Theorem 2.8], the fundamental step to prove Theorem 2.3 is to study the resolvent problem of (2.1),

$$\begin{cases} \lambda \eta + \gamma_1 \operatorname{div} \boldsymbol{u} = d & \text{in} \quad \Omega, \\ \gamma_1 \lambda \boldsymbol{u} - \operatorname{Div}\big(\mathbb{S}(\boldsymbol{u}) - \gamma_2 \eta \mathbb{I}\big) = \boldsymbol{F} & \text{in} \quad \Omega, \\ \big(\mathbb{S}(\boldsymbol{u}) - \gamma_2 \eta \mathbb{I}\big) \boldsymbol{n}_{\Gamma_0} + \sigma(m - \Delta_{\Gamma_0}) h \, \boldsymbol{n}_{\Gamma_0} = \boldsymbol{G} & \text{on} \quad \Gamma_0, \\ \lambda h - \boldsymbol{u} \cdot \boldsymbol{n}_{\Gamma_0} = K & \text{on} \quad \Gamma_0. \end{cases} \tag{2.3}$$

To state the result of (2.3), we need some preparations. Firstly, we recall the definition of $\mathcal{R}-$boundedness and the Fourier multiplier theory proved by L. Weis in [11] for the convenience.

Definition 2.4. *Let X, Y be two Banach spaces and $\mathcal{L}(X; Y)$ be the collection of all bounded linear operators from X to Y. We say that a family of bounded operators $\tau \subset \mathcal{L}(X, Y)$ is $\mathcal{R}$-bounded if for any $N \in \mathbb{N}$, $T_j \in \tau$, $x_j \in X$ and the Rademacher functions $r_j(t) := \operatorname{sign}(\sin 2^j \pi t)$ defined for $t \in [0, 1]$, the following inequality holds,*

$$\Big\| \sum_{j=1}^N r_j T_j x_j \Big\|_{L_p([0,1]; Y)} \leq C_p \Big\| \sum_{j=1}^N r_j x_j \Big\|_{L_p([0,1]; X)} \quad \text{for some } p \in [1, \infty[.$$

Above the choice of C_p depends only on p but not on N, T_j, x_j, r_j and $1 \leq j \leq N$. The smallest C_p is called $\mathcal{R}$-bound of τ, denoted by $R_{\mathcal{L}(X;Y)}(\tau)$.

Theorem 2.5 (Weis)**.** *Let X and Y be two UMD Banach spaces and $1 < p < \infty$. Let $M(\cdot)$ be a mapping in $C^1\big(\mathbb{R} \backslash \{0\}; \mathcal{L}(X; Y)\big)$ such that*

$$\mathcal{R}_{\mathcal{L}(X;Y)}\big(\{(\tau \partial_\tau)^\ell M(\tau) : \tau \in \mathbb{R} \backslash \{0\}\}\big) \leq r_b \quad (\ell = 0, 1),$$

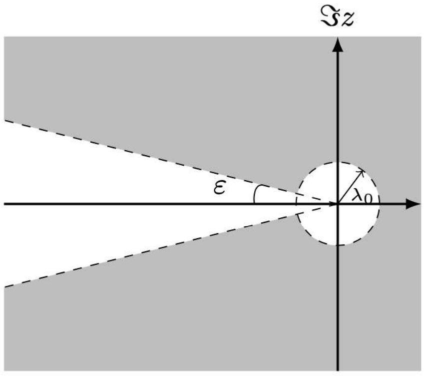
with some constant $r_b > 0$. Then the multiplier operator $T_M(\varphi) := \mathcal{F}^{-1}\big[M\mathcal{F}[\varphi]\big]$ for any $\varphi \in \mathcal{S}(\mathbb{R};X)$ can be uniquely extended to a bounded linear operator from $L_p(\mathbb{R};X)$ into $L_p(\mathbb{R};Y)$ with the bound

$$\|T_M\|_{\mathcal{L}\left(L_p(\mathbb{R};X);L_p(\mathbb{R};Y)\right)} \leq C_{p,X,Y} r_b.$$

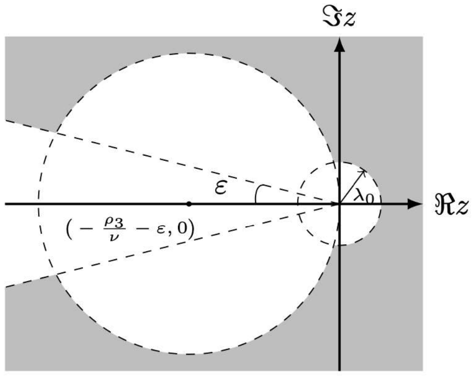
Secondly, for any $\nu, \lambda_0 > 0$ and $0 < \varepsilon < \pi/2$, we introduce the sectors

$$\Sigma_\varepsilon := \big\{z \in \mathbb{C}\backslash\{0\} : |\arg z| \leq \pi - \varepsilon\big\}, \quad \Sigma_{\varepsilon,\lambda_0} := \{z \in \Sigma_\varepsilon : |z| \geq \lambda_0\},$$
$$\Lambda_{\varepsilon,\lambda_0} := \Big\{z \in \Sigma_{\varepsilon,\lambda_0} : \big(\Re z + \frac{\rho_3}{\nu} + \varepsilon\big)^2 + (\Im z)^2 \geq \big(\frac{\rho_3}{\nu} + \varepsilon\big)^2\Big\}.$$

One may refer to the following graphs for $\Sigma_{\varepsilon,\lambda_0}$ and $\Lambda_{\varepsilon,\lambda_0}$.



(A) $\Sigma_{\varepsilon,\lambda_0}$



(B) $\Lambda_{\varepsilon,\lambda_0}$

FIGURE 1. Sectorial regions $\Sigma_{\varepsilon,\lambda_0}$ and $\Lambda_{\varepsilon,\lambda_0}$

With above definitions and comments, our main result for the model problem (2.3) is as follows.

Theorem 2.6. *Let $0 < \varepsilon < \pi/2$, $\sigma, \mu, \nu > 0$, $1 < q, q' := q/(q-1) < \infty$, $N < r < \infty$ and $r \geq \max\{q, q'\}$. Assume that the exterior domain Ω is of type $W_r^{3-1/r}$, $m \geq \lambda_1(\varepsilon, \Gamma_0)$ by Proposition 2.2, and (2.2) is satisfied. Set that*

$$X_q(\Omega) := H_q^1(\Omega) \times L_q(\Omega)^N \times H_q^1(\Omega)^N \times W_q^{2-1/q}(\Gamma_0),$$
$$\mathcal{X}_q(\Omega) := H_q^1(\Omega) \times L_q(\Omega)^N \times L_q(\Omega)^N \times H_q^1(\Omega)^N \times W_q^{2-1/q}(\Gamma_0).$$

For any $(d, \boldsymbol{F}, \boldsymbol{G}, K) \in X_q(\Omega)$, *there exist constants* $\lambda_0, r_b \geq 1$ *and operator families*

$$\begin{aligned}
&\mathcal{P}(\lambda,\Omega) \in \mathrm{Hol}\Big(\Lambda_{\varepsilon,\lambda_0}; \mathcal{L}\big(\mathcal{X}_q(\Omega); H^1_q(\Omega)\big)\Big),\\
&\mathcal{A}(\lambda,\Omega) \in \mathrm{Hol}\Big(\Lambda_{\varepsilon,\lambda_0}; \mathcal{L}\big(\mathcal{X}_q(\Omega); H^2_q(\Omega)^N\big)\Big),\\
&\mathcal{H}(\lambda,\Omega) \in \mathrm{Hol}\Big(\Lambda_{\varepsilon,\lambda_0}; \mathcal{L}\big(\mathcal{X}_q(\Omega); W^{3-1/q}_q(\Gamma_0)\big)\Big),
\end{aligned}$$

such that $(\eta, \boldsymbol{u}, h) := \big(\mathcal{P}(\lambda,\Omega), \mathcal{A}(\lambda,\Omega), \mathcal{H}(\lambda,\Omega)\big)(d, \boldsymbol{F}, \lambda^{1/2}\boldsymbol{G}, \boldsymbol{G}, K)$ *is the unique solution of* (2.3)*. Moreover, we have*

$$\begin{aligned}
\mathcal{R}_{\mathcal{L}\big(\mathcal{X}_q(\Omega); H^1_q(\Omega)\big)}\Big(\Big\{(\tau\partial_\tau)^\ell\big(\lambda\mathcal{P}(\lambda,\Omega)\big) : \lambda \in \Lambda_{\varepsilon,\lambda_0}\Big\}\Big) &\leq r_b,\\
\mathcal{R}_{\mathcal{L}\big(\mathcal{X}_q(\Omega); H^{2-j}_q(\Omega)^N\big)}\Big(\Big\{(\tau\partial_\tau)^\ell\big(\lambda^{j/2}\mathcal{A}(\lambda,\Omega)\big) : \lambda \in \Lambda_{\varepsilon,\lambda_0}\Big\}\Big) &\leq r_b,\\
\mathcal{R}_{\mathcal{L}\big(\mathcal{X}_q(\Omega); W^{3-1/q-j'}_q(\Gamma_0)\big)}\Big(\Big\{(\tau\partial_\tau)^\ell\big(\lambda^{j'}\mathcal{H}(\lambda,\Omega)\big) : \lambda \in \Lambda_{\varepsilon,\lambda_0}\Big\}\Big) &\leq r_b,
\end{aligned}$$

for $\ell, j' = 0, 1,\ j = 0, 1, 2,$ *and* $\tau := \Im\lambda$*. Above the choices of* λ_0 *and* r_b *depend solely on the parameters* ε, σ, m, μ, ν, q, r, N, ρ_1, ρ_2, ρ_3 *and* Ω.

By Theorem 2.6, we can also prove that the linearized model problem (1.2) is characterized by a semigroup structure. Thus we consider the following homogeneous system

$$\begin{cases}
\partial_t\eta + \gamma_1 \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega \times \mathbb{R}_+,\\
\gamma_1\partial_t\boldsymbol{u} - \operatorname{Div}\big(\mathbb{S}(\boldsymbol{u}) - \gamma_2\eta\mathbb{I}\big) = \boldsymbol{0} & \text{in} \quad \Omega \times \mathbb{R}_+,\\
\big(\mathbb{S}(\boldsymbol{u}) - \gamma_2\eta\mathbb{I}\big)\boldsymbol{n}_{\Gamma_0} + \sigma(m - \Delta_{\Gamma_0})h\,\boldsymbol{n}_{\Gamma_0} = \boldsymbol{0} & \text{on} \quad \Gamma_0 \times \mathbb{R}_+,\\
\partial_t h - \boldsymbol{u}\cdot\boldsymbol{n}_{\Gamma_0} = 0 & \text{on} \quad \Gamma_0 \times \mathbb{R}_+,\\
(\eta, \boldsymbol{u}, h)|_{t=0} = (\eta_0, \boldsymbol{u}_0, h_0) & \text{in} \quad \Omega.
\end{cases} \tag{2.4}$$

Note that the free boundary condition in (2.4) is equivalent to

$$\big(\mathbb{S}(\boldsymbol{u})\boldsymbol{n}_{\Gamma_0}\big)_\tau|_{\Gamma_0} = \boldsymbol{0}, \quad \mathbb{S}(\boldsymbol{u})\boldsymbol{n}_{\Gamma_0}\cdot\boldsymbol{n}_{\Gamma_0} - \gamma_2\eta + \sigma(m - \Delta_{\Gamma_0})h|_{\Gamma_0} = 0, \tag{2.5}$$

where $\boldsymbol{f}_\tau := \boldsymbol{f} - (\boldsymbol{f}\cdot\boldsymbol{n}_{\Gamma_0})\boldsymbol{n}_{\Gamma_0}$ stands for the tangential component of $\boldsymbol{f}$ along Γ_0. Then we set

$$\begin{aligned}
&\mathfrak{X}_q(\Omega) := H^1_q(\Omega) \times L_q(\Omega)^N \times W^{2-1/q}_q(\Gamma_0),\\
&\mathcal{D}_q(\mathcal{A}) := \Big\{(\eta, \boldsymbol{u}, h) \in \mathfrak{X}_q(\Omega) : \boldsymbol{u} \in H^2_q(\Omega)^N,\ h \in W^{3-1/q}_q(\Gamma_0),\ (2.5) \text{ holds}\Big\},
\end{aligned}$$

endowed with the norms

$$\begin{aligned}
\|(\eta, \boldsymbol{u}, h)\|_{\mathfrak{X}_q(\Omega)} &:= \|\eta\|_{H^1_q(\Omega)} + \|\boldsymbol{u}\|_{L_q(\Omega)} + \|h\|_{W^{2-1/q}_q(\Gamma_0)},\\
\|(\eta, \boldsymbol{u}, h)\|_{\mathcal{D}_q(\mathcal{A})} &:= \|\eta\|_{H^1_q(\Omega)} + \|\boldsymbol{u}\|_{H^2_q(\Omega)} + \|h\|_{W^{3-1/q}_q(\Gamma_0)}.
\end{aligned}$$

Furthermore, define the linear operator

$$\mathcal{A}\boldsymbol{U} := \begin{bmatrix} -\gamma_1 \operatorname{div} \boldsymbol{u} \\ \gamma_1^{-1} \operatorname{Div}\big(\mathbb{S}(\boldsymbol{u}) - \gamma_2 \eta \mathbb{I}\big) \\ \boldsymbol{u}\cdot \boldsymbol{n}_{\Gamma_0} \end{bmatrix} \quad \text{for} \quad \boldsymbol{U} := (\eta, \boldsymbol{u}, h) \in \mathcal{D}_q(\mathcal{A}),$$

and the following functional space by the real interpolation theory,

$$\mathcal{D}_{q,p}(\Omega) := \big(\mathfrak{X}_q(\Omega), \mathcal{D}_q(\mathcal{A})\big)_{1-1/p,p} \subset H^1_q(\Omega) \times B^{2(1-1/p)}_{q,p}(\Omega) \times B^{3-1/q-1/p}_{q,p}(\Gamma_0),$$

with $\|(\eta, \boldsymbol{u}, h)\|_{\mathcal{D}_{q,p}(\Omega)} := \|\eta\|_{H^1_q(\Omega)} + \|\boldsymbol{u}\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|h\|_{B^{3-1/q-1/p}_{q,p}(\Gamma_0)}$. Thanks to above settings, (2.4) can be regarded as the abstract Cauchy problem

$$\partial_t \boldsymbol{U} - \mathcal{A}\boldsymbol{U} = \boldsymbol{0} \ \text{ for } \ t > 0, \quad \boldsymbol{U}|_{t=0} = (\eta_0, \boldsymbol{u}_0, h_0),$$

whose resolvent problem is formulated as follows

$$\lambda \boldsymbol{U} - \mathcal{A}\boldsymbol{U} = \boldsymbol{F} \ \text{ for } \ \lambda \in \mathbb{C} \ \text{ and } \ \boldsymbol{F} = (d, \boldsymbol{f}, k) \in \mathfrak{X}_q(\Omega).$$

Then we can furnish the following results from Theorem 2.6 and the standard semigroup theory.

Theorem 2.7 (Generation of the C^0 semigroup)**.** *Let* $0 < \varepsilon < \pi/2$, $\sigma, \mu, \nu, \rho_1, \rho_2, \rho_3 > 0$, $1 < q, q' := q/(q-1) < \infty$, $N < r < \infty$ *and* $r \geq \max\{q, q'\}$. *Assume that the exterior domain* Ω *is of type* $W^{3,2}_r$, $m \geq \lambda_1(\varepsilon, \Gamma_0)$ *by Proposition 2.2, and* (2.2) *is satisfied. Denote that* $\boldsymbol{U}_0 := (\eta_0, \boldsymbol{u}_0, h_0)$. *Then there exist positive constants* γ_0, C *such that the following assertions hold true.*

1. *The operator* $\mathcal{A}$ *generates a* C^0 *semigroup* $\{T(t)\}_{t\geq 0}$ *in* $\mathfrak{X}_q(\Omega)$, *which is analytic. Moreover, we have*

$$\|\boldsymbol{U}\|_{\mathfrak{X}_q(\Omega)} + t\big(\|\partial_t \boldsymbol{U}\|_{\mathfrak{X}_q(\Omega)} + \|\boldsymbol{U}\|_{\mathcal{D}_q(\mathcal{A})}\big) \leq C e^{\gamma_0 t} \|\boldsymbol{U}_0\|_{\mathfrak{X}_q(\Omega)},$$

$$\|\partial_t \boldsymbol{U}\|_{\mathfrak{X}_q(\Omega)} + \|\boldsymbol{U}\|_{\mathcal{D}_q(\mathcal{A})} \leq C e^{\gamma_0 t} \|\boldsymbol{U}_0\|_{\mathcal{D}_q(\mathcal{A})},$$

   with $\boldsymbol{U} := T(t)\boldsymbol{U}_0$.

2. *For any* $\boldsymbol{U}_0 \in \mathcal{D}_{q,p}(\Omega)$, (2.4) *admits a unique solution*

$$e^{-\gamma_0 t}(\eta, \boldsymbol{u}, h) \in H^1_p\big(\mathbb{R}_+; \mathfrak{X}_q(\Omega)\big) \cap L_p\big(\mathbb{R}_+; \mathcal{D}_q(\mathcal{A})\big),$$

   satisfying the estimates

$$\|e^{-\gamma_0 t}\partial_t(\eta, \boldsymbol{u}, h)\|_{L_p(\mathbb{R}_+;\mathfrak{X}_q(\Omega))} + \|e^{-\gamma_0 t}(\eta, \boldsymbol{u}, h)\|_{L_p(\mathbb{R}_+;\mathcal{D}_q(\mathcal{A}))} \leq C \|(\eta_0, \boldsymbol{u}_0, h_0)\|_{\mathcal{D}_{q,p}(\Omega)}.$$

3 Boundary estimates

To obtain the resolvent (or elliptic type) estimates in Theorem 2.6, we observe that (2.3) can be reduced to some generalized model problem. More precisely, let us introduce some parameter ζ fulfilling $|\zeta| \le \zeta_0$ and either of the following cases

$$\text{(C1) } \zeta = \lambda^{-1}; \quad \text{(C2) } \zeta \in \Sigma_\varepsilon \text{ and } \Re\zeta < 0; \quad \text{(C3) } \Re\zeta \ge 0.$$

Then set that

$$\Gamma_{\varepsilon,\lambda_0,\zeta} := \begin{cases} \Lambda_{\varepsilon,\lambda_0} & \text{for (C1)}, \\ \{\lambda \in \mathbb{C} : \Re\lambda \ge |\frac{\Re\zeta}{\Im\zeta}||\Im\lambda|,\ \Re\lambda \ge \lambda_0\} & \text{for (C2)}, \\ \{\lambda \in \mathbb{C} : \Re\lambda \ge \lambda_0\} & \text{for (C3)}. \end{cases} \tag{3.1}$$

For $\lambda \in \Gamma_{\varepsilon,\lambda_0,\zeta}$, we consider the model problem

$$\begin{cases} \lambda \boldsymbol{v} - \gamma_1^{-1} \operatorname{Div}\big(\mathbb{S}(\boldsymbol{v}) + \zeta\gamma_3 \operatorname{div} \boldsymbol{v}\mathbb{I}\big) = \boldsymbol{f} & \text{in} \quad \Omega, \\ \big(\mathbb{S}(\boldsymbol{v}) + \zeta\gamma_3 \operatorname{div} \boldsymbol{v}\mathbb{I}\big)\boldsymbol{n}_{\Gamma_0} + \sigma(m - \Delta_{\Gamma_0})h\,\boldsymbol{n}_{\Gamma_0} = \boldsymbol{g} & \text{on} \quad \Gamma_0, \\ \lambda h - \boldsymbol{v}\cdot\boldsymbol{n}_{\Gamma_0} = k & \text{on} \quad \Gamma_0, \end{cases} \tag{3.2}$$

where γ_1 and γ_3 are uniformly continuous functions on $\overline{\Omega}$ such that

$$0 < \rho_1 \le \gamma_1(x) \le \rho_2, \quad 0 < \gamma_3(x) \le \rho_3, \quad \forall\, x \in \overline{\Omega}, \quad \|(\nabla\gamma_1, \nabla\gamma_3)\|_{L_r(\Omega)} \le \rho_3, \tag{3.3}$$

for some constants $\rho_1, \rho_2, \rho_3 > 0$ and $N < r < \infty$. Note that (3.2) under the assumption (C1) can be derived from (2.3) by eliminating η.

Theorem 3.1. *Let $0 < \varepsilon < \pi/2$, $\sigma, \mu, \nu > 0$, $1 < q < \infty$, $N < r < \infty$ and $r \ge q$. Assume that the exterior domain Ω is of type $W_r^{3-1/r}$, $m \ge \lambda_1(\varepsilon, \Gamma_0)$ by Proposition 2.2, and* (3.3) *is satisfied. Set that*

$$Y_q(\Omega) := L_q(\Omega)^N \times H_q^1(\Omega)^N \times H_q^2(\Omega), \quad \mathcal{Y}_q(\Omega) := L_q(\Omega)^N \times Y_q(\Omega).$$

For any $(\boldsymbol{f}, \boldsymbol{g}, k) \in Y_q(\Omega)$, there exist constants $\lambda_0, r_b \ge 1$ and operator families

$$\mathcal{A}_0(\lambda, \Omega) \in \operatorname{Hol}\Big(\Gamma_{\varepsilon,\lambda_0,\zeta}; \mathcal{L}\big(\mathcal{Y}_q(\Omega); H_q^2(\Omega)^N\big)\Big),$$
$$\mathcal{H}_0(\lambda, \Omega) \in \operatorname{Hol}\Big(\Gamma_{\varepsilon,\lambda_0,\zeta}; \mathcal{L}\big(\mathcal{Y}_q(\Omega); H_q^3(\Omega)\big)\Big),$$

such that $(\boldsymbol{v}, h) := \big(\mathcal{A}_0(\lambda,\Omega), \mathcal{H}_0(\lambda,\Omega)\big)(\boldsymbol{f}, \lambda^{1/2}\boldsymbol{g}, \boldsymbol{g}, k)$ is a solutions of (3.2)*. Moreover, we have*

$$\mathcal{R}_{\mathcal{L}\left(\mathcal{Y}_q(\Omega); H_q^{2-j}(\Omega)^N\right)}\Big(\Big\{(\tau\partial_\tau)^\ell\big(\lambda^{j/2}\mathcal{A}_0(\lambda,\Omega)\big) : \lambda \in \Gamma_{\varepsilon,\lambda_0,\zeta}\Big\}\Big) \le r_b,$$
$$\mathcal{R}_{\mathcal{L}\left(\mathcal{Y}_q(\Omega); H_q^{3-j'}(\Omega)\right)}\Big(\Big\{(\tau\partial_\tau)^\ell\big(\lambda^{j'}\mathcal{H}_0(\lambda,\Omega)\big) : \lambda \in \Gamma_{\varepsilon,\lambda_0,\zeta}\Big\}\Big) \le r_b,$$

for $\ell, j' = 0,1$, $j = 0,1,2$, *and* $\tau := \Im\lambda$. *Above the constants* λ_0 *and* r_b *depend solely on* ε, σ, m, μ, ν, ζ_0, q, r, N, ρ_1, ρ_2, ρ_3 *and* Ω.

It is nor hard to see that Theorem 2.6 is immediate from Theorem 3.1. The complicated part of the proof of Theorem 3.1 is the boundary estimate as (3.2) is elliptic type system. The idea for the boundary estimates is very classical. Firstly, we state the model problem in the half space via the technique from Fourier analysis. Then we review the bent half space case which characterizes the behaviour of the boundary points on Γ_0. Finally, one can conclude Theorem 3.1 by the boundary estimates and interior estimates, which we will omit. The interested reader may refer to the last part of [12] for more details.

3.1 Model problem in the half space

In [12], we consider the following model problem in $\mathbb{R}^N_+$,

$$\begin{cases} \lambda \boldsymbol{u} - \gamma_1^{-1}\,\mathrm{Div}\,\big(\mathbb{S}(\boldsymbol{u}) + \zeta\gamma_3 \,\mathrm{div}\,\boldsymbol{u}\mathbb{I}\big) = \boldsymbol{F} & \text{in} \quad \mathbb{R}^N_+, \\ \big(\mathbb{S}(\boldsymbol{u}) + \zeta\gamma_3\,\mathrm{div}\,\boldsymbol{u}\mathbb{I}\big)\boldsymbol{n}_0 + \sigma(m-\Delta')h\,\boldsymbol{n}_0 = \boldsymbol{G} & \text{on} \quad \mathbb{R}^N_0, \\ \lambda h - \boldsymbol{u}\cdot\boldsymbol{n}_0 = K & \text{on} \quad \mathbb{R}^N_0, \end{cases} \tag{3.4}$$

where $\mathbb{R}^N_0 := \{x = (x', x_N) \in \mathbb{R}^N : x_N = 0\}$, $\boldsymbol{n}_0 := (0, \dots, 0, -1)^\top$ and $\Delta' := \sum_{j=1}^{N-1} \partial_j^2$. On $\mathbb{R}^N_0$, the Laplace-Beltrami operator degenerates to the classical operator Δ'. Moreover, the parameter ζ and the constants γ_1, γ_3 fulfil the conditions

$$|\zeta| \leq \zeta_0, \;\; 0 < \rho_1 \leq \gamma_1 \leq \rho_2, \;\; 0 < \gamma_3 \leq \rho_3 \tag{3.5}$$

for some $\rho_1, \rho_2, \rho_3 > 0$. Then recalling the definition of $\Gamma_{\varepsilon,\lambda_0,\zeta}$ in (3.1), our main result for (3.4) reads:

Theorem 3.2. *[12, Theorem 3.1] Assume that* $0 < \varepsilon < \pi/2$, $\sigma, m, \mu, \nu > 0$, $1 < q < \infty$ *and* (3.5) *is satisfied. Set that*

$$Y_q(\mathbb{R}^N_+) := L_q(\mathbb{R}^N_+)^N \times H^1_q(\mathbb{R}^N_+)^N \times H^2_q(\mathbb{R}^N_+), \quad \mathcal{Y}_q(\mathbb{R}^N_+) := L_q(\mathbb{R}^N_+)^N \times Y_q(\mathbb{R}^N_+).$$

For any $(\boldsymbol{F}, \boldsymbol{G}, K) \in Y_q(\mathbb{R}^N_+)$, *there exist constants* $\lambda_0, r_b \geq 1$ *and operator families*

$$\mathcal{A}_0(\lambda, \mathbb{R}^N_+) \in \mathrm{Hol}\,\Big(\Gamma_{\varepsilon,\lambda_0,\zeta}; \mathcal{L}\big(\mathcal{Y}_q(\mathbb{R}^N_+); H^2_q(\mathbb{R}^N_+)^N\big)\Big),$$

$$\mathcal{H}_0(\lambda, \mathbb{R}^N_+) \in \mathrm{Hol}\,\Big(\Gamma_{\varepsilon,\lambda_0,\zeta}; \mathcal{L}\big(\mathcal{Y}_q(\mathbb{R}^N_+); H^3_q(\mathbb{R}^N_+)\big)\Big),$$

such that $(\boldsymbol{u}, h) := \big(\mathcal{A}_0(\lambda, \mathbb{R}^N_+), \mathcal{H}_0(\lambda, \mathbb{R}^N_+)\big)(\boldsymbol{F}, \lambda^{1/2}\boldsymbol{G}, \boldsymbol{G}, K)$ *is a solution of* (3.4). *Moreover, we have*

$$\mathcal{R}_{\mathcal{L}\left(\mathcal{Y}_q(\mathbb{R}^N_+); H^{2-j}_q(\mathbb{R}^N_+)^N\right)}\Big(\Big\{(\tau\partial_\tau)^\ell\big(\lambda^{j/2}\mathcal{A}_0(\lambda, \mathbb{R}^N_+)\big) : \lambda \in \Gamma_{\varepsilon,\lambda_0,\zeta}\Big\}\Big) \leq r_b,$$

$$\mathcal{R}_{\mathcal{L}\left(\mathcal{Y}_q(\mathbb{R}^N_+); H^{3-j'}_q(\mathbb{R}^N_+)\right)}\Big(\Big\{(\tau\partial_\tau)^\ell\big(\lambda^{j'}\mathcal{H}_0(\lambda, \mathbb{R}^N_+)\big) : \lambda \in \Gamma_{\varepsilon,\lambda_0,\zeta}\Big\}\Big) \leq r_b,$$

for $\ell, j' = 0, 1,\ j = 0, 1, 2,$ *and* $\tau := \Im\lambda$. *Above the choices of* λ_0 *and* r_b *depend solely on* $\varepsilon,\ \sigma,\ m,\ \mu,\ \nu,\ q,\ N,\ \zeta_0,\ \rho_1,\ \rho_2,\ \rho_3$.

3.2 Model problem in the bent half space

To describe the result for the bent half space, we need to introduce some notations on the domains. Let $\boldsymbol{\Phi}$ be a C^1 diffeomorphism from $\mathbb{R}^N_\xi$ onto $\mathbb{R}^N_x$ and $\boldsymbol{\Phi}^{-1}$ be the inverse of $\boldsymbol{\Phi}$. Assume that

$$\nabla_\xi \boldsymbol{\Phi}^\top(\xi) := \mathbb{A} + \mathbb{B}(\xi) = \Big[a_{ij}\Big]_{N\times N} + \Big[b_{ij}(\xi)\Big]_{N\times N},$$
$$\mathscr{A}_\Phi := \nabla_x(\boldsymbol{\Phi}^{-1})^\top|_{x=\boldsymbol{\Phi}(\xi)} = \mathbb{A}_- + \mathbb{B}_-(\xi) = \Big[\bar{a}_{ij}\Big]_{N\times N} + \Big[\bar{b}_{ij}(\xi)\Big]_{N\times N},$$

for some constant orthogonal matrices $\mathbb{A}$ and $\mathbb{A}_-$. Moreover, denote $\Omega_+ := \boldsymbol{\Phi}(\mathbb{R}^N_+)$ and $\Gamma_+ := \partial\Omega_+ = \boldsymbol{\Phi}(\mathbb{R}^N_0)$. Then Γ_+ is characterized by the equation $(\Phi^{-1})_N(x) = 0$, and the unit outer normal $\boldsymbol{n}_+$ to Γ_+ is given by

$$\boldsymbol{n}_+\big(\boldsymbol{\Phi}(\xi)\big) = \frac{\mathscr{A}_\Phi \boldsymbol{n}_0}{|\mathscr{A}_\Phi \boldsymbol{n}_0|} = -\frac{(\nabla_x \Phi_N^{-1})\big(\boldsymbol{\Phi}(\xi)\big)}{\big|(\nabla_x \Phi_N^{-1})\big(\boldsymbol{\Phi}(\xi)\big)\big|} = -\frac{\big(\bar{a}_{1N} + \bar{b}_{1N}(\xi), \dots, \bar{a}_{NN} + \bar{b}_{NN}(\xi)\big)^\top}{\Big(\sum_{j=1}^N \big(\bar{a}_{Nj} + \bar{b}_{Nj}(\xi)\big)^2\Big)^{1/2}},$$

with $\boldsymbol{n}_0 = (0, \dots, 0, -1)^\top$.

For such Γ_+ characterized by $H^3_r(\mathbb{R}^N)$ mapping, we consider the following model problem,

$$\begin{cases} \lambda \boldsymbol{v} - \gamma_1^{-1}\operatorname{Div}\big(\mathbb{S}(\boldsymbol{v}) + \zeta\gamma_3 \operatorname{div}\boldsymbol{v}\mathbb{I}\big) = \boldsymbol{f} & \text{in} \quad \Omega_+, \\ \mathbb{S}(\boldsymbol{v})\boldsymbol{n}_+ + \zeta\gamma_3 \operatorname{div}\boldsymbol{v}\,\boldsymbol{n}_+ + \sigma(m - \Delta_{\Gamma_+})h\,\boldsymbol{n}_+ = \boldsymbol{g} & \text{on} \quad \Gamma_+, \\ \lambda h - \boldsymbol{v}\cdot\boldsymbol{n}_+ = k & \text{on} \quad \Gamma_+. \end{cases} \tag{3.6}$$

In (3.6), γ_1 and γ_3 are uniformly continuous functions on $\overline{\Omega}_+$, and there exist some constants $\widehat{\gamma}_1, \widehat{\gamma}_3$ such that

$$\begin{gathered} 0 < \rho_1 \le \gamma_1(x), \widehat{\gamma_1} \le \rho_2, \quad 0 < \gamma_3(x), \widehat{\gamma_3} \le \rho_3, \ \ \forall\, x \in \overline{\Omega}_+, \\ \sum_{\mathfrak{a}=1,3} \|\gamma_{\mathfrak{a}} - \widehat{\gamma_{\mathfrak{a}}}\|_{L_\infty(\Omega_+)} \le M_1 < 1, \quad \sum_{\mathfrak{a}=1,3} \|\nabla\gamma_{\mathfrak{a}}\|_{L_r(\Omega_+)} \le CM_2, \end{gathered} \tag{3.7}$$

for some constants $\rho_1, \rho_2, \rho_3 > 0$. The main result for (3.6) reads:

Theorem 3.3. *[12, Theorem 4.1] Let* $0 < \varepsilon < \pi/2$, $\sigma, m, \mu, \nu, \zeta_0 > 0$, $1 < q < \infty$, $N < r < \infty$ *and* $r \ge q$. *Assume that* (3.7) *is satisfied. For* Ω_+ *given above, we set that*

$$Y_q(\Omega_+) := L_q(\Omega_+)^N \times H^1_q(\Omega_+)^N \times H^2_q(\Omega_+), \quad \mathcal{Y}_q(\Omega_+) := L_q(\Omega_+)^N \times Y_q(\Omega_+).$$

Then for any $(\boldsymbol{f}, \boldsymbol{g}, k) \in Y_q(\Omega_+)$, *there exist constants* $\lambda_0, r_b \geq 1$ *and operator families*

$$\mathcal{A}_0(\lambda, \Omega_+) \in \mathrm{Hol}\Big(\Gamma_{\varepsilon,\lambda_0,\zeta}; \mathcal{L}\big(\mathcal{Y}_q(\Omega_+); H^2_q(\Omega_+)^N\big)\Big),$$
$$\mathcal{H}_0(\lambda, \Omega_+) \in \mathrm{Hol}\Big(\Gamma_{\varepsilon,\lambda_0,\zeta}; \mathcal{L}\big(\mathcal{Y}_q(\Omega_+); H^3_q(\Omega_+)\big)\Big),$$

such that $(\boldsymbol{v}, h) := \big(\mathcal{A}_0(\lambda, \Omega_+), \mathcal{H}_0(\lambda, \Omega_+)\big)(\boldsymbol{f}, \lambda^{1/2}\boldsymbol{g}, \boldsymbol{g}, k)$ *is a solution of* (3.6)*. Moreover, we have*

$$\mathcal{R}_{\mathcal{L}\left(\mathcal{Y}_q(\Omega_+); H^{2-j}_q(\Omega_+)^N\right)}\Big(\Big\{(\tau\partial_\tau)^\ell\big(\lambda^{j/2}\mathcal{A}_0(\lambda, \Omega_+)\big) : \lambda \in \Gamma_{\varepsilon,\lambda_0,\zeta}\Big\}\Big) \leq r_b,$$
$$\mathcal{R}_{\mathcal{L}\left(\mathcal{Y}_q(\Omega_+); H^{3-j'}_q(\Omega_+)\right)}\Big(\Big\{(\tau\partial_\tau)^\ell\big(\lambda^{j'}\mathcal{H}_0(\lambda, \Omega_+)\big) : \lambda \in \Gamma_{\varepsilon,\lambda_0,\zeta}\Big\}\Big) \leq r_b,$$

for $\ell, j' = 0, 1,\ j = 0, 1, 2,$ *and* $\tau := \Im\lambda$*. Above the constants* λ_0 *and* r_b *depend solely on* ε, σ, m, μ, ν, q, r, N, ζ_0, ρ_1, ρ_2, ρ_3.

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