The Weisfeiler–Leman stability: the case of trees

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$M_X(\mathbb{C})$: the full matrix algebra $/\mathbb{C}$
rows + columns indexed by X

w.r.t. the ordinary matrix product

$$(AB)_{xy} = \sum_{z \in X} A_{xz} B_{zy}$$

$M_X(\mathbb{C})^\circ$: the commutative alg. $/\mathbb{C}$

w.r.t. the Hadamard product $\circ$

$M_X(\mathbb{C})^\circ = M_X(\mathbb{C})$ as vector spaces

$$(A \circ B)_{xy} = A_{xy} B_{xy}$$

$M_X(\mathbb{C}) \supseteq \mathcal{M}$ <u>coherent algebra</u>

$\overset{\text{def}}{\iff}$ (0) $\mathcal{M}$: a subspace as a vector space

(i) closed under transpose - conjugate

$$A \in \mathcal{M} \implies {}^t\overline{A} \in \mathcal{M}$$

(ii) closed under $\cdot$ & $\circ$

$$A, B \in \mathcal{M} \implies AB \in \mathcal{M}$$
$$A \circ B \in \mathcal{M}$$

i.e. $\mathcal{M} \leq M_X(\mathbb{C})$ subalg.

and $\mathcal{M} \leq M_X(\mathbb{C})^{\circ}$ subalg.

$$\mathrm{Span}\{I, J\} \subseteq \mathcal{M} \subseteq M_X(\mathbb{C})$$

smallest coherent alg. | coherent alg. | largest coherent alg.

$$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} : \text{identity of } M_X(\mathbb{C})$$

$$J = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \\ 1 & \cdots & 1 \end{pmatrix} : \text{identity of } M_X(\mathbb{C})^{\circ}$$

<u>coherent algebra</u>

$$\mathcal{M} = \mathrm{Span}\{A_\alpha \mid \alpha \in \Lambda\}$$

A_α : (0,1)-matrix

$$I = \sum_{\alpha \in \Lambda_0} A_\alpha \qquad (\text{some } \Lambda_0 \subseteq \Lambda)$$

$$J = \sum_{\alpha \in \Lambda} A_\alpha$$

$${}^{t}A_\alpha = A_{\alpha'} \qquad \text{for some } \alpha' \in \Lambda$$

$$A_\alpha A_\beta = \sum_{\gamma \in \Lambda} p^{\gamma}_{\alpha\beta} A_\gamma$$

<u>coherent configuration</u> (D. G. Higman) 1974

$$\mathfrak{X} = \left(X , \{R_\alpha\}_{\alpha \in \Lambda} \right)$$

If $M = \operatorname{Span}\{ A_\alpha \mid \alpha \in \Lambda \}$

is a coherent configuration $\subseteq M_X(\mathbb{C})$

$$\begin{array}{ccc} X \times X & & M_X(\mathbb{C}) \\ \cup & & \in \\ R_\alpha & \longleftrightarrow & A_\alpha \end{array}$$ adjacency matrix

$$(A_\alpha)_{xy} = \begin{cases} 1 & \text{if } (x,y) \in R_\alpha \\ 0 & \text{otherwise} \end{cases}$$

coherent configuration $\longleftrightarrow$ coherent algebra

Example

$G \leq \mathrm{Sym}(X)$

$G \longrightarrow X \times X \qquad a\cdot(x,y) = (ax, ay)$

$\{R_\alpha\}_{\alpha \in \Lambda}$: the G-orbits on $X \times X$

$\mathcal{X} = (X, \{R_\alpha\}_{\alpha\in\Lambda})$ coherent conf.

<u>Schurian</u>

Aut $\mathcal{X}$ is transitive on each R_α $(\alpha \in \Lambda)$.

$$\mathrm{Aut}\,\mathcal{X} = \left\{ a \in \mathrm{Sym}(X) \;\middle|\; \begin{array}{c} (x,y) \in R_\alpha \\ \Updownarrow \\ (ax, ay) \in R_\alpha \\ \text{for all } \alpha \in \Lambda \end{array} \right\}$$

$\Gamma = (\underset{\substack{\text{vertex}\\ \text{set}}}{X}, \underset{\substack{\text{edge}\\ \text{set}}}{R})$ graph

$$R \subseteq X \times X - \underset{\text{diagonal}}{\Delta}$$

$M_X(\mathbb{C}) \ni A$

adjacency matrix of R

$M_X(\mathbb{C}) \geq \mathcal{A}_0 = \langle A, {}^tA \rangle \ni I$

subalg. generated by A, ${}^tA$

$M_X(\mathbb{C})^\circ \geq \mathcal{A}_1 = \langle \mathcal{A}_0 \rangle^\circ \ni J$

subalg. generated by $\mathcal{A}_0$

$M_X(\mathbb{C}) \geq \mathcal{A}_2 = \langle \mathcal{A}_1 \rangle$

subalg. generated by $\mathcal{A}_1$

$M_X(\mathbb{C})^\circ \geq \mathcal{A}_3 = \langle \mathcal{A}_2 \rangle^\circ$

subalg. generated by $\mathcal{A}_2$

$\vdots$

$$\mathcal{A}_0 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_4 \subseteq \cdots \subseteq M_X(\mathbb{C})$$

sequence of subalgebras

$$\mathcal{A}_1 \subseteq \mathcal{A}_3 \subseteq \mathcal{A}_5 \subseteq \cdots \subseteq M_X(\mathbb{C})^{\circ}$$

sequence of subalgebras.

<u>Weisfeiler – Lehman stabilization</u>

late 60's

$\exists\, r$ s.t.

$$\mathcal{A}_0 \subsetneq \mathcal{A}_1 \subsetneq \cdots \subsetneq \mathcal{A}_{r-1} \subsetneq \mathcal{A}_r = \mathcal{A}_{r+1}$$

$$\mathcal{A} = \bigcup_{i=0}^{\infty} \mathcal{A}_i = \mathcal{A}_r \subseteq M_X(\mathbb{C})$$

the smallest coherent algebra $\supseteq \mathcal{A}_0$

<u>the coherent closure</u>

$$\mathcal{A} = \bigcap_{\substack{\mathcal{A}_0 \subseteq \mathcal{M} \subseteq M_X(\mathbb{C}) \\ \text{coherent} \\ \text{alg.}}} \mathcal{M}$$

$r = r(\Gamma)$: <u>coherent length</u>

$\mathcal{A} \longleftrightarrow \mathcal{X} = (X, \{R_\alpha\}_{\alpha \in \Lambda})$

coherent closure of $\mathcal{A}_0 = \langle A, {}^tA \rangle$

coherent conf.

coherent closure of Γ

Fact $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\mathcal{X})$

Remark $\mathcal{X}$ is NOT Schurian in general, i.e.,

$\mathrm{Aut}(\mathcal{X})$ is NOT transitive on each R_α in general

Theorem (joint with 徐静, 李双东)

Γ : tree

Then $r \leq 7$.

$$A_0 \subseteq A_2 \subseteq A_4 \subseteq \quad \cdots \quad \subseteq M_X(\mathbb{C})$$

sequence of subalgebras

semi-simple alg. (representations)

$$A_1 \subseteq A_3 \subseteq A_5 \subseteq \quad \cdots \quad \subseteq M_X(\mathbb{C})^{o}$$

sequence of subalgebras

semi-simple alg. (combinatorics)

$X \ni \exists!\, x_0$ the centre of Γ

$$D(x_0) < D(x) = \max\{\partial(x,y) \mid y \in X\}$$

all $x \in X$, $x \neq x_0$

or $\exists!\, x_0, x_1$ the adjacent centre of Γ

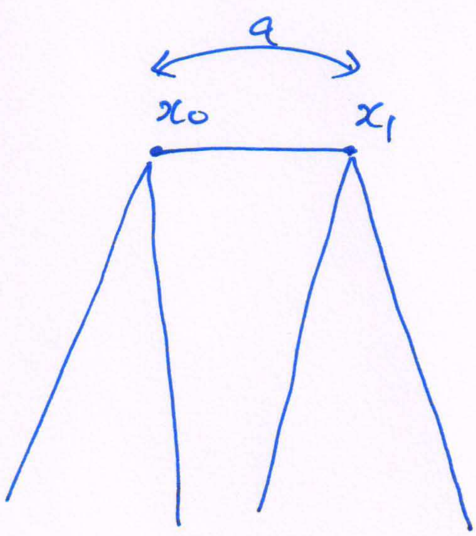
$$D(x_0) = D(x_1) < D(x).$$

all $x \in X$, $x \neq x_0, x_1$

$G = \mathrm{Aut}(\Gamma)$

Set $X_0 = \{x_0\}$ if $a x_0 = x_0$ all $a \in G$

$X_0 = \{x_0, x_1\}$ if $a x_0 = x_1$, $a x_1 = x_0$ (some $a \in G$)



Terwilliger algebra $T = T(x_0)$

$$X_i = \{ x \in X \mid \partial(x_0, x) = i \}, \quad 0 \leq i \leq D$$

$V = \mathbb{C}X$ (X : orthonormal basis)

standard module

$$= \bigoplus_{i=0}^{D} V_i^*, \quad V_i^* = \mathbb{C}X_i$$

$E_i^* : V \longrightarrow V_i^*$ orthogonal projection

$$T = \langle A, E_i^* \mid 0 \leq i \leq D \rangle \subseteq M_X(\mathbb{C})$$

A: adjacency alg. of Γ

$S = \mathrm{Hom}_G(V, V)$

$= \{ f : V \to V$ linear mapping $|$

$f(av) = a f(v)$, all $a \in G$, all $v \in V \}$

the centralizer algebra of G

$\mathcal{A}_0 = \langle A \rangle \subseteq T \subseteq S \subseteq M_X(\mathbb{C})$

Want to show

$$T \subseteq \mathcal{A}_6$$

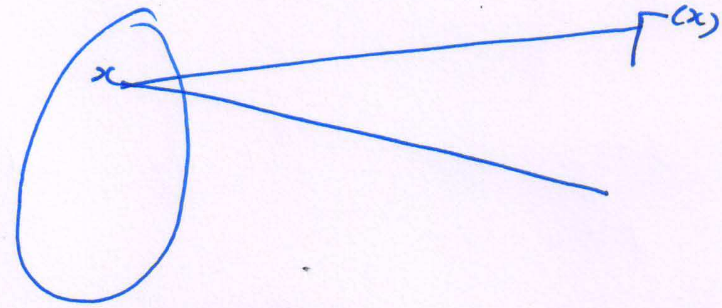
$$T \circ T = S = \mathcal{A}_7$$

particularly, the coherent (configuration) closure

$\mathfrak{X} = (X, \{R_\alpha\}_{\alpha \in \Lambda})$ is Schurian.

Isomorphism classes of Irreducible T-modules

X_0

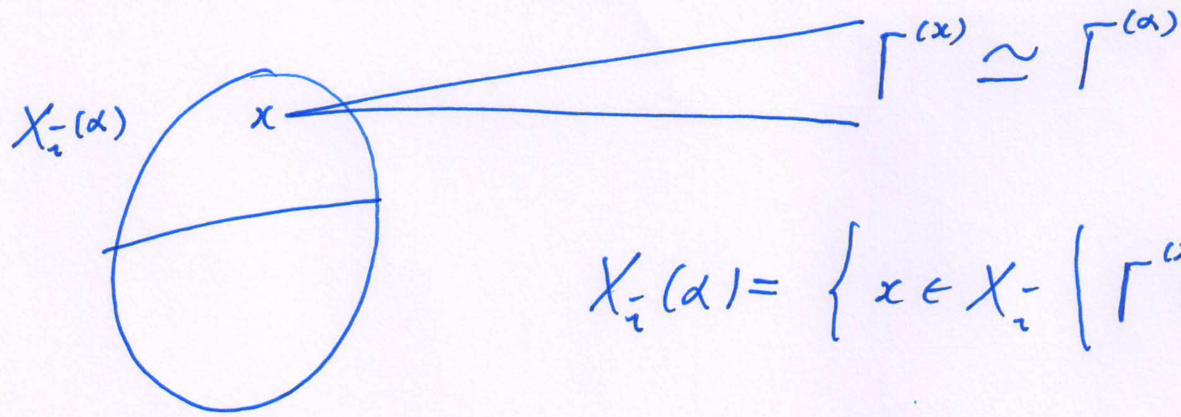


$X_i \ni x, y$

$x \sim y \iff \Gamma^{(x)} \simeq \Gamma^{(y)}$ as rooted trees

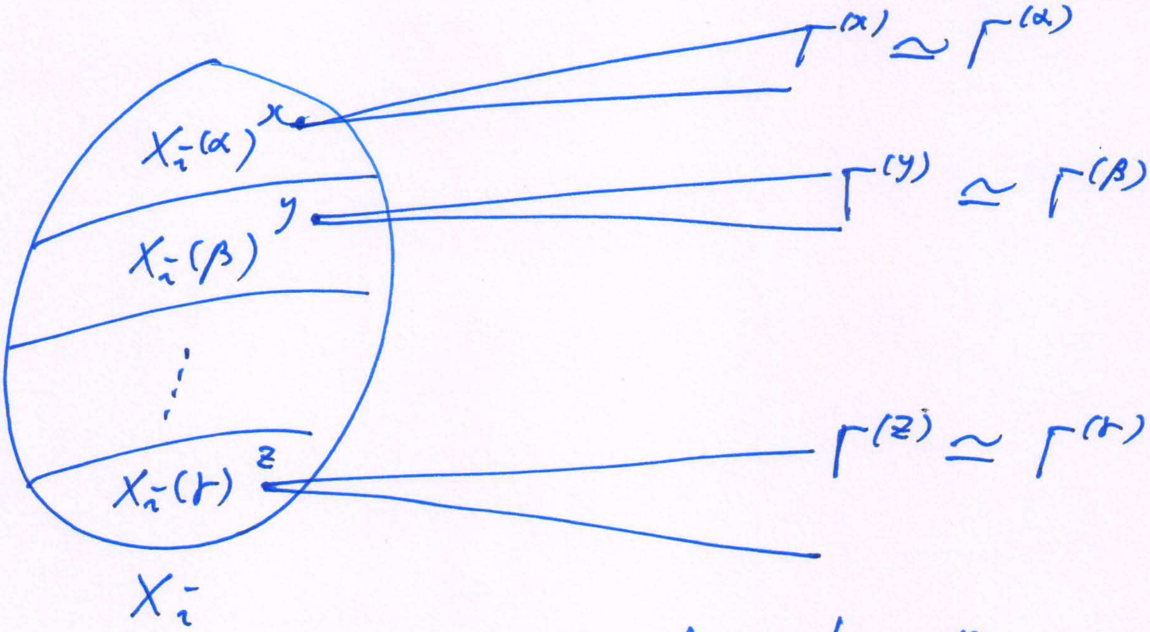
$\{\Gamma^{(\alpha)} \mid \alpha \in \Lambda_i\}$ representatives of the equivalence classes $\Gamma^{(x)}, \ x \in X_i$

$= \{\Gamma^{(x)} \mid x \in X_i\} / \sim$



X_i

$X_i(\alpha) = \{x \in X_i \mid \Gamma^{(x)} \simeq \Gamma^{(\alpha)}\}$



$$\Lambda_i = \{\alpha, \beta, \cdots, \gamma\}$$

$$X_i = X_i(\alpha) \cup X_i(\beta) \cup \cdots \cup X_i(\gamma)$$

$$V_i^* = \mathbb{C}X_i = V_i^*(\alpha) \oplus V_i^*(\beta) \oplus \cdots \oplus V_i^*(\gamma)$$

<u>Def</u>

$$V_i^*(\alpha) = V_i^{*(0)}(\alpha) \oplus V_i^{*(1)}(\alpha) \quad \text{orthogonal sum}$$

$$V_i^{*(1)}(\alpha) = \ker\, E_{i-1}^* A E_i^* \Big|_{V_i^*(\alpha)}$$

<u>Remark</u> $V_i^{*(1)}(\alpha) = 0$ may happen!

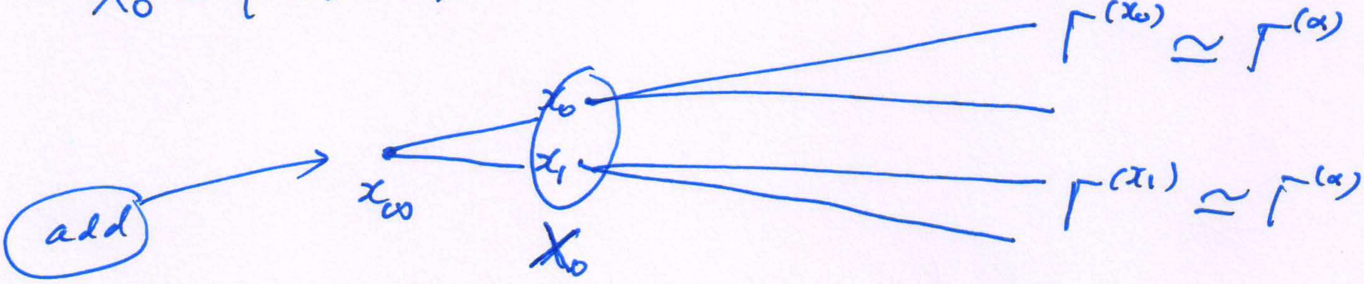
In this case, $V_i^{*(0)}(\alpha) = V_i^*(\alpha)$.

$\underline{i=0}$ $|\Lambda_0| = 1$. $\Lambda_0 = \{\alpha\}$

$X_0 = \{ x_0 \}$. $V_0^{*(0)}(\alpha) = 0$

$V_0^{*(1)}(\alpha) = V_0^{*}$

$X_0 = \{ x_0 , x_1 \}$



$X_{-1} = \{ x_\infty \}$

$$V_0^{*(1)}(\alpha) \underset{def}{=} \mathbb{C}(x_0 - x_1)$$

Set $V_{-1}^{*(1)}(\alpha) = V_0^{*(0)}(\alpha) \underset{def}{=} \mathbb{C}(x_0 + x_1)$

Classification of Irreducible T-modules

Theorem

(i) $V_i^{*(1)}(\alpha) \ni w \neq 0 \implies W = Tw$ irred. T-module

(ii) $V_i^{*(1)}(\alpha) \ni w \neq 0, \quad W = Tw$

$V_j^{*(1)}(\beta) \ni w' \neq 0, \quad W' = Tw'$

Then

$W \simeq W'$ as T-modules

$\iff \quad i = j, \quad \alpha = \beta$

(iii) $V \supseteq {}^{\forall} W$ irred. T-module

$\exists\, 0 \neq w \in V_i^{*(1)}(\alpha)$ s.t. $W = Tw$.

Cor

$$T = \bigoplus_{\substack{(i,\alpha) \\ V_i^{*(1)}(\alpha) \neq 0}} E_i^{*(1)}(\alpha) \otimes M_{\Gamma^{(\alpha)}/\sim}(\mathbb{C})$$

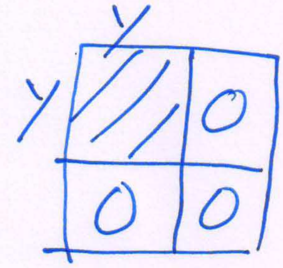
semi simple

direct sum of simple algebras

$$\subseteq M_X(\mathbb{C})$$

Notation

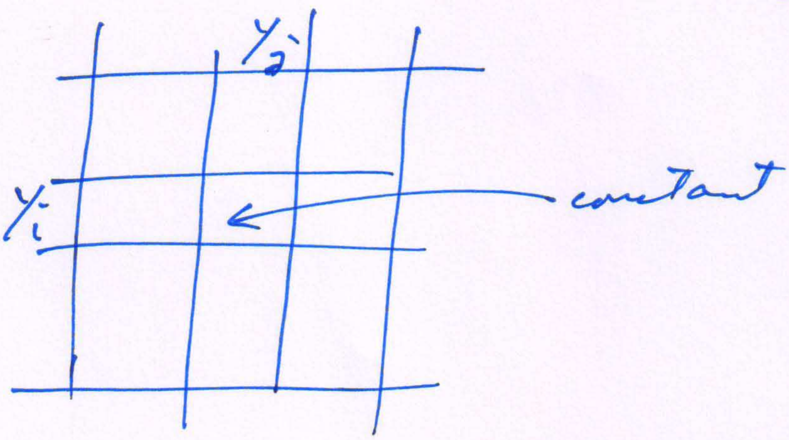
(i) $X \geq Y$

$M_X(\mathbb{C}) \geq M_Y(\mathbb{C}) =$ 

(ii) $Y/\sim \; = Y_1 \cup Y_2 \cup \cdots \cup Y_r$

partition

$$M_{Y/\sim}(\mathbb{C}) = \left\{ a \in M_Y(\mathbb{C}) \;\middle|\; \begin{array}{l} a \text{ is constant} \\ \text{on } Y_i \times Y_j,\ \text{all } i,j \end{array} \right\}$$



(iii) $\Gamma^{(\alpha)}/\sim$: orbits of $\mathrm{Aut}\,\Gamma^{(\alpha)}$ on $\Gamma^{(\alpha)}$

(iv) $\displaystyle\bigcup_{x \in X_i^-(\alpha)} \Gamma^{(x)} = X_i^-(\alpha) \times \Gamma^{(\alpha)}$

identified

(v)

$V_i^*(\alpha) = \mathbb{C} X_i^-(\alpha) = V_i^{*(0)}(\alpha) \oplus V_i^{*(1)}(\alpha)$

$E_i^{*(1)}(\alpha) : V_i^*(\alpha) \longrightarrow V_i^{*(1)}(\alpha)$

orthogonal projection

$\cap$

$M_{X_i(\alpha)}(\mathbb{C})$

So $E_i^{*(1)}(\alpha) \otimes M_{\Gamma^{(\alpha)}/\sim}(\mathbb{C})$

$\subseteq M_{X_i^-(\alpha) \times \Gamma^{(\alpha)}}(\mathbb{C}) \subseteq M_X(\mathbb{C})$

$S = \mathrm{Hom}_G(V, V)$ centralizer algebra

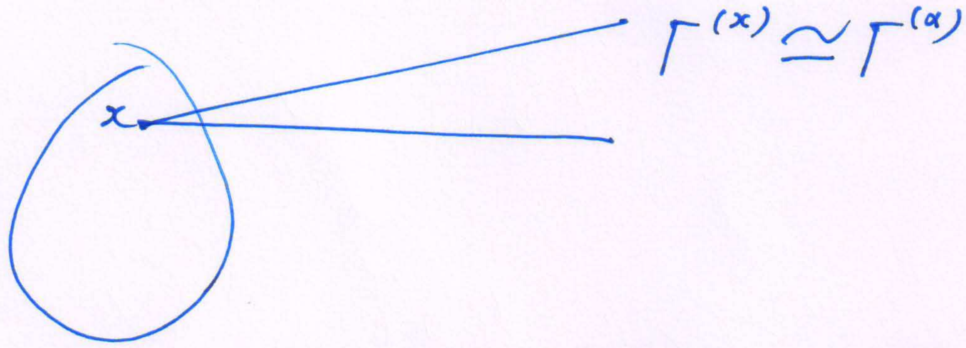
$G = \mathrm{Aut}\,\Gamma$

$V = \mathbb{C}X$

$T \subseteq S$

$X/\sim$: the G-orbits

$= \bigcup_{i,\alpha} X_i(\alpha)/\sim$



$X_i(\alpha)$ $\underline{G\text{-inv}}$

$X_i(\alpha)/\sim \ \ni Y$ G-orbit

$V_Y = \mathbb{C}Y$

$= V_Y^{(0)} \oplus V_Y^{(1)}$ orthogonal sum

$V_Y^{(1)} = \ker E_{i-1}^* A E_i^* \big|_{V_Y}$

Then

$$V_i^*(\alpha) = \mathbb{C}X_i(\alpha) = \bigoplus_{Y \in X_i(\alpha)/\sim} V_Y$$

$$\begin{cases} V_i^{*(0)}(\alpha) = \bigoplus_{Y \in X_i(\alpha)/\sim} V_Y^{(0)} \\ V_i^{*(1)}(\alpha) = \bigoplus_{Y \in X_i(\alpha)/\sim} V_Y^{(1)} \end{cases}$$

Classification of Irreducible S-modules

Theorem

(i) $V_y^{(1)} \ni w \neq 0 \implies W = Sw$
$(y \in X/\sim)$ irred. S-module

(ii) $V_y^{(1)} \ni w \neq 0, \quad W = Sw \quad (y \in X/\sim)$

$V_z^{(1)} \ni w' \neq 0, \quad W' = Sw' \quad (z \in X/\sim)$

Then $W \simeq W'$ as S-modules

$\iff y = z$

(iii) $V \supseteq {}^{\forall} W$ irred. S-module

$\exists_{0 \neq} w \in V_y^{(1)}$ s.t. $W = Sw$

<u>Cor</u>

$$S = \bigoplus_{\substack{Y \in X_{\frac{-}{2}}(\alpha)/\sim \\ V_Y^{(1)} \neq 0}} E_Y^{(1)} \otimes M_{\Gamma^{(\alpha)}/\sim}(\mathbb{C})$$

direct sum of simple algebras

where $E_Y^{(1)} \;:\; V_Y = \mathbb{C}Y \longrightarrow V_Y^{(1)}$ orthogonal proj.

$\quad\;\;\cap$

$\quad M_Y(\mathbb{C})$

$$\underline{X_0} = \begin{cases} x_0 & \text{if } \quad X_0 = \{x_0\} \\ x_0 + x_1 & \text{if } \quad X_0 = \{x_0, x_1\} \end{cases}$$

$$V_0 = T\,\underline{X_0} = S\,X_0 = \underbrace{\mathrm{Span}\left\{ \underline{Y} \mid Y \in X/\!\sim \right\}}_{\text{principal module}}$$

irreducible T-module
unique

Then

$$T \supseteq T|_{V_0} = S|_{V_0} = M_{X/\sim}$$

Observe

$$Y \times \Gamma^{(\alpha)}/\!\sim \;\subseteq\; X/\!\sim$$

$$Y \times \Gamma^{(\alpha)} \;:\; \text{union of } G\text{-orbits}$$

$$J_{Y\times\Gamma^{(\alpha)}} = \begin{array}{c|c} & Y\times\Gamma^{(\alpha)} \\ \hline Y\times\Gamma^{(\alpha)} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \in T_0 = T|_{V_0}$$

$$J_{Y\times\Gamma^{(\alpha)}} \circ T = E_Y^{(1)} \otimes M_{\Gamma^{(\alpha)}/\sim}(\mathbb{C})$$

So

$$(T_0) \circ T = S$$