SOP_2 AND ANTICHAINS

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Abstract. In this paper, we pay our attention into 2-strong order property, namely SOP_2, and its relatives. We first observe strongly indiscernible trees and its modeling property, then find the equivalent conditions for each SOP_2 and weak-TP_1. As a main theorem, we prove that the definitions of SOP_2 and TP_1 can be more generalized, by using the notion of strong similarity and antichains.

1. introduction

A formula $\phi(x, y)$ has 2-strong order property(SOP_2) if there is a tree of parameters $(a_\eta : \eta \in {}^{\omega>}2)$ such that the types in which parameters consists of a path in tree is consistent, but any two formulas having parameters of incomparable pairs are contradictory.

SOP_2 has similar definition of the tree property of the first kind(TP_1), except TP_1 uses parameters indexed on ${}^{\omega>}\omega$. It is easily proved that SOP_2 is equivalent to the tree property of the first kind(TP_1), but as it is still too strong to deal with, there have been many attempts to weakening the conditions.

In [3], suggesting the notions k-TP_1 and weak k-TP_1, Kim and Kim proved that k-TP_1 is equivalent to TP_1 using the tree indiscernibility called 1-fully-tree-indiscernible. The equivalence of TP_1 and weak k-TP_1 was later proved in [1] by using strongly indiscernible trees. We aim to generalize the notions of tree properties by introducing $\bar{\nu}^{str}$-SOP_2(Definition 3.6), and then prove that SOP_2 is equivalent to $\bar{\nu}^{str}$-SOP_2 for any antichain tuple $\bar{\nu}$.

2. Preliminaries on Strongly Indiscernible Trees and SOP_2

Consider a tree ${}^{\lambda>}\kappa$ of height λ which has κ many branches. Each element in tree can be considered as a string. We denote $\langle\rangle$ as an empty string, 0^α as a string of α many zeros, and α as a string $\langle\alpha\rangle$ of length one.

Definition 2.1. Let $\eta, \nu, \xi \in {}^{\lambda>}\kappa$.

(1) (Ordering) $\eta \lhd \nu$ if $\nu\lceil\alpha = \eta$ for some ordinal $\alpha \in \mathrm{dom}(\nu)$.
(2) (Meet) $\xi = \eta \wedge \nu$ if ξ is the meet of η and ν, i.e., $\xi = \eta\lceil\beta$, when $\beta = \bigcup\{\alpha \leq \mathrm{dom}(\eta)\cap\mathrm{dom}(\nu) \mid \eta\lceil\alpha = \nu\lceil\alpha\}$. For $\bar{\eta} \in {}^{\lambda>}\kappa$, $\bar{\nu}$ is the meet closure of $\bar{\eta}$ if $\bar{\nu} = \{\eta_1 \wedge \eta_2 | \eta_1, \eta_2 \in \bar{\eta}\}$
(3) (Incomparability) $\eta \perp \nu$ if they are $\trianglelefteq$-incomparable, i.e., $\neg(\eta \trianglelefteq \nu)$ and $\neg(\nu \trianglelefteq \eta)$.
(4) (Lexicographic order) $\eta <_{lex} \nu$ if
   (a) $\eta \lhd \nu$, or
   (b) $\eta \perp \nu$ and for ordinal $\alpha = \mathrm{dom}(\eta \wedge \nu)$, $\eta(\alpha) < \nu(\alpha)$

Definition 2.2. A *strong language* L_0 is defined by the collection $\{\lhd, \wedge, <_{lex}\}$

We may view the tree ${}^{\lambda>}\kappa$ as an L_0-structure.

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Fix a complete first order theory T(with language L). Let $\mathfrak{C} \models T$ be a monster model. From now on, we will work in this $\mathfrak{C}$. Note that we distinguish an index structure from $\mathfrak{C}$. We visit [4] to introduce generalized indiscernibility and modeling property.

Definition 2.3. Let L_0-structure ${}^{\lambda>}\kappa$ be an index structure. For a tree $(b_\eta | \eta \in {}^{\lambda>}\kappa)$ in $\mathfrak{C}$, we say it is *strongly indiscernible* if for any finite tuple $\bar{\eta}$ and $\bar{\nu}$ in ${}^{\lambda>}\kappa$,

$$\mathrm{qftp}_{L_0}(\bar{\eta}) = \mathrm{qftp}_{L_0}(\bar{\nu}) \Rightarrow (b_\eta)_{\eta \in \bar{\eta}} \equiv (b_\nu)_{\nu \in \bar{\nu}}.$$

We say $\bar{\eta}$ and $\bar{\nu}$ are strongly similar, $\bar{\eta} \sim_{str} \bar{\nu}$, to denote $\mathrm{qftp}_{L_0}(\bar{\eta}) = \mathrm{qftp}_{L_0}(\bar{\nu})$.

Definition 2.4. Let $\mathcal{I}$ be an index structure. A set $B = \{b_\eta \mid \eta \in \mathcal{I}\}$ is *based on* a set $A = \{a_\nu \mid \nu \in \mathcal{I}\}$ if for all $\varphi(x_{i_1}, \ldots, x_{i_n})$ in L and for all $\eta_1, \ldots, \eta_n \in \mathcal{I}$, there exists some $\nu_1, \cdots, \nu_n \in \mathcal{I}$ such that

(a) $\nu_1 \ldots \nu_n \equiv^{qf}_{\mathcal{I}} \eta_1 \ldots \eta_n$, and
(b) $b_{\eta_1} \ldots b_{\eta_n} \equiv_\varphi a_{\nu_1} \ldots a_{\nu_n}$

In particular, when $\mathcal{I}$ is L_0-structure ${}^{<\lambda}\kappa$, we say B is *strongly based on* A whenever B is based on A.

Definition 2.5. For an index structure $\mathcal{I}$, we say $\mathcal{I}$-indexed indiscernibles have the *modeling property* if given any $A = \{a_\nu | \nu \in \mathcal{I}\}$, there is and $\mathcal{I}$-indexed indiscernible $B = \{b_\eta | \eta \in \mathcal{I}\}$ such that B is based on A.

Fact 2.6. [5] Let ${}^{<\omega}\omega$ be the universe of the index structure. The strong indiscernibles have the modeling property.

Note that we cannot use the above fact when the index structure is a binary tree.

Now we introduce SOP_2 and its relatives.

Definition 2.7. [2, 4, 5] Fix $k \geq 2$.

(1) $\varphi(x; y)$ has SOP_2 is there is a $(a_\eta \mid \eta \in {}^{\omega>}2)$ such that
  (a) For all $\eta \in {}^{\omega}2$, $\{\varphi(x; a_{\eta\lceil\alpha}) \mid \alpha < \omega\}$ is consistent,
  (b) For all $\xi, \nu \in {}^{\omega>}2$, if $\xi \perp \nu$, then $\{\varphi(x; a_\xi), \varphi(x; a_\nu)\}$ is inconsistent.
(2) $\varphi(x; y)$ has the tree property of the first kind (TP_1) if there is $(a_\eta \mid \eta \in {}^{\omega>}\omega)$ such that
  (a) For all $\eta \in {}^{\omega}\omega$, $\{\varphi(x; a_{\nu\lceil\alpha}) | \alpha < \omega\}$ is consistent,
  (b) For all $\eta \perp \nu \in {}^{\omega>}\omega$, $\{\varphi(x; a_\eta), \varphi(x; a_\nu)\}$ is inconsistent.
(3) $\varphi(x; y)$ has weak k-TP_1 if there is $(a_\eta \mid \eta \in {}^{\omega>}\omega)$ such that
  (a) For all $\eta \in {}^{\omega}\omega$, $\{\varphi(x; a_{\nu\lceil\alpha}) | \alpha < \omega\}$ is consistent,
  (b) For any $\eta, \eta_0, \ldots, \eta_{k-1} \in {}^{\omega>}\omega$ and $i_0 < \cdots < i_{k-1} < \omega$, if $\eta^\frown\langle i_l\rangle \trianglelefteq \eta_i$ for each $l < k$, then $\{\varphi(x; a_{\eta_i}) \mid i < k\}$ is inconsistent.
(4) We say T has SOP_2 (resp. TP_1) if there is a formula having SOP_2 (resp. TP_1). If not, we say T is NSOP_2 (resp. NTP_1). We say T has weak-TP_1 if there is a formula having k-TP_1 for some k. If not, we say T is weak-NTP_1.

In [1], we see that all the notions in Definition 2.7 for theories are equivalent.

By modeling property, any formula having TP_1 or weak k-TP_1, we have a strongly indiscernible tree $(a_\eta)_{\eta \in {}^{<\omega}2}$ that has the same conditions. On the other hand, we cannot use the modeling property on SOP_2, though there is some trick to obtain strongly indiscernible binary tree. See [1, lemma 4.3 (3)] for more details.

3. Main result

We first modify [2, lemma 2.20] to an argument about ω-branched trees.

Lemma 3.1. *Suppose κ is a regular cardinal and we color ${}^{\kappa>}\omega$ by $\theta < \kappa$ colors. Let c be the given coloring.*

(1) There is ν^* in ${}^{\kappa>}\omega$ and $j < \theta$ such that for any $\nu \trianglerighteq \nu^*$ we can find $\rho \trianglerighteq \nu$ the color of which is j.

(2) There is an embedding $h : {}^{\omega>}\omega \to {}^{\kappa>}\omega$ such that

- *$h(\eta)^\frown\langle i\rangle \trianglelefteq h(\eta^\frown\langle i\rangle)$ for each $i < \omega$*
- *$Ran(h)$ is monochromatic.*

Proof. (1) Suppose not. Then for $i < \theta$, we inductively choose $\eta_i \in {}^{\kappa>}\omega$ such that

- if $i < j$, then $\eta_i \trianglelefteq \eta_j$, and
- for $\rho \in {}^{\kappa>}\omega$, if $\eta_{i+1} \trianglelefteq \rho$, then $c(\rho) \neq i$.

Let $\nu = \bigcup_{i<\theta} \eta_i$. Since $\theta < \mathrm{cf}(\kappa)$, $\nu \in {}^{\kappa>}\omega$. But this contradicts that $c(\nu)$ has no color.

(2) Use (1). $\square$

Before the propositions, we give a useful notation.

For each $\eta \in {}^{\omega_1>}2$, $m < \omega$, $\alpha \leq \omega_1$, we say $K_{\eta,m,\alpha}$ to denote the set $\{\eta^\frown\nu^\frown 0^\beta : \nu \in {}^m 2, \beta < \alpha\}$, and O_η to denote the set $\{\eta^\frown 0^\beta : \beta < \omega_1\}$.

Proposition 3.2. *The following are equivalent.*

1. *T is NSOP$_2$.*
2. *For all $\phi(x, y)$ and strongly indiscernible tree $(a_\eta : \eta \in {}^{\omega_1>}2)$, if $\{\phi(x, a_\nu) : \nu \in O_{\langle\rangle}\}$ is consistent, then for each $\eta \in {}^{\omega_1>}2$ and $m < \omega$, $\{\phi(x, a_\nu) : \nu \in K_{\eta,m,\omega_1}\}$ is consistent.*

Proof. (Sketch)

($1 \Leftarrow 2$) Suppose ϕ has SOP$_2$. By [1, lemma 4.3 (3)], we may assume there is a strongly indiscernible tree $(a_\eta : \eta \in {}^{\omega>}2)$ witnessing SOP$_2$. Use compactness to obtain a tree where (2) does not hold.

($1 \Rightarrow 2$) Suppose not. Fix ϕ and $(a_\eta : \eta \in {}^{\omega_1>}2)$. We inductively choose a finite subset $w_\eta \subseteq {}^{\omega_1>}2$ and $\nu_\eta \in {}^{\omega_1>} 2$ for each $\eta \in {}^{\omega_1>}2$ so that the following conditions holds after the construction;

(a) for each $i = 0, 1$, the union of $\{\phi(x, a_\nu) : \nu \in \bigcup\{w_{\eta\lceil\alpha} : \alpha \leq \mathrm{len}(\eta)\}\}$ and $\{\phi(x, a_\nu) : \nu \in O_{\nu_\eta}\}$ is consistent,

(b) the union of $\{\phi(x, a_\nu) : \nu \in \bigcup\{w_{\eta\lceil\alpha} : \alpha \leq \mathrm{len}(\eta)\}\}$ and $\{\phi(x, a_\nu) : \nu \in w_{\eta^\frown 0} \cup w_{\eta^\frown 1}\}$ is inconsistent.

Set $w_{\langle\rangle} = \varnothing$ and $\nu_{\langle\rangle} = \langle\rangle$. At limit case, $w_\eta = \varnothing$ and $\nu_\eta = \bigcup_{\xi\triangleleft\eta} \nu_\xi$.

Assume $w_{\eta\lceil\alpha}, \nu_{\eta\lceil\alpha}$ is chosen for all $\alpha \leq \mathrm{len}(\eta)$. Let $p_\eta = \{\phi(x, a_\nu) : \nu \in \bigcup\{w_{\eta\lceil\alpha} : \alpha \leq \mathrm{len}(\eta)\}\}$. We take the least $m_\eta < \omega$ where $p_\eta \cup \{\phi(x, a_\nu) : \nu \in K_{\nu_\eta,m_\eta,\omega_1}\}$ is inconsistent.

By minimiality of m_η and strong indiscernibility, $p_\eta \cup \{\phi(x, a_\nu) : \nu \in K_{\nu_\eta^\frown i,m_\eta-1,\omega_1}\}$ is consistent for $i = 0, 1$.

By compactness and strong indiscernibility, we have $l_\eta < \omega$ such that $p_\eta \cup \{\phi(x, a_\nu) : \nu \in K_{\nu_\eta,m_\eta,l_\eta}\}$ is inconsistent.

Take $w_{\eta^\frown i} = K_{\nu_\eta^\frown i,m_\eta-1,l_\eta}$ and $\nu_{\eta^\frown i} = \nu_\eta^\frown i^\frown 0^{m_\eta-1\frown} 0^{l_\eta+1}$ for $i = 0, 1$.

Having done the construction, we choose a finite subset $q_\eta \subseteq p_\eta$ for each η such that $q_\eta \cup \{\phi(x, a_\nu) : \nu \in w_{\eta^\frown 0} \cup w_{\eta^\frown 1}\}$ is inconsistent.

Let $\tau_\eta = \{a_\nu : \phi(x, a_\nu) \in q_\eta\}$. We may assume τ_η is a finite collection of $K_{\nu,m,\alpha}$s. Considering τ_η as a tuple, the number of $\sim_{str}$-equivalent classes in $\{\bar{\tau}_\eta : \eta \in {}^{\kappa>}2\}$ and $\{\bar{w}_{\eta^\frown i} : \eta \in {}^{\kappa>}2, i = 0, 1\}$ are both countable.

By [2, lemma 2.20], we have an embedding $h : {}^{\omega>}2 \to {}^{\kappa>}2$ whose range is monochromatic.

Define a formula $\psi(x, y)$ and a tree $(b_\eta : \eta \in {}^{\omega>}2)$ such that $\psi(x, b_\eta) = \bigwedge q_{h(\langle\rangle)} \wedge \bigwedge\{\phi(x, a_\nu) : \nu \in w_{h(0^\frown\eta)}\}$. Then $\psi(x, y)$ and $(b_\eta : \eta \in {}^{\omega>}2)$ witness SOP$_2$. $\square$

We analogously give another proposition about weak-TP_1.

For each $\eta \in {}^{\omega_1>}\omega$, $k \leq \omega$, $m < \omega$, $\alpha \leq \omega_1$, we say $K_{\eta,k,m,\alpha}$ to denote the set $\{\eta^\frown\nu^\frown 0^\beta : \nu \in {}^m k, \beta < \alpha\}$, and O_η to denote the set $\{\eta^\frown 0^\beta : \beta < \omega_1\}$.

Proposition 3.3. *The following are equivalent.*

(1) T does not have weak-TP_1.

(2) For all $\phi(x,y)$ and strongly indiscernible tree $(a_\eta : \eta \in {}^{\omega_1>}\omega)$, if $\{\phi(x,a_\nu) : \nu \in O_{\langle\rangle}\}$ is consistent, then for each $\eta \in {}^{\omega_1>}\omega$ and $m < \omega$, $\{\phi(x,a_\nu) : \nu \in K_{\eta,m,\omega_1}\}$ is consistent.

Proof. (Sketch)

($1 \Leftarrow 2$) Use modeling property and compactness.

($1 \Rightarrow 2$) Suppose not. Fix $\phi(x,y)$ and $(a_\eta : \eta \in {}^{\omega_1>}\omega)$. We inductively choose a finite subset $w_\eta \subseteq {}^{\omega_1>}\omega$ and $\nu_\eta \in^{\omega_1>} \omega$ for each $\eta \in {}^{\omega_1>}\omega$ so that the following conditions holds after the construction;

(a) the union of $\{\phi(x,a_\nu) : \nu \in \bigcup\{w_{\eta\lceil\alpha} : \alpha \leq \text{len}(\eta)\}\}$ and $\{\phi(x,a_\nu) : \nu \in O_{\nu_\eta}\}$ is consistent,

(b) the union of $\{\phi(x,a_\nu) : \nu \in \bigcup\{w_{\eta\lceil\alpha} : \alpha \leq \text{len}(\eta)\}\}$ and $\{\phi(x,a_\nu) : \nu \in \bigcup_{i<k} w_{\eta^\frown i}\}$ is inconsistent.

Set $w_{\langle\rangle} = \varnothing$ and $\nu_{\langle\rangle} = \langle\rangle$. At limit case, $w_\eta = \varnothing$ and $\nu_\eta = \bigcup_{\xi \trianglelefteq \eta} \nu_\xi$.

Assume $w_{\eta\lceil\alpha}, \nu_{\eta\lceil\alpha}$ is chosen for all $\alpha \leq \text{len}(\eta)$. Let $p_\eta = \{\phi(x,a_\nu) : \nu \in \bigcup\{w_{\eta\lceil\alpha} : \alpha \leq \text{len}(\eta)\}\}$. We take the least $m_\eta < \omega$ where $p_\eta \cup \{\phi(x,a_\nu) : \nu \in K_{\nu_\eta,\omega,m_\eta,\omega_1}\}$ is inconsistent.

By minimiality of m_η and strong indiscernibility, $p_\eta \cup \{\phi(x,a_\nu) : \nu \in K_{\nu_\eta^\frown i,\omega,m_\eta-1,\omega_1}\}$ is consistent for any $i < \omega$.

To argue inconsistency, we need an observation on strongly indiscernible trees.

Observation 3.4. *Let $(a_\eta : \eta \in {}^{\omega_1>}\omega)$ be strongly indiscernible. If $\{\phi(x,a_\nu) : \nu \in K_{\eta,\omega,m,\omega_1}\}$ is inconsistent for some $m < \omega$ and $\eta \in {}^{\omega_1>}\omega$, then there is some $k,l < \omega$ such that $\{\phi(x,a_\nu) : \nu \in K_{\eta,k,m,l}\}$ is inconsistent.*

By the above observation, we have $k_\eta, l_\eta < \omega$ such that $p_\eta \cup \{\phi(x,a_\nu) : \nu \in K_{\eta,k_\eta,m_\eta,l_\eta}\}$ is inconsistent. Take $w_{\eta^\frown i} = K_{\eta^\frown i,k_\eta,m_\eta-1,l_\eta}$ and $\nu_{\eta^\frown i} = \nu_\eta^\frown i^\frown 0^{m_\eta-1\frown} 0^{l_\eta+1}$ for $i < \omega$. Note that for any $i_0 < \cdots < i_{k_\eta-1} < \omega$, $p_\eta \cup \bigcup_{j<k_\eta}\{\phi(x,a_\nu) : \nu \in w_{\eta^\frown i_j}\}$ is inconsistent by strong indiscernibility.

Having done the construction, we choose a finite subset $q_\eta \subseteq p_\eta$ for each η such that $q_\eta \cup \bigcup_{j<k_\eta}\{\phi(x,a_\nu) : \nu \in w_{\eta^\frown j}\}$ is inconsistent.

Let $\tau_\eta = \{a_\nu : \phi(x,a_\nu) \in q_\eta\}$. By observation again, we may assume τ_η is a finite collection of $K_{\nu,k,m,l}$s. Considering τ_η as a tuple, the number of $\sim_{str}$-equivalent classes of $\{\bar{\tau}_\eta : \eta \in {}^{\kappa>}\omega\}$ and $\{\bar{w}_{\eta^\frown i} : \eta \in {}^{\kappa>}\omega, i < \omega\}$ are both countable. By lemma 3.1, we have an embedding $h : {}^{\omega>}\omega \to {}^{\omega_1>}\omega$ whose range is monochromatic.

Define a formula $\psi(x,y)$ and a tree $(b_\eta : \eta \in {}^{\omega>}\omega)$ such that $\psi(x,b_\eta) = \bigwedge q_{h(\langle\rangle)} \wedge \bigwedge\{\phi(x,a_\nu) : \nu \in w_{h(0^\frown\eta)}\}$. Then $\psi(x,y)$ and $(b_\eta : \eta \in {}^{\omega>}\omega)$ witness weak-TP_1. □

Now we turn our intention to antichains.

Definition 3.5. A subset $A \subseteq {}^{\kappa>}\lambda$ is called an antichain if for all $\eta,\nu \in A$, $\eta \perp \nu$.

Definition 3.6. (1) Let A be a set of tuples in ${}^{\omega>}2$. We say ϕ has A^{str}-SOP_2 if there is $(a_\eta : \eta \in {}^{\omega>}2)$ such that

(a) for all $\eta \in {}^\omega 2$, $\{\phi(x,a_{\eta\lceil m}) : m < \omega\}$ is consistent, and

(b) for all $\bar{\nu} \in {}^{\omega>}2$, if $\bar{\nu} \sim_{str} \bar{\xi}$ for some $\bar{\xi} \in A$, then $\{\phi(x,a_\nu) : \nu \in \bar{\nu}\}$ is inconsistent.

(2) Let A be a set of tuples in ${}^{\omega>}\omega$. We say ϕ has A^{str}-TP_1 if there is $(a_\eta : \eta \in {}^{\omega>}\omega)$ such that

(a) for all $\eta \in {}^{\omega}\omega$, $\{\phi(x, a_{\eta\lceil m}) : m < \omega\}$ is consistent, and
(b) for all $\bar{\nu} \in {}^{\omega>}\omega$, if $\bar{\nu} \sim_{str} \bar{\xi}$ for some $\bar{\xi} \in A$, then $\{\phi(x, a_\nu) : \nu \in \bar{\nu}\}$ is inconsistent.

(3) We say T has A^{str}-SOP_2 (resp. A^{str}-TP_1) if it has a A^{str}-SOP_2 (resp. A^{str}-TP_1) formula. If $A = \{\bar{\nu}\}$, then we say ϕ (or T) has $\bar{\nu}^{str}$-SOP_2 (resp. $\bar{\nu}^{str}$-TP_1).

Remark 3.7. (1) ϕ has SOP_2 if and only if ϕ has $\langle\langle 0\rangle, \langle 1\rangle\rangle^{str}$-$SOP_2$.
(2) ϕ has TP_1 if and only if ϕ has $\langle\langle 0\rangle, \langle 1\rangle\rangle^{str}$-$TP_1$.
(3) ϕ has weak k-TP_1 if and only if ϕ has $\langle\langle 0\rangle, \dots, \langle k-1\rangle\rangle^{str}$-$TP_1$.

Theorem 3.8. *Let $1 < k < \omega$ be given.*
(1) Let $\bar{\nu} \in {}^{\omega>}\omega$ be any antichain of size k. Then T has TP_1 if and only if T has $\bar{\nu}^{str}$-TP_1.
(2) Let $\bar{\nu} \in {}^{\omega>}2$ be any antichain of size k. Then T has SOP_2 if and only if T has $\bar{\nu}^{str}$-SOP_2.

Proof. (1) Suppose ϕ has TP_1. Since any antichain tuple contains $\trianglelefteq$-incomparable pairs, ϕ has $\bar{\nu}^{str}$-TP_1 for any antichain $\bar{\nu}$.

The converse is clear by proposition 3.3, and that T has TP_1 if and only if T has weak-TP_1.

(2) Note that T has SOP_2 if and only if T has TP_1. Then by (1), T has $\bar{\nu}^{str}$-TP_1. Let $(a_\eta : \eta \in {}^{\omega>}\omega)$ be the witness of $\bar{\nu}^{str}$-TP_1-ness. Then ϕ with the subtree $(a_\eta : \eta \in {}^{\omega>}2)$ satisfies $\bar{\nu}^{str}$-SOP_2.

The converse is clear by proposition 3.2. □

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