Expressing Dung's Extensions as FO-Formulas to Enumerate Them with an SMT Solver

Keishi Okamoto* Hiroyuki Kido† Toshinori Takai‡

Abstract

It is useful to express constraints for Dung's extensions as FO-formulas so that we can enumerate extensions with an SMT solver. We can extract an extension with a naive SMT solver if we naively express constraints for extensions as FO-formulas. But the definitions of some extensions require maximality/minimality conditions that can not be expressed as FO-formulas. On the other hand, a naive expression is readable but sometimes hard to solve with an SMT solver. Moreover, we need to improve a naive SMT solver to enumerate extensions since an enumeration is an iteration of an extraction.

In this paper, we propose a method to enumerate Dung's extensions by solving a Partial Maximal Satisfiable Subsets Enumeration problem, which is an extension of a Maximal Satisfiable Subsets Enumeration problem[1]. In particular, we express hard constraints and soft constraints, which are required for a Partial Maximal Satisfiable Subsets Enumeration problem, with FO-formulas.

1 Introduction

As products have become more complicated and complex, it has become more difficult to produce high-quality products. On the other hand, it is known that the cause of a product defect largely depends on the quality of its specifications document. Therefore, it is very important to improve the quality of the specification documents.

In IEEE830-1998, eight attributes for the quality of the software requirement specification, namely correctness, unambiguity, completeness, consistency, ranked for importance and/or stability, verifiability, modifiability, and traceability are introduced. Many research activities have been conducted to improve these attributes. In our project "Development of a Tool to Resolve Inconsistencies in the Specification Documents based on Mathematical Argumentation Theory", we are conducting a project to propose a method to support resolving "inconsistencies" in a specification document.

*National Institute of Technology, Sendai College
†Cardiff University
‡Nara Institute of Science and Technology

The first part of our proposal is to suggest acceptable subsets of claims in the specification document. Since an acceptable subset of claims is consistent, if we decided to accept an acceptable subset then it is enough to correct claims not contained in the acceptable subset. This suggestion is based on an argumentation framework, which can be a model of a given specification document, of Mathematical Argumentation Theory.

The second part of our proposal is to extract an attack relation from a specification document since an argumentation framework is a directed graph whose vertex is an argument, which is a chunk of descriptions in the given specification document, and edge relation is an attack relation. If we can extract an attack relation that represents pairwise inconsistencies, then we can resolve an inconsistency as a whole based on Mathematical Argumentation Theory. We will extract an attack relation based on Mathematical Argumentation Theory and Natural Language Processing but this is ongoing work.

In this paper, we show a first step of the first part. As an acceptable subset of an argumentation framework can be represented as a Dung's extension, we propose a method to enumerate extensions. Furthermore, we express constraints for extensions as FO-formulas so that we can enumerate extensions with an SMT solver.

In Section 2, we introduce definitions of Satisfiability Modulo Theories. These definitions are used to formulate a problem to enumerate extensions. In Section 3, we define hard constraints and soft constraints for Dung's extensions as FO-formulas so that we can enumerate extensions with an SMT solver. In Section 4, we conclude with several remarks.

2 Satisfiability Modulo Theories

In this section, we introduce definitions[2] of SMT problems that are used for enumerating extensions. First, we define a satisfiable subset and a maximal satisfiable subset of a set of constraints, and then extend it to a partial maximal satisfiable subset of a set of constraints by declaring constraints to be hard and soft. Second, we define a Partial Maximal Satisfiable Subsets Enumeration problem (a Partial MSSEn problem) that is used for enumerating extensions from an argumentation framework in Section 3.

Definition 1 (Satisfiability)

1. *A first-order formula φ is* satisfiable *if there is a model of φ. A first-order formula φ is* unsatisfiable *if it is not satisfiable.*
2. *A set C of first-order formulas is* satisfiable *if there is a model of the conjunction of all the first-order formula in C, namely a model of C. A set C of first-order formulas is* unsatisfiable *if it is not satisfiable.*

We note that a set C of first-order formulas is satisfiable if and only if there is a model that satisfies all the formulas in C.

We define a maximal satisfiable subset and a partial maximal satisfiable subset. A maximal satisfiable subset is defined in [1], and it can be extended to a partial maximal satisfiable subset as in [3]. The definition of a partial maximal satisfiable subset requires hard constraints that must be satisfied and soft constraints that should be satisfied as possible.

From now on, we use a word "constraint" instead of an FO-formula, because a constraint can represent both an FO-formula and a propositional logic formula.

Definition 2 (Maximal Satisfiable Subset, Partial MSS)

1. *Let C be a set of constraints. A satisfiable subset S of C is a* maximal satisfiable subset *(MSS) if $\forall c \in C \setminus S.\ S \cup \{c\}$ is unsatisfiable.*

2. *Let C be a set of constraints in which some constraints are declared to be hard and the rest are declared to be soft. A satisfiable subset S of C is a* partial maximal satisfiable subset *(Partial MSS) if S is a maximal satisfiable subset and includes all the hard constraints.*

We define an extraction problem and an enumeration problem. We extract the grounded extension from a given argumentation framework (A, R) since there is unique grounded extension in (A, R). And we enumerate other extensions in (A, R) since, for instance, there are possibly many preferred extensions in (A, R).

Definition 3 (Partial MSS Enumeration Problem) *Let C be a set of constraints in which some constraints are declared to be hard and the rest are declared to be soft.*

1. *The Partial MSS Extraction problem (a Partial MSSEx problem) for C is the problem of extracting a partial MSS of C.*

2. *The Partial MSS Enumeration problem (a Partial MSSEn problem) for C is the problem of enumerating partial MSS's of C.*

We can consider an extraction problem as a special case of an enumeration problem.

3 Expressing Extensions with FO-formulas

In this section, we express extensions as FO-formulas. Then we show that we can enumerate extensions of an argumentation framework (A, R) by solving a Partial MSSEn problem. For instance, a preferred extension of (A, R) is a maximal subset S of A satisfying that S is an admissible subset of A. In other words, a preferred extension is a maximal subset in $\{S' \subseteq A \mid S' \text{ is admissible}\}$. We show that we can enumerate preferred extensions by enumerating partial maximal satisfiable subsets of A with a hard constraint that "S is admissible" and soft constraints that "S is maximal under the hard constraint". Moreover, the other extensions can be enumerated in a similar way.

In 3.1, we define hard constraints for extensions as FO-formulas. In 3.2, we define soft constraints for extensions to enumerate extensions by solving a Partial MSSEn problem.

3.1 Hard Constraints for Extensions

In this subsection, we define hard constraints for Dung's extensions [4, 5] as FO-formulas. More precisely, we define a hard constraint $\mathbf{SE}(S)$ (respectively $\mathbf{CE}(S)$) for a stable extension (a complete extension) so that we can enumerate stable extensions (complete extensions) by enumerating subsets satisfying $\mathbf{SE}(S)$ (respectively $\mathbf{CE}(S)$). And we define a hard constraint $\mathbf{PE}(S)$ for a preferred extension so that we can enumerate preferred extensions by enumerating maximal subsets satisfying $\mathbf{PE}(S)$. Finally, we define a hard constraint $\mathbf{GE}(S)$ for the grounded extension so that we can enumerate (extract) the grounded extension by enumerating the minimal subset satisfying $\mathbf{GE}(S)$.

We model an argument framework (A, R) as a first-order structure of a directed graph i.e., A is a set and R is a binary relation on A.

Definition 4 (Argument Framework) *An argument framework (A, R) is a pair of a set A of arguments and a binary relation R on A. R represents an attack relation between arguments, namely $(a, b) \in R$ (or $R(a, b)$ holds) means that an argument a attacks an argument b. A subset S of A attacks an argument a if there is an argument $b \in S$ such that $(b, a) \in R$*

We model an extension S in (A, R) as a uninterpreted unary predicate $S(x)$. Since a subset of A can be represented as a set $\{a \in A \mid (A, R) \models S(a)\}$ for an appropriate unary predicate $S(x)$, we can consider an extension S as an uninterpreted unary predicate $S(x)$. In a first-order structure (A, R), a unary predicate $S(x)$ is interpreted by assigning a subset S of A to $S(x)$ (we write $S(x)^{(A,R)} = S$ for the assignment). A uninterpreted unary predicate $S(x)$ is not assigned to a specific subset of A. Thus solving a satisfiability problem to a FO-formula with a uninterpreted unary predicate $S(x)$ is assigning a specific subset S to $S(x)$[2].

We express the definition of a conflict-free subset (a stable extension) in [4, 5] as a FO-formula.

Definition 5 (Conflict-Free Subset, Stable Extension) *A subset S of A is* conflict-free *if there are no two arguments a and b such that $(a, b) \in R$. A subset S of A is a* stable extension *if it is conflict-free and, for each argument $a \in A$, if $a \notin S$ then there exists an argument $b \in S$ such that $(b, a) \in R$.*

By Definition 5, the definition that "a subset S is conflict-free" can be expressed as a FO-formula with a uninterpreted unary predicate $S(x)$ ([5]):

$$\forall x \forall y ((S(x) \land S(y)) \to \neg R(x, y)) \quad (\mathbf{CF}(S)).$$

Again by Definition 5, a part of the definition that "a subset S is a stable extension" can be expressed as a FO-formula [5]:

$$\forall x(\neg S(x) \rightarrow \exists y(S(y) \wedge R(y, x)))$$

which is logically equivalent to a FO-formula in prenex normal form:

$$\forall x \exists y(\neg S(x) \rightarrow (S(y) \wedge R(y, x))) \quad (\mathbf{SE}(S))$$

Moreover, we use Skolem functions[6, 7] to solve the satisfiability of a formula in prenex normal form:

$$\forall x \exists y \varphi(x, y).$$

The above formula is equisatisfiable to the following formula with a Skolem function f.

$$\forall x \varphi(x, f(x))$$

Then the satisfiability of $\mathbf{SE}(S)$ is equivalent to the satisfiability of the FO-formula $\mathbf{SE}(S, f_1)$ with a uninterpreted unary predicate $S(x)$ and a Skolem function $f_1(x)$ where

$$\forall x(\neg S(x) \rightarrow (S(f_1(x)) \wedge R(f_1(x), x))) \quad (\mathbf{SE}(S, f_1)).$$

Then a stable extension of (A, R) is a subset S of A satisfying a hard constraint $\mathbf{CF}(S) \wedge \mathbf{SE}(S, f_1)$.

We express the definition that "an argument x is defend by a set of arguments S" in [4, 5] as FO-formulas $D(S, x)$ with a uninterpreted unary predicate $S(x)$ and a uninterpreted constant x.

Definition 6 (Defend) *An argument a is* defended by *a set $S \subseteq A$ (or S* defends *a) if and only if for any argument $b \in A$, if b attacks a then S attacks b.*

We say that "a is defended by S" instead of saying that "a is acceptable with respect to S" in [4].

By Definition 6, the definition of "x is defended by S" can be defined as a FO-formula[5]:

$$\forall y\, (R(y, x) \rightarrow \exists z\, (S(z) \wedge R(z, y)))$$

which is logically equivalent to a FO-formula in prenex normal form:

$$\forall y \exists z(R(y, x) \rightarrow (S(z) \wedge R(z, y))) \quad (\mathbf{D}(S, x)).$$

We express the definition of an admissible subset in [4, 5] as a FO-formula $\mathbf{AS}(S)$ with a uninterpreted unary predicate $S(x)$. Then an admissible subset of (A, R) is a subset S of A satisfying a [hard] constraint $\mathbf{CF}(S) \wedge \mathbf{AS}(S)$.

Definition 7 (Admissible Subset) *A subset S of A is* admissible *if it is conflict-free and, for each argument $a \in A$, if $a \in S$ then a is defended by S.*

By Definition 7, a part of the definition that ”a subset S is an admissible subset” can be expressed as a first-order formula[5]:

$$\forall x\,(S(x) \rightarrow D(S,x))$$

which is logically equivalent to the following FO-formula in prenex normal form:

$$\forall x \forall y \exists z[(S(x) \wedge R(y,x)) \rightarrow (S(z) \wedge R(z,y))] \quad (\mathbf{AS}(S))$$

Moreover, the satisfiability of $\mathbf{CF}(S) \wedge \mathbf{AS}(S)$ is equivalent to the satisfiability of the FO-formula $\mathbf{CF}(S) \wedge \mathbf{AS}(S, f_2)$ with a uninterpreted unary predicate $S(x)$ and a Skolem function $f_2(x,y)$ where

$$\forall x \in A \forall y \in A[(S(x) \wedge R(y,x)) \rightarrow (S(f_2(x,y)) \wedge R(f_2(x,y),y))] \ (\mathbf{AS}(S, f_2)).$$

Then an admissible subset of (A,R) is a subset S of A satisfying a hard constraint $\mathbf{CF}(S) \wedge \mathbf{AS}(S, f_2)$.

We express a part of the definition of a preferred extension in [4, 5] as a FO-formula.

Definition 8 (Preferred Extension) *A subset S of A is a* preferred extension *if it is a maximal admissible subset of* (A,R).

By Definition 8, a preferred extension of (A,R) is a maximal subset of A satisfying a hard constraint $\mathbf{CF}(S) \wedge \mathbf{AS}(S, f_2)$ with a uninterpreted unary predicate $S(x)$ and a Skolem function f_2.

We express the definition of a complete extension in [4, 5] as an FO-formula.

Definition 9 (Complete Extension) *A subset S of A is a* complete extension *if it is an admissible subset of* (A,R) *and, for each argument $a \in A$, if a is defended by S then $a \in S$.*

By Definition 9, a part of the definition that “a subset S is a complete extension” can be expressed as a FO-formula[5]:

$$\forall x \in A\{(D(S,x)) \rightarrow S(x)\}$$

which is logically equivalent to the following FO-formula in prenex normal form:

$$\forall x \in A \exists y \in A \forall z \in A[\{(R(y,x) \rightarrow (S(z) \wedge R(z,y))\} \rightarrow S(x)] \ (\mathbf{CE}(S)).$$

Moreover, the satisfiability of $\mathbf{AS}(S, f_2) \wedge \mathbf{CE}(S)$ is equivalent to the satisfiability of the FO-formula $\mathbf{AS}(S, f_2) \wedge \mathbf{CE}(S, f_3)$ with a Skolem function $f_3(x)$ where

$$\forall x \in A \forall z \in A[\{(R(f_3(x),x) \rightarrow (S(z) \wedge R(z, f_3(x)))\} \rightarrow S(x)] \ (\mathbf{CE}(S, f_3)).$$

Thus a complete extension of (A,R) is a subset of A satisfying a hard constraint $\mathbf{AS}(S, f_2) \wedge \mathbf{CE}(S, f_3)$ with uninterpreted unary predicate $S(x)$ and Skolem functions f_2 and f_3.

We express a part of the definition of the grounded extension in [4, 5] as FO-formulas.

Definition 10 (Grounded Extension) *A subset S of A is* the grounded extension *if it is the least complete extension of* (A, R).

By Definition 10, the grounded extension of (A, R) is a minimal subset S of A satisfying a hard constraint $\mathbf{AS}(S, f_2) \wedge \mathbf{CE}(S, f_3)$. Indeed the solution S for $\mathbf{AS}(S, f_2) \wedge \mathbf{CE}(S, f_3)$ is unique.

3.2 Soft Constraints for Extensions

In this subsection, we define soft constraints for extensions to enumerate extensions by solving Partial MSSEn problems. We define three kinds of soft constraints, namely soft constraints to enumerate maximal satisfiable subsets, soft constraints to enumerate minimal satisfiable subsets and soft constraints to enumerate satisfiable subsets. By using these three kinds of soft constraints, we can uniformly enumerate extensions. However, the third kind is redundant, and it is more efficient to simply solve it as a problem to enumerate subsets satisfying hard constraints.

Let $G(A, R)$ be the set of FO-formulas representing an argument framework (A, R), namely an FO-formula representing an enumeration of A and the set $\{R(a, b) \mid (a, b) \in R\}$ of FO-formulas.

3.2.1 Preferred Extensions

In general, we can extract a maximal subset S of A satisfying an FO-formula $\Psi(S)$ by extracting a MSS of a hard constraint $\Psi(S)$ and soft constraints $S(a_1), S(a_2), \ldots, S(a_n)$ where $A = \{a_1, a_2, \ldots, a_n\}$. Since a preferred extension is a maximal subset S of A satisfying an FO-formula $\mathbf{CF}(S) \wedge \mathbf{AS}(S, f_2)$, we can enumerate preferred extensions by enumerating MSS's for

- Hard Constraints: $\mathbf{CF}(S) \wedge \mathbf{AS}(S, f_2)$, $G(A, R)$ and
- Soft Constraints: $S(a_1), S(a_2), \ldots, S(a_n)$ where $A = \{a_1, a_2, \ldots, a_n\}$.

We show an example of enumerating preferred extensions whose soft constraints are $S(a), S(b), S(c), S(d), S(e)$. Assume that the union of the hard constraints and a subset $\{S(a), S(d)\}$ is a solution for the Partial MSSEn problem. This shows that the subset $\{a, d\}(= S)$ of A is a maximal subset satisfying the hard constraint $\mathbf{CF}(S) \wedge \mathbf{AS}(S, f_2)$. Thus the set $\{a, d\}$ is a maximal admissible subset, namely a preferred extension.

3.2.2 The Grounded Extension

The grounded extension is a minimal subset of A satisfying an FO-formula $\mathbf{AS}(S, f_2) \wedge \mathbf{CE}(S, f_3)$. Let A be a set $\{a_1, a_2, \ldots, a_n\}$. Then, the grounded extension is the complement of $\{a_{i1}, a_{i2}, \ldots, a_{im}\}(\subseteq A)$ such that $\{\neg S(a_{i1}), \neg S(a_{i2}), \ldots, \neg S(a_{im}), \mathbf{AS}(S, f_2) \wedge \mathbf{CE}(S, f_3)\}$ is a maximal satisfiable subset for hard constraints including $\mathbf{AS}(S, f_2) \wedge \mathbf{CE}(S, f_3)$ and soft constraints $\{\neg S(a_1), \neg S(a_2),$

$\ldots, \neg S(a_n)\}$. Thus we can extract the grounded extension from (A, R) by enumerating MSS's for the following constraints:

- Hard Constraints: $\mathbf{AS}(S, f_2) \wedge \mathbf{CE}(S, f_3)$, $G(A, R)$ and
- Soft Constraints: $\neg S(a_1), \neg S(a_2), \ldots, \neg S(a_n)$ where $A = \{a_1, a_2, \ldots, a_n\}$.

We note that the result of enumeration of MSS's is unique by the definition of the grounded extension.

We show an example of extracting the grounded extension whose soft constraints are $\neg S(a), \neg S(b), \neg S(c), \neg S(d), \neg S(e)$. Assume that the union of the hard constraints and a subset $\{\neg S(b), \neg S(c), \neg S(d), \neg S(e)\}$ is a solution for the Partial MSSEn problem. The resulting subset shows that the subset $\{a\}(= S)$ of A is a minimal subset satisfying the hard constraint $\mathbf{AS}(S, f_2) \wedge \mathbf{CE}(S, f_3)$. Thus the set $\{a\}$ is a minimal complete extension, namely the grounded extension.

3.2.3 Stable Extensions and Complete Extensions

Since a stable extension is a subset S of A satisfying an FO-formula $\mathbf{CF}(S) \wedge \mathbf{SE}(S, f_1)$, we can enumerate stable extensions by enumerating MSS's of

- Hard Constraints: $\mathbf{CF}(S) \wedge \mathbf{SE}(S, f_1)$, $G(A, R)$ and
- Soft Constraints: $S(a_1), S(a_2), \ldots, S(a_n)$, $\neg S(a_1), \neg S(a_2), \ldots, \neg S(a_n)$ where $A = \{a_1, a_2, \ldots, a_n\}$.

We can enumerate complete extensions in a similar way since a complete extension is a subset S of A satisfying an FO-formula $\mathbf{AS}(S, f_2) \wedge \mathbf{CE}(S, f_3)$.

We show an example of enumerating stable extensions whose soft constraints are $S(a), S(b), S(c), S(d), S(e), \neg S(a), \neg S(b), \neg S(c), \neg S(d), \neg S(e)$. Assume that the union of all the hard constraints and a subset $\{S(a), \neg S(b), \neg S(c), S(d), \neg S(e)\}$ of the soft constraints is a solution for the Partial MSSEn problem. The resulting subset shows that the subset $\{a, d\}(= S)$ of A is a subset satisfying the hard constraint $\mathbf{CF}(S) \wedge \mathbf{SE}(S, f_1)$. Thus the set $\{a, d\}$ is a stable extension.

4 Concluding Remarks

In this paper, we propose a method to enumerate Dung's extensions by solving Partial MSSEn problems. In particular, we define hard constraints and soft constraints for extensions as FO-formulas so that we can enumerate extensions with an SMT solver.

Our future work is implementing a Partial MSSEn solver that is an extension of a (naive) SMT solver. In [1], authors propose a method to enumerate MUS's based on an SMT solver. The method can also enumerate MSS's, then we will extend it to enumerate Partial MSS's. Moreover, comparison with other extension enumeration methods is necessary.

We introduce FO-formulas to express Dung's extensions. These FO-formulas are optimized to solve corresponding Partial MSSEn problems with an SMT solver. In general, the expressed constraints tend to be complicated if we express constraints naively, and then we may not solve them with an SMT solver. Depending on the application of a argumentation framework, constraints other than those introduced in this paper may be added, so other constraints must also be concisely expressed.

Acknowledgments

This work was supported by JSPS KAKENHI Grant Number JP19K11914. This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University

References

[1] Enumerating Infeasibility: Finding Multiple MUSes Quickly, Liffiton M.H., Malik A., In: Gomes C., Sellmann M. (eds) Integration of AI and OR Techniques in Constraint Programming for Combinatorial Optimization Problems. CPAIOR 2013. Lecture Notes in Computer Science, vol 7874. Springer, Berlin, Heidelberg (2013)

[2] Handbook of Satisfiability, Armin Biere, Marijn Heule, Hans Van Maaren, Toby Walsh (Ed.), IOS Press (2009)

[3] Constraint Hierarchies, Alan Borning, Bjorn Freeman-Benson, and Molly Wilson, Lisp and Symbolic Computation, Vol. 5 No. 3, pages 223-270 (1992)

[4] On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games, Phan Minh Dung, Artificial Intelligence 77 321-357 (1995)

[5] Checking the acceptability of a set of arguments, Philippe Besnard and Sylvie Doutre

[6] Handbook of Practical Logic and Automated Reasoning, John Harrison, Cambridge University Press (2009)

[7] Model Theory (Encyclopedia of Mathematics and its Applications), Wilfrid Hodges, Cambridge University Press (1993)