New results for the Oseen problem with applications to the Navier–Stokes equations in exterior domains

Thomas Eiter and Giovanni P. Galdi

We prove new existence and uniqueness results in full Sobolev spaces for the steady-state Oseen problem in a smooth exterior domain of $\mathbb{R}^n$, $n \geq 2$. These results are then employed, on the one hand, in the study of analogous properties for the corresponding (linear) time-periodic case and, on the other hand and more significantly, to prove analogous properties for their nonlinear counterpart, at least for small data.

1 Introduction

As is well known, the steady-state Oseen problem consists in solving the following set of equations

$$
\begin{cases}
-\Delta u + \lambda \partial_1 u + \nabla \mathfrak{p} = f & \text{in } \Omega, \\
\operatorname{div} u = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega, \\
\lim_{|x| \to \infty} u(x) = 0,
\end{cases}
\tag{1.1}
$$

where Ω is an exterior domain of $\mathbb{R}^n$, $f \colon \Omega \to \mathbb{R}^n$ and $\lambda \ (> 0)$ are given external force and dimensionless (Reynolds) number, whereas $u \colon \Omega \to \mathbb{R}^n$ and $\mathfrak{p} \colon \Omega \to \mathbb{R}$ are unknown velocity and pressure fields, respectively.

Problem (1.1) has been investigated by a number of authors, beginning with the pioneering work [4]; for a rather detailed, yet incomplete, list of contributors and corresponding contributions we refer the reader to [5, Chapter VII], [2] and the bibliography there included. The peculiarity of these results is due to the circumstance that the function space where u belongs is *not* a full Sobolev space but, instead, a homogeneous

Sobolev space. The latter means that for a given f in the Lebesgue space $\mathrm{L}^{q_3}(\Omega)$ (suitable $q_3 \in (1,\infty)$), the associated velocity-field solution u, its first derivatives and its second derivatives belong, in the order, to *different* L^{q_i}-spaces, $i = 1, 2, 3$, with

$$q_1 > q_2 > q_3. \tag{1.2}$$

These findings are sharp, in the sense that, under the stated assumptions on f, it can be shown by means of counterexamples that the numbers q_1, q_2, q_3 must, in general, be different and satisfy (1.2).

However, particularly motivated by the recent approaches to the study of time-periodic well-posedness [9, 8] and time-periodic bifurcation [6, 7], we would like to investigate which *further* assumptions (if any) f must satisfy in order to ensure that u belongs to the full Sobolev space $\mathrm{W}^{2,q}(\Omega)$ ($q \equiv q_3$). A positive answer to this question would, for example, allow to frame time-periodic bifurcation in a full Sobolev space and hopefully, analyze the phenomenon of secondary bifurcation in a similar way as that employed for flow in bounded domains [10]. Notice that the rigorous interpretation of this phenomenon in the case of an exterior domain is, to date, an entirely open problem.

The main objective of this paper is to show that if, in addition to being in L^q, f is in the dual, $\mathrm{D}_0^{-1,r}(\Omega)$, of a suitable homogeneous Sobolev space[1], with $r = r(q,n)$, then there exists a unique $(u, \mathfrak{p})$ solving (1.1) with, in particular, $u \in \mathrm{W}^{2,q}(\Omega), \nabla\mathfrak{p} \in \mathrm{L}^q(\Omega)$. Moreover, the solution depends continuously on f, uniformly in $\lambda \in (0, \lambda_0)$ for arbitrarily fixed $\lambda_0 > 0$; see Theorem 2.1.

In view of the results established in [9], Theorem 2.1 produces an immediate corollary that ensures that a similar property holds also for time-periodic solutions of period $\mathcal{T} > 0$ to the (linear) Oseen problem (see (2.7)), provided f is periodic of the same period $\mathcal{T}$; see Theorem 2.3.

Finally, combining Theorem 2.1 and Theorem 2.3 with the contraction mapping theorem, we may extend the results proved there to the fully nonlinear Navier–Stokes case (see (2.10), (2.12)), on condition that the "size" of f and λ is suitably restricted, and $n \geq 3$. We thus show existence in full Sobolev space for both steady-state (Theorem 2.5) and time-periodic (Theorem 2.7) Navier–Stokes problems in exterior domain.

The plan of the paper is as follows. In Section 2, after recalling some basic notation used in the paper, we state and comment our main results. In the subsequent Section 3, we provide the proof of well-posedness for the steady-state (Theorem 2.1) and time-periodic (Theorem 2.3) Oseen problem. Finally, in Section 4 we extend the results of the preceding section to the fully nonlinear case; see Theorem 2.5 and Theorem 2.7.

2 Statement of the Main Results

We begin to introduce our principal notation. Unless otherwise stated, by the symbol Ω we mean a (smooth) exterior domain of $\mathbb{R}^n$, i.e., the complement of a (smooth) compact set Ω_0. With the origin of coordinates in the interior of Ω_0 we put $\mathrm{B}_R := \{x \in \mathbb{R}^n : |x| < R\}$, $\Omega_R := \Omega \cap \mathrm{B}_R$, and $\Omega^R := \Omega \setminus \overline{\mathrm{B}^R}$.

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[1]See the next section for the precise definition of $\mathrm{D}_0^{-1,r}(\Omega)$.

For $(t,x) \in \mathbb{R} \times \Omega$, we set $\partial_t := \partial/\partial t$, $\partial_j := \partial/\partial x_j$, $j = 1, \ldots, n$, and, as customary, for $\alpha \in \mathbb{N}_0^n$ we set $\mathrm{D}^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, and denote by $\nabla^k u$ the collection of all spatial derivatives $\mathrm{D}^\alpha u$ of u of order $|\alpha| = k$.

For $\mathcal{A}$ an open set of $\mathbb{R}^n$ and $q \in [1, \infty]$, we denote by $\mathrm{L}^q(\mathcal{A})$ and $\mathrm{W}^{k,q}(\mathcal{A})$ the classical Lebesgue and Sobolev spaces of order $k \in \mathbb{N}$, equipped with norms $\|\cdot\|_q = \|\cdot\|_{q;\mathcal{A}}$ and $\|\cdot\|_{k,q} = \|\cdot\|_{k,q;\mathcal{A}}$, respectively. We also consider homogeneous Sobolev spaces:

$$\mathrm{D}^{k,q}(\mathcal{A}) := \{u \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{A}) \mid \nabla^k u \in \mathrm{L}^q(\mathcal{A})\},$$

with corresponding seminorm

$$|u|_{k,q} = |u|_{k,q;\mathcal{A}} := \|\nabla^k u\|_{q;\mathcal{A}} := \sum_{|\alpha|=k} \|\mathrm{D}^\alpha u\|_{q;\mathcal{A}},$$

and $\mathrm{D}_0^{k,q}(\mathcal{A})$ obtained by (Cantor) completing $\mathrm{C}_0^\infty(\mathcal{A})$ in the norm $|\cdot|_{k,q}$. We indicate the latter's dual space by $\mathrm{D}_0^{-k,q'}(\mathcal{A})$, where $q' = q/(q-1)$, $q \in (1, \infty)$, with norm $|\cdot|_{-k,q'}$.

Let X be a seminormed vector space, $\mathcal{T} > 0$, and $q \in [1, \infty]$. Then $\mathrm{L}^q_{\mathrm{per}}(\mathbb{R}; X)$ is the space of all measurable $f \colon \mathbb{R} \to X$ such that $f(t + \mathcal{T}) = f(t)$ for almost all $t \in \mathbb{R}$ and $\|f\|_{\mathrm{L}^q_{\mathrm{per}}(\mathbb{R};X)} < \infty$, where

$$\|f\|_{\mathrm{L}^q_{\mathrm{per}}(\mathbb{R};X)} := \left(\frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} \|f(t)\|_X^q \, \mathrm{d}t\right)^{\frac{1}{q}} \quad \text{if } q < \infty, \qquad \|f\|_{\mathrm{L}^\infty_{\mathrm{per}}(\mathbb{R};X)} := \operatorname*{ess\,sup}_{t \in \mathbb{R}} \|f(t)\|_X.$$

For simplicity, we set $\mathrm{L}^q_{\mathrm{per}}(\mathbb{R} \times \mathcal{A}) := \mathrm{L}^q_{\mathrm{per}}(\mathbb{R}; \mathrm{L}^q(\mathcal{A}))$ and $\|f\|_q := \|f\|_{\mathrm{L}^q_{\mathrm{per}}(\mathbb{R};\mathrm{L}^q(\mathcal{A}))}$ for $f \in \mathrm{L}^q_{\mathrm{per}}(\mathbb{R} \times \mathcal{A})$. Moreover, we introduce the "maximal regularity space"

$$\mathrm{W}^{1,2,q}_{\mathrm{per}}(\mathbb{R} \times \mathcal{A}) := \left\{u \in \mathrm{L}^q_{\mathrm{per}}(\mathbb{R} \times \mathcal{A}) \mid u \in \mathrm{L}^q_{\mathrm{per}}(\mathbb{R}; \mathrm{W}^{2,q}(\mathcal{A})), \ \partial_t u \in \mathrm{L}^q_{\mathrm{per}}(\mathbb{R} \times \mathcal{A})\right\},$$

equipped with the norm

$$\|u\|_{1,2,q} := \|u\|_{\mathrm{W}^{1,2,q}_{\mathrm{per}}(\mathbb{R} \times \mathcal{A})} := \|u\|_{\mathrm{L}^q_{\mathrm{per}}(\mathbb{R};\mathrm{W}^{2,q}(\mathcal{A}))} + \|\partial_t u\|_{\mathrm{L}^q(\mathbb{R} \times \mathcal{A})}.$$

For functions $f \in \mathrm{L}^q_{\mathrm{per}}(\mathbb{R}; X)$ we introduce the projections

$$\mathcal{P}f := \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} f(t) \, \mathrm{d}t, \qquad \mathcal{P}_\perp f := f - \mathcal{P}f,$$

and we call $\mathcal{P}f \in X$ the *steady-state* part and $\mathcal{P}_\perp f \in \mathrm{L}^q_{\mathrm{per}}(\mathbb{R}; X)$ the *purely oscillatory* part of f. Setting $\mathrm{L}^q_{\mathrm{per},\perp}(\mathbb{R}; X) := \mathcal{P}_\perp \mathrm{L}^q_{\mathrm{per}}(\mathbb{R}; X)$, we obtain the decomposition

$$\mathrm{L}^q_{\mathrm{per}}(\mathbb{R}; X) = X \oplus \mathrm{L}^q_{\mathrm{per},\perp}(\mathbb{R}; X).$$

We shall also use the notation

$$\mathrm{L}^q_{\mathrm{per},\perp}(\mathbb{R} \times \mathcal{A}) := \mathcal{P}_\perp \mathrm{L}^q_{\mathrm{per}}(\mathbb{R} \times \mathcal{A}), \qquad \mathrm{W}^{1,2,q}_{\mathrm{per},\perp}(\mathbb{R} \times \mathcal{A}) := \mathcal{P}_\perp \mathrm{W}^{1,2,q}_{\mathrm{per}}(\mathbb{R} \times \mathcal{A}).$$

Unless otherwise stated, we do not distinguish between the real-valued function space X and its $\mathbb{R}^n$-valued analogue X^n, $n \in \mathbb{N}$.

We use the letter C to denote generic positive constants in our estimates. The dependence of a constant C on quantities $a, b, \ldots$, will be emphasized by writing $C(a, b, \ldots)$.

We are now in a position to state our main findings. We begin with the following theorem that, in fact, represents the key result upon which all the others rely.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and let $q \in (1, \infty)$, $r \in (\frac{n+1}{n}, n+1)$, $0 < \lambda \leq \lambda_0$. Set $s := \frac{(n+1)r}{n+1-r}$. Then, for every $f \in \mathrm{L}^q(\Omega)^n \cap \mathrm{D}_0^{-1,r}(\Omega)^n$ there exists a solution*

$$u \in \mathrm{D}^{2,q}(\Omega)^n \cap \mathrm{D}^{1,r}(\Omega)^n \cap \mathrm{L}^s(\Omega)^n, \qquad \mathfrak{p} \in \mathrm{D}^{1,q}(\Omega)$$

to (1.1). This solution satisfies $\partial_1 u \in \mathrm{L}^q(\Omega)^n$ and obeys the estimates

$$|u|_{1,r} + \lambda^{\frac{1+\delta}{n+1}} \|u\|_{\frac{(n+1)r}{n+1-r}} \leq C\lambda^{-\frac{M}{n+1}} |f|_{-1,r}, \tag{2.1}$$

$$|u|_{2,q} + \lambda \|\partial_1 u\|_q + \|\nabla \mathfrak{p}\|_q \leq C\big(\|f\|_q + \lambda^{-\frac{M}{n+1}} |f|_{-1,r}\big) \tag{2.2}$$

for some constant $C = C(n, q, r, \Omega, \lambda_0) > 0$, where

$$M = \begin{cases} 2 & \text{if } \frac{n+1}{n} < r \leq \frac{n}{n-1}, \\ 0 & \text{if } \frac{n}{n-1} < r < n, \\ 1 & \text{if } n \leq r < n+1, \end{cases} \qquad \delta = \begin{cases} 1 & \text{if } n = r = 2, \\ 0 & \text{else.} \end{cases} \tag{2.3}$$

In particular, if $s \leq q$, then $u \in \mathrm{W}^{2,q}(\Omega)^n$ and

$$\lambda^{\frac{(1+\delta)\theta}{n+1}} \|u\|_q + \lambda^{\frac{(1+\delta)\theta}{2(n+1)}} |u|_{1,q} + |u|_{2,q} \leq C\big(\|f\|_q + \lambda^{-\frac{M}{n+1}} |f|_{-1,r}\big) \tag{2.4}$$

where

$$\theta := \frac{qs}{n(q-s) + qs} = \frac{(n+1)qr}{n(n+1)(q-r) + qr} \in [0, 1]. \tag{2.5}$$

Moreover, if $(u_1, \mathfrak{p}_1)$ is another solution to (1.1) that belongs to the same function class as $(u, \mathfrak{p})$, then $u = u_1$ and $\mathfrak{p} = \mathfrak{p}_1 + c$ for some constant $c \in \mathbb{R}$.

Additionally, if $r > \frac{n}{n-1}$, we can choose $\mathfrak{p}$ such that $\mathfrak{p} \in \mathrm{L}^r(\Omega)$. Then $\partial_1 u \in \mathrm{D}^{-1,r}(\Omega)$ and it holds

$$\|\mathfrak{p}\|_r + \lambda |\partial_1 u|_{-1,r} \leq C\lambda^{-\frac{M}{n+1}} |f|_{-1,r}. \tag{2.6}$$

Remark 2.2. Note that the condition $s \leq q$ is equivalent to $\frac{1}{q} \leq \frac{1}{r} - \frac{1}{n+1}$. Therefore, the assumption $r > \frac{n+1}{n}$ in Theorem 2.1 implies the necessary condition $q > \frac{n+1}{n-1}$.

The first, important consequence of this theorem is presented in the next one concerning the corresponding linear time-periodic problem

$$
\begin{cases}
\partial_t u - \Delta u + \lambda \partial_1 u + \nabla \mathfrak{p} = f & \text{in } \mathbb{R} \times \Omega, \\
\operatorname{div} u = 0 & \text{in } \mathbb{R} \times \Omega, \\
u = 0 & \text{on } \mathbb{R} \times \partial\Omega, \\
\lim_{|x| \to \infty} u(t, x) = 0, & t \in \mathbb{R},
\end{cases}
\tag{2.7}
$$

with $f \colon \mathbb{R} \times \Omega \to \mathbb{R}^n$ a given time-periodic external force. Precisely, we will prove the following.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and let $q \in (1, \infty)$, $r \in (\frac{n+1}{n}, n+1)$ and $0 < \lambda \leq \lambda_0$. Set $s := \frac{(n+1)r}{n+1-r}$. Then, for every $f \in \mathrm{L}^q(\mathbb{R} \times \Omega)^n$ with $\mathcal{P}f \in \mathrm{D}^{-1,r}(\Omega)^n$ there is a solution $(u, \mathfrak{p}) = (v + w, p + \mathfrak{q})$ to (2.7) with*

$$
\begin{aligned}
v &\in \mathrm{D}^{2,q}(\Omega)^n \cap \mathrm{D}^{1,r}(\Omega)^n \cap \mathrm{L}^s(\Omega)^n, & p &\in \mathrm{D}^{1,q}(\Omega), \\
w &\in \mathrm{W}^{1,2,q}_{\mathrm{per},\perp}(\mathbb{R} \times \Omega)^n, & \mathfrak{q} &\in \mathrm{L}^q_{\mathrm{per},\perp}(\mathbb{R}; \mathrm{D}^{1,q}(\Omega)),
\end{aligned}
$$

which satisfies

$$
\begin{aligned}
|v|_{1,r} + \lambda^{\frac{1+\delta}{n+1}} \|v\|_s &\leq C \lambda^{-\frac{M}{n+1}} |\mathcal{P}f|_{-1,r}, \\
|v|_{2,q} + \lambda \|\partial_1 v\|_q + \|\nabla p\|_q &\leq C \big(\|\mathcal{P}f\|_q + \lambda^{-\frac{M}{n+1}} |\mathcal{P}f|_{-1,r} \big), \\
\|w\|_{1,2,q} + \|\nabla \mathfrak{q}\|_q &\leq C \|\mathcal{P}_\perp f\|_q
\end{aligned}
\tag{2.8}
$$

for a constant $C = C(q, r, \Omega, \lambda_0) > 0$ and M and δ as in (2.3). Moreover, if $(u_1, \mathfrak{p}_1)$ is another solution to (2.7) that belongs to the same function class as $(u, \mathfrak{p})$, then $u = u_1$ and $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_0$ for some $\mathcal{T}$-periodic function $\mathfrak{p}_0 \colon \mathbb{R} \to \mathbb{R}$.

In particular, if $s \leq q$, then $u \in \mathrm{W}^{1,2,q}_{\mathrm{per}}(\mathbb{R} \times \Omega)^n$ and

$$
\lambda^{\frac{(1+\delta)\theta}{n+1}} \|v\|_q + \lambda^{\frac{(1+\delta)\theta}{2(n+1)}} |v|_{1,q} + |v|_{2,q} \leq C \big(\|\mathcal{P}f\|_q + \lambda^{-\frac{M}{n+1}} |\mathcal{P}f|_{-1,r} \big)
\tag{2.9}
$$

where $\theta \in [0, 1]$ is given in (2.5).

Remark 2.4. The observation made in Remark 2.2 applies to Theorem 2.3 as well.

Next, combining the above theorems with the contraction mapping theorem, we are able to extend analogous results to the nonlinear case, under the assumption of "small" data.

More specifically, let us begin to consider the steady-state problem

$$
\begin{cases}
-\Delta v + \lambda \partial_1 v + \nabla p + v \cdot \nabla v = f & \text{in } \Omega, \\
\operatorname{div} v = 0 & \text{in } \Omega, \\
v = -\lambda \, e_1 & \text{on } \partial\Omega, \\
\lim_{|x| \to \infty} v(x) = 0.
\end{cases}
\tag{2.10}
$$

We shall prove the following result.

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Theorem 2.5. *Let* $\Omega \subset \mathbb{R}^n$, $n \geq 3$, *and let* $q, r \in (1, \infty)$ *with*

$$q \geq \frac{n}{3}, \quad \frac{1}{3q} + \frac{1}{n+1} \leq \frac{1}{r}, \quad \frac{2}{q} - \frac{4}{n} \leq \frac{1}{r}, \quad \frac{2}{n+1} \leq \frac{1}{r} < \begin{cases} \frac{n-1}{n} & \text{if } n = 3, 4, \\ \frac{n}{n+1} & \text{if } n \geq 5. \end{cases} \tag{2.11}$$

Then there is $\lambda_0 > 0$ *such that for all* $0 < \lambda \leq \lambda_0$ *we may find* $\varepsilon > 0$ *such that for all* $f \in \mathrm{L}^q(\Omega) \cap \mathrm{D}_0^{-1,r}(\Omega)$ *satisfying* $\|f\|_q + |f|_{-1,r} \leq \varepsilon$ *there exists a pair* (v, p) *with*

$$v \in \mathrm{D}^{2,q}(\Omega) \cap \mathrm{D}^{1,r}(\Omega) \cap \mathrm{L}^{\frac{(n+1)r}{n+1-r}}(\Omega), \quad \partial_1 v \in \mathrm{L}^q(\Omega), \quad p \in \mathrm{D}^{1,q}(\Omega)$$

satisfying (2.10). *In particular, if* $s \leq q$, *then* $v \in \mathrm{W}^{2,q}(\Omega)^n$.

Remark 2.6. As in Remark 2.2, the additional assumption $s \leq q$ is equivalent to $\frac{1}{q} \leq \frac{1}{r} - \frac{1}{n+1}$. Therefore, the upper bound in (2.11) leads to the necessary conditions

$$q > \frac{n(n+1)}{n^2 - n - 1} \quad \text{if } n = 3, 4, \qquad q > \frac{n+1}{n-1} \quad \text{if } n \geq 5.$$

Likewise, consider the problem

$$\begin{cases} \partial_t v - \Delta v + \lambda \partial_1 v + \nabla p + v \cdot \nabla v = f & \text{in } \mathbb{R} \times \Omega, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R} \times \Omega, \\ v = -\lambda \, \mathrm{e}_1 & \text{on } \mathbb{R} \times \partial\Omega, \\ \lim_{|x| \to \infty} v(t, x) = 0, & t \in \mathbb{R}. \end{cases} \tag{2.12}$$

where f is a suitably prescribed time-periodic function. We shall prove the following.

Theorem 2.7. *Let* $\Omega \subset \mathbb{R}^n$, $n \geq 3$, *and let* $q, r \in (1, \infty)$ *with*

$$\frac{n+2}{3} \leq q \leq n+1, \qquad \frac{n(n+1)}{n^2 - n - 1} < q, \tag{2.13}$$

$$\frac{2}{q} - \frac{4}{n} \leq \frac{1}{r} \leq \frac{2}{q}, \qquad \frac{1}{q} + \frac{1}{n+1} \leq \frac{1}{r} < \begin{cases} \frac{n-1}{n} & \text{if } n = 3, 4, \\ \frac{n}{n+1} & \text{if } n \geq 5. \end{cases} \tag{2.14}$$

Then there is $\lambda_0 > 0$ *such that for all* $0 < \lambda \leq \lambda_0$ *we can find* $\varepsilon > 0$ *such that for all* $f \in \mathrm{L}^q_{\mathrm{per}}(\mathbb{R} \times \Omega)^n$ *with* $\mathcal{P}f \in \mathrm{D}_0^{-1,r}(\Omega)^n$ *satisfying* $\|f\|_q + |\mathcal{P}f|_{-1,r} \leq \varepsilon$ *there exists a unique solution*

$$(v, p) \in \mathrm{W}^{1,2,q}_{\mathrm{per}}(\mathbb{R} \times \Omega)^n \times \mathrm{L}^q_{\mathrm{per}}(\mathbb{R}; \mathrm{D}^{1,q}(\Omega)), \quad \mathcal{P}p \in \mathrm{L}^r(\Omega)$$

to (2.12).

Remark 2.8. For $n = 3$, condition (2.13) yields $q \in (\frac{12}{5}, 4]$. For $n \geq 4$, the second restriction in (2.13) is redundant and it simplifies to $q \in (\frac{n+2}{3}, n+1]$.

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3 Proofs of Theorem 2.1 and Theorem 2.3

In order to prove Theorem 2.1, we first establish the following density result, which enables us to consider problem (1.1) only for right-hand sides $f \in C_0^\infty(\Omega)^n$.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain and let $q, r \in (1, \infty)$. Then $C_0^\infty(\Omega)$ is a dense subset of $L^q(\Omega) \cap D_0^{-1,r}(\Omega)$.*

Proof. The space $L^q(\Omega) \cap D_0^{-1,r}(\Omega)$ can be identified with the dual space of $L^{q'}(\Omega) + D_0^{1,r'}(\Omega)$, where $s' = s/(s-1)$. Identifying elements of $C_0^\infty(\Omega)$ with the corresponding functionals, we consider $g \in L^{q'}(\Omega) + D_0^{1,r'}(\Omega)$ that is an element of the kernel of each functional in $C_0^\infty(\Omega)$, i.e.,

$$\int_\Omega \varphi\, g \,\mathrm{d}x = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. This implies $g = 0$. Consequently, by a standard duality argument, $C_0^\infty(\Omega)$ is dense in $L^q(\Omega) \cap D_0^{-1,r}(\Omega)$. $\qquad\square$

We recall the notion of weak solutions: A pair $(u, \mathfrak{p}) \in D_0^{1,r}(\Omega)^n \times L_{\mathrm{loc}}^r(\Omega)$ is called *weak solution* to (1.1) if $\operatorname{div} u = 0$ and

$$\int_\Omega \nabla u : \nabla\varphi + \lambda\partial_1 u \cdot \varphi \,\mathrm{d}x = \int_\Omega \mathfrak{p}\operatorname{div}\varphi + f \cdot \varphi \,\mathrm{d}x$$

for all $\varphi \in C_0^\infty(\Omega)^n$ with $\operatorname{div}\varphi = 0$. We show that weak solutions have better regularity when f is sufficiently regular.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^n$ be an exterior domain of class C^2. Let $q, r, s \in (1, \infty)$, $f \in C_0^\infty(\Omega)^n$, and let $(u, \mathfrak{p}) \in \big(D^{1,r}(\Omega)^n \cap L^s(\Omega)^n\big) \times L_{\mathrm{loc}}^r(\Omega)$ be a weak solution to (1.1). Then $u \in D^{2,q}(\Omega)^n$, $\partial_1 u \in L^q(\Omega)^n$ and $\mathfrak{p} \in D^{1,q}(\Omega)$, and for each $R > 0$ with $\partial\Omega \subset B_R$ there exists $C = C(n, q, \Omega, R) > 0$ such that*

$$|u|_{2,q} + \lambda\|\partial_1 u\|_q + |\mathfrak{p}|_{1,q} \le C(1 + \lambda^4)\big(\|f\|_q + \|u\|_{q;\Omega_R} + \|\mathfrak{p}\|_{q;\Omega_R}\big). \tag{3.1}$$

Proof. By [5, Theorem VII.1.1], we have $u \in W_{\mathrm{loc}}^{2,q}(\Omega) \cap C^\infty(\Omega)$ and $\mathfrak{p} \in W_{\mathrm{loc}}^{1,q}(\Omega) \cap C^\infty(\Omega)$. Let $0 < R_0 < R_1 < R$ such that $\partial B_{R_0} \subset \Omega$, and let $\chi \in C_0^\infty(B_{R_1})$ with $\chi \equiv 1$ on B_{R_0}. We set $v := (1-\chi)u + \mathfrak{B}(u \cdot \nabla\chi)$, where $\mathfrak{B}$ denotes the Bogovskiĭ operator, and $p := (1-\chi)\mathfrak{p}$. Then $v \in W_{\mathrm{loc}}^{2,q}(\mathbb{R}^n) \cap D^{1,r}(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$ and $p \in W_{\mathrm{loc}}^{1,q}(\mathbb{R}^n)$ satisfy

$$\begin{cases} -\Delta v + \lambda\partial_1 v + \nabla p = F & \text{in } \mathbb{R}^n, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^n \end{cases} \tag{3.2}$$

with

$$F := (1-\chi)f + 2\nabla\chi \cdot \nabla u + \Delta\chi u - \lambda\partial_1\chi u + \mathfrak{p}\nabla\chi + \big[-\Delta + \lambda\partial_1\big]\mathfrak{B}(u \cdot \nabla\chi).$$

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By [5, Theorem VII.7.1], there exists a solution $(v_1, p_1) \in D^{2,q}(\Omega) \times D^{1,q}(\Omega)$ to (3.2) that satisfies

$$|v_1|_{2,q} + \lambda \|\partial_1 v_1\|_q + |p_1|_{1,q} \le C\|F\|_q. \tag{3.3}$$

Now set $w := v - v_1$. Then w is a solution to the homogeneous Oseen system in the whole space. Therefore, $w = v - v_1$ is a polynomial, which can be readily shown with the help of Fourier transform. From [5, Theorem VII.6.1] and $f \in C_0^\infty(\Omega)$ we conclude $D^\alpha u(x) \to 0$ and thus $D^\alpha v(x) \to 0$ as $|x| \to \infty$ for each $\alpha \in \mathbb{N}_0^n$. In virtue of $D^\alpha v_1 \in L^q(\Omega)$ for $|\alpha| = 2$, the polynomial $D^\alpha w = D^\alpha v - D^\alpha v_1$ must thus be zero, i.e., $D^\alpha w = 0$ for $|\alpha| = 2$. In the same way we conclude $\partial_1 w = 0$ and, in consequence, $\nabla \mathfrak{q} = 0$. Hence we can replace (v_1, p_1) with (v, p) in estimate (3.3). Since $u = v$ and $\mathfrak{p} = p$ on Ω^{R_1}, estimate (3.3) thus implies

$$|u|_{2,q;\Omega^{R_1}} + \lambda \|\partial_1 u\|_{q;\Omega^{R_1}} + |\mathfrak{p}|_{1,q;\Omega^{R_1}} \le |v|_{2,q} + \lambda \|\partial_1 v\|_q + |p|_{1,q}$$
$$\le C\big(\|f\|_q + (1+\lambda)\|u\|_{1,q;\Omega_{R_1}} + \|\mathfrak{p}\|_{q;\Omega_{R_1}}\big). \tag{3.4}$$

To derive the estimate near the boundary, we use another cut-off function $\chi_1 \in C_0^\infty(B_R)$ with $\chi_1 \equiv 1$ on B_{R_1}, and we set $v := \chi_1 u$ and $p := \chi_1 \mathfrak{p}$. Then $(v, p) \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$ is a solution to

$$\begin{cases} -\Delta v + \nabla p = \chi_1 f - 2\nabla\chi_1 \cdot \nabla u - \Delta\chi_1 u - \chi_1 \lambda \partial_1 u + \mathfrak{p}\nabla\chi_1 & \text{in } \Omega_R, \\ \operatorname{div} v = u \cdot \nabla\chi_1 & \text{in } \Omega_R, \\ v = 0 & \text{on } \partial\Omega_R. \end{cases}$$

It is well known (see [5, Exercise IV.6.3] for example) that then (v, p) is subject to the estimate

$$\|v\|_{2,q} + \|\nabla p\|_q \le C\big(\|f\|_{q;\Omega_R} + (1+\lambda)\|u\|_{1,q;\Omega_R} + \|\mathfrak{p}\|_{q;\Omega_R}\big).$$

Since $u = v$ and $\mathfrak{p} = p$ on Ω_{R_1}, a combination of this estimate with (3.4) yields

$$|u|_{2,q} + \lambda\|\partial_1 u\|_q + |\mathfrak{p}|_{1,q} \le C(1+\lambda)\big(\|f\|_q + (1+\lambda)\|u\|_{1,q;\Omega_R} + \|\mathfrak{p}\|_{q;\Omega_R}\big).$$

Finally, an application of Ehrling's inequality

$$|u|_{1,q;\Omega_R} \le C(\varepsilon^{-1}\|u\|_{q;\Omega_R} + \varepsilon|u|_{2,q;\Omega_R})$$

(see [1, Theorem 5.2]) for $\varepsilon > 0$ sufficiently small leads to (3.1). $\qquad \square$

Proof of Theorem 2.1. For the moment, consider $f \in C_0^\infty(\Omega)$. The existence of a weak solution $(u, \mathfrak{p})$ to (2.1) with $u \in D^{1,r}(\Omega) \cap L^s(\Omega)$ satisfying (2.1) follows from [11, Theorem 2.2]. In the case $q > \frac{n}{n-1}$, one shows $\mathfrak{p} \in L^r(\Omega)$ and $\|\mathfrak{p}\|_r \le C\lambda^{-\frac{M}{n+1}}|f|_{-1,r}$ in the same way as in [5, Proof of Theorem VII.7.2]. Then, for $\varphi \in C_0^\infty(\Omega)^n$ we have

$$\lambda \int_\Omega \partial_1 u \cdot \varphi \, dx = \int_\Omega \big(f \cdot \varphi + \mathfrak{p}\operatorname{div}\varphi - \nabla u : \nabla\varphi\big) dx \le \big(|f|_{-1,r} + \|\mathfrak{p}\|_r + \|\nabla u\|_r\big)\|\nabla\varphi\|_{r'},$$

where $r' = r/(r-1)$, which implies $\partial_1 u \in D_0^{-1,r}(\Omega)$ and

$$\lambda |\partial_1 u|_{-1,r} \leq C\big(|f|_{-1,r} + \|\mathfrak{p}\|_r + \|\nabla u\|_r\big) \leq C\lambda^{-\frac{M}{n+1}} |f|_{-1,r}.$$

This shows (2.6) if $r > \frac{n}{n-1}$. If this is not the case, we instead obtain a local estimate in the following way: First of all, by [5, Lemma VII.1.1] we have $\mathfrak{p} \in L^r(\Omega_R)$ for all $R > 0$ with $\partial B_R \subset \Omega$. For fixed R, we can add a constant to $\mathfrak{p}$ such that $\int_{\Omega_R} \mathfrak{p} = 0$. Now let $\psi \in W_0^{1,r'}(\Omega_R)^n$, $r' = r/(r-1)$, be a solution to the problem

$$\operatorname{div}\psi = |\mathfrak{p}|^{r-2}\mathfrak{p} - \frac{1}{|\Omega_R|}\int_{\Omega_R} |\mathfrak{p}|^{r-2}\mathfrak{p}\,\mathrm{d}x =: g \qquad \text{in } \Omega_R,$$

which exists since g has vanishing mean value and satisfies $g \in L^{r'}(\Omega_R)$ (see [5, Theorem III.3.6] for example). Moreover, we have

$$\|\psi\|_{1,r';\Omega_R} \leq C\|g\|_{r';\Omega_R} \leq C\|\mathfrak{p}\|_{r;\Omega_R}^{r-1}.$$

Since $(u,\mathfrak{p})$ is a weak solution and $\mathfrak{p}$ has vanishing mean value on Ω_R, we deduce

$$\begin{aligned}
\|\mathfrak{p}\|_{r;\Omega_R}^r &= \int_{\Omega_R} \mathfrak{p}\operatorname{div}\psi\,\mathrm{d}x + \int_{\Omega_R}\mathfrak{p}\,\mathrm{d}x\,\frac{1}{|\Omega_R|}\int_{\Omega_R}|\mathfrak{p}|^{r-2}\mathfrak{p}\,\mathrm{d}x \\
&= \int_{\Omega_R} \nabla u : \nabla\psi - \lambda\partial_1 u \cdot \psi - f \cdot \psi\,\mathrm{d}x \\
&\leq C(1+\lambda_0)\big(\|\nabla u\|_r + \|f\|_{-1,r;\Omega_R}\big)\|\psi\|_{1,r';\Omega_R} \\
&\leq C(1+\lambda_0)\big(\|\nabla u\|_r + |f|_{-1,r}\big)\|\mathfrak{p}\|_{r;\Omega_R}^{r-1}.
\end{aligned}$$

Using estimate (2.1), this leads to

$$\|\mathfrak{p}\|_{r;\Omega_R} \leq C\lambda^{-\frac{M}{n+1}} |f|_{-1,r}. \tag{3.5}$$

Next, by Lemma 3.2, from $f \in C_0^\infty(\Omega)$ we conclude $u \in D^{2,q}(\Omega)$ and $\mathfrak{p} \in D^{1,q}(\Omega)$ and the validity of (3.1). We apply the estimate

$$\|u\|_{q;\Omega_R} \leq C(\varepsilon)\|u\|_{\sigma;\Omega_R} + \varepsilon |u|_{1,q;\Omega_R}$$

for $\varepsilon > 0$ and $\sigma \in (1,\infty)$ several times to deduce

$$\begin{aligned}
\|u\|_{q;\Omega_R} &\leq C(\varepsilon)\|u\|_s + C(\varepsilon)|u|_{1,r} + \varepsilon|u|_{2,q}, \\
\|\mathfrak{p}\|_{q;\Omega_R} &\leq C(\varepsilon)\|\mathfrak{p}\|_{r;\Omega_R} + \varepsilon|\mathfrak{p}|_{1,q}.
\end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small and combining these with the estimates (3.1), (2.1) and (3.5), we conclude (2.2) for $f \in C_0^\infty(\Omega)$. Employing the above estimates and Proposition 3.1, we can finally extend the result to general $f \in L^q(\Omega) \cap D_0^{-1,r}(\Omega)$ by a standard density argument.

Moreover, the additional assumption $s \leq q$ yields the embedding

$$\mathrm{L}^s(\Omega) \cap \mathrm{D}^{2,q}(\Omega) \hookrightarrow \mathrm{W}^{2,q}(\Omega),$$

so that $u \in \mathrm{W}^{2,q}(\Omega)$ in this case, and the Gagliardo–Nirenberg inequality (see [3]) implies

$$\|u\|_q \leq C\|u\|_s^\theta |u|_{2,q}^{1-\theta} \leq C\lambda^{-\frac{(1+\delta)\theta}{n+1}} \left(\|f\|_q + \lambda^{-\frac{M}{n+1}} |f|_{-1,r}\right)$$

and

$$|u|_{1,q} \leq C\|u\|_q^{1/2} |u|_{2,q}^{1/2} \leq C\lambda^{-\frac{(1+\delta)\theta}{2(n+1)}} \left(\|f\|_q + \lambda^{-\frac{M}{n+1}} |f|_{-1,r}\right),$$

where we used (2.2). This shows estimate (2.4) and completes the proof. $\qquad \square$

Now let us turn to the time-periodic Oseen problem (2.7). We recall the following result, which treats the case $\mathcal{P}f = 0$.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an exterior domain of class C^2, $q \in (1, \infty)$ and $\lambda \in [0, \lambda_0]$, $\lambda_0 > 0$. For any $f \in \mathrm{L}^q_{\mathrm{per},\perp}(\mathbb{R} \times \Omega)^n$ there is a solution*

$$(u, \mathfrak{p}) \in \mathrm{W}^{1,2,q}_{\mathrm{per},\perp}(\mathbb{R} \times \Omega)^n \times \mathrm{L}^q_{\mathrm{per},\perp}(\mathbb{R}; \mathrm{D}^{1,q}(\Omega))$$

to (2.7), which satisfies

$$\|u\|_{1,2,q} + \|\nabla \mathfrak{p}\|_q \leq C\|f\|_q \qquad (3.6)$$

for a constant $C = C(n, q, \Omega, \lambda_0) > 0$. If $(u_1, \mathfrak{p}_1) \in \mathrm{W}^{1,2,q}_{\mathrm{per},\perp}(\mathbb{R} \times \Omega)^n \times \mathrm{L}^q_{\mathrm{per},\perp}(\mathbb{R}; \mathrm{D}^{1,q}(\Omega))$ is another solution to (2.7), then $u = u_1$ and $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_0$ for some $\mathcal{T}$-periodic function $\mathfrak{p}_0 \colon \mathbb{R} \to \mathbb{R}$.

Proof. The result for $n = 3$ has been established in [9, Theorem 5.1]. The general case $n \geq 2$ is proved along the same lines. $\qquad \square$

A combination of Theorem 2.1 and Theorem 3.3 allows us to treat general time-periodic forcing terms and immediately leads to a proof of Theorem 2.3.

Proof of Theorem 2.3. Set $f_1 := \mathcal{P}f$ and $f_2 := \mathcal{P}_\perp f$. Let $(v, p) \in \left(\mathrm{L}^s \cap \mathrm{D}^{2,q}(\Omega)\right) \times \mathrm{D}^{1,q}(\Omega)$ be a solution to (1.1) with right-hand side $f = f_1$ that exists due to Theorem 2.1. Moreover, let $(w, \mathfrak{q}) \in \mathrm{W}^{1,2,q}_{\mathrm{per},\perp}(\mathbb{R} \times \Omega) \times \mathrm{L}^q_{\mathrm{per},\perp}(\mathbb{R}; \mathrm{D}^{1,q}(\Omega))$ be a solution to (2.7) with right-hand side $f = f_2$ that exists due to Theorem 3.3. Then $(u, \mathfrak{p}) := (v + w, p + \mathfrak{q})$ is a solution to (2.7) with the asserted properties. The uniqueness statement is deduced in a similar way. $\qquad \square$

4 Proofs of Theorem 2.5 and Theorem 2.7

In the following we focus on the time-periodic case and the proof of Theorem 2.7. The proof of Theorem 2.5 is very similar but less involved, and we will sketch it at the end of this section.

First, we reformulate (2.12) as a problem with homogeneous boundary conditions. For this purpose, let $R > 0$ with $\partial B_R \subset \Omega$, and let $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi \equiv 1$ on B_R. We define the function $V \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$V(x) = \frac{\lambda}{2}\big[-\Delta + \nabla \operatorname{div}\big](\varphi(x)x_2^2\, e_1).$$

Then $\operatorname{div} V \equiv 0$ and $V(x) = -\lambda\, e_1$ for $x \in \partial\Omega$, and V obeys the estimate

$$\|-\Delta V + \lambda\partial_1 V\|_q + |-\Delta V + \lambda\partial_1 V|_{-1,r} \leq C\lambda(1 + \lambda). \tag{4.1}$$

We set $u(t,x) := v(t,x) - V(x)$, $\mathfrak{p} := p$. Then (v,p) is a $\mathcal{T}$-time-periodic solution to (2.12) if and only if $(u,\mathfrak{p})$ is a $\mathcal{T}$-time-periodic solution to

$$\begin{cases} \partial_t u - \Delta u + \lambda\partial_1 u + \nabla\mathfrak{p} = f + \mathcal{N}(u) & \text{in } \mathbb{R}\times\Omega, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}\times\Omega, \\ u = 0 & \text{on } \mathbb{R}\times\partial\Omega, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{4.2}$$

where

$$\mathcal{N}(u) = -u\cdot\nabla u - u\cdot\nabla V - V\cdot\nabla u - V\cdot\nabla V + \Delta V - \lambda\partial_1 V.$$

We will show existence of a solution to (4.2) in the function space

$$X_\lambda^{q,r} := \big\{u \in W_{\text{per}}^{1,2,q}(\mathbb{R}\times\Omega)^n \ \big| \ \operatorname{div} u = 0,\ u\big|_{\mathbb{R}\times\partial\Omega} = 0,\ \|\mathcal{P}u\|_\lambda < \infty\big\},$$

$$\|v\|_\lambda := |v|_{2,q} + |v|_{1,r} + \lambda^{\frac{1}{n+1}}\|v\|_s, \qquad s := \frac{(n+1)r}{n+1-r},$$

which we equip with the norm

$$\|u\|_{X_\lambda^{q,r}} := \|\mathcal{P}u\|_\lambda + \|\mathcal{P}_\perp u\|_{1,2,q}.$$

Then $X_\lambda^{q,r}$ is a Banach space since $s \leq q$ by (2.14). The following lemma enables us to derive suitable estimates for $\mathcal{N}(u)$ when $u \in X_\lambda^{q,r}$.

Lemma 4.1. *Let $q,r \in (1,\infty)$ satisfy (2.13) and (2.14), $0 < \lambda \leq \lambda_0$, and let $u_1, u_2 \in X_\lambda^{q,r}$. Set $v_j := \mathcal{P}u_j$, $w_j := \mathcal{P}_\perp u_j$ for $j = 1, 2$. Then*

$$\|v_1\cdot\nabla v_2\|_q \leq C\lambda^{-\frac{\theta}{n+1}}\|v_1\|_\lambda\|v_2\|_\lambda, \tag{4.3}$$

$$|v_1\cdot\nabla v_2|_{-1,r} \leq C\lambda^{-\frac{\eta}{n+1}}\|v_1\|_\lambda\|v_2\|_\lambda, \tag{4.4}$$

$$\|w_1\cdot\nabla w_2\|_q \leq C\|w_1\|_{1,2,q}\|w_2\|_{1,2,q}, \tag{4.5}$$

$$|\mathcal{P}(w_1\cdot\nabla w_2)|_{-1,r} \leq C\|w_1\|_{1,2,q}\|w_2\|_{1,2,q}, \tag{4.6}$$

$$\|v_1\cdot\nabla w_2\|_q \leq C\lambda^{-\frac{\zeta}{n+1}}\|v_1\|_\lambda\|w_2\|_{1,2,q}, \tag{4.7}$$

$$\|w_1\cdot\nabla v_2\|_q \leq C\lambda^{-\frac{\zeta}{n+1}}\|w_1\|_{1,2,q}\|v_2\|_\lambda, \tag{4.8}$$

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where $\zeta \in [0,1)$ and $\theta, \eta \in [0,2]$. Moreover, $\eta = 2$ if and only if $r = (n+1)/2$.

Proof. Due to (2.14), the Gagliardo–Nirenberg inequality (see [3]) implies

$$\|v_1\|_{3q} \le C|v_1|_{2,q}^{\theta_1}\|v_1\|_s^{1-\theta_1}, \qquad \|v_2\|_{3q/2} \le C|v_2|_{2,q}^{\theta_2}\|v_2\|_s^{1-\theta_2},$$

for $\theta_1, \theta_2 \in [0,1]$. An application of Hölder's inequality thus yields

$$\|v_1 \cdot \nabla v_2\|_q \le \|v_1\|_{3q}\|\nabla v_2\|_{3q/2} \le C|v_1|_{2,q}^{\theta_1}\|v_1\|_s^{1-\theta_1}|v_2|_{2,q}^{\theta_2}\|v_2\|_s^{1-\theta_2} \le C\lambda^{-\frac{\theta}{n+1}}\|v_1\|_\lambda\|v_2\|_\lambda,$$

which is (4.3) with $\theta = 2 - \theta_1 - \theta_2$. Since $\frac{1}{q} - \frac{2}{n} \le \frac{1}{2r} \le \frac{1}{s}$, in the same way one shows (4.4) by estimating

$$|v_1 \cdot \nabla v_2|_{-1,r} = |\mathrm{div}(v_1 \otimes v_2)|_{-1,r} \le C\|v_1\|_{2r}\|v_2\|_{2r} \le C\lambda^{-\frac{\eta}{n+1}}\|v_1\|_\lambda\|v_2\|_\lambda.$$

Note that (2.14) implies $\eta \in [0,2]$. To derive (4.5), we distinguish two different cases. On the one hand, if $q > \max\{2, n/2\}$, Hölder's inequality and the embedding theorem from [9, Theorem 4.1] yield

$$\|w_1 \cdot \nabla w_2\|_q \le \|w_1\|_{\mathrm{L}_{\mathrm{per}}^q(\mathbb{R};\mathrm{L}^\infty(\Omega))}\|\nabla w_2\|_{\mathrm{L}_{\mathrm{per}}^\infty(\mathbb{R};\mathrm{L}^q(\Omega))} \le C\|w_1\|_{1,2,q}\|w_2\|_{1,2,q}.$$

On the other hand, if $(n+2)/3 \le q < (n+1)/2$, we conclude in the same way

$$\|w_1 \cdot \nabla w_2\|_q \le \|w_1\|_{\mathrm{L}_{\mathrm{per}}^{2q}(\mathbb{R};\mathrm{L}^{\frac{nq}{n+1-2q}}(\Omega))}\|\nabla w_2\|_{\mathrm{L}_{\mathrm{per}}^{2q}(\mathbb{R};\mathrm{L}^{\frac{nq}{2q-1}}(\Omega))} \le C\|w_1\|_{1,2,q}\|w_2\|_{1,2,q}.$$

This yields (4.5). Since (2.13) and (2.14) imply $\frac{1}{r} \ge \frac{2(n+2)}{nq} - \frac{6}{n}$, and we have $\frac{1}{q} - \frac{2}{n} \le \frac{1}{2r} \le \frac{1}{q}$, for the derivation of (4.6) we can again use Hölder's inequality and [9, Theorem 4.1] to deduce

$$|\mathcal{P}(w_1 \cdot \nabla w_2)|_{-1,r} = |\mathrm{div}\,\mathcal{P}(w_1 \otimes w_2)|_{-1,r} \le C\|w_1 \otimes w_2\|_{\mathrm{L}_{\mathrm{per}}^1(\mathbb{R};\mathrm{L}^r(\Omega))}$$
$$\le C\|v\|_{\mathrm{L}_{\mathrm{per}}^2(\mathbb{R};\mathrm{L}^{2r}(\Omega))}\|w\|_{\mathrm{L}_{\mathrm{per}}^2(\mathbb{R};\mathrm{L}^{2r}(\Omega))} \le C\|v\|_{1,2,q}\|w\|_{1,2,q}.$$

The remaining estimates (4.7) and (4.8) follow in a similar fashion. $\qquad\square$

Proof of Theorem 2.7. It suffices to show existence of a solution to (4.2). Consider the solution operator

$$\mathcal{S}_\lambda \colon \left(\mathrm{L}^q(\Omega)^n \cap \mathrm{D}_0^{-1,r}(\Omega)^n\right) \oplus \mathrm{L}_{\mathrm{per},\perp}^q(\mathbb{R} \times \Omega)^n \to \mathrm{X}_\lambda^{q,r}, \quad f \mapsto u,$$

where u is the unique velocity field of the solution $(u, \mathfrak{p}) \in \mathrm{X}_\lambda^{q,r} \times \mathrm{L}_{\mathrm{per}}^q(\mathbb{R};\mathrm{D}^{1,q}(\Omega))$ to (2.7) that exists due to Theorem 2.3. This yields a family of continuous linear operators with

$$\|\mathcal{S}_\lambda f\|_{\mathrm{X}_\lambda^{q,r}} \le C\big(\|f\|_q + \lambda^{-\frac{M}{n+1}}|\mathcal{P}f|_{-1,r}\big), \tag{4.9}$$

with M as in (2.3), compare estimate (2.8). Then $(u, \mathfrak{p})$ is a solution to (4.2) if u is a fixed point of the mapping

$$\mathcal{F}\colon X_\lambda^{q,r} \to X_\lambda^{q,r}, \quad u \mapsto \mathcal{S}_\lambda(f + \mathcal{N}(u)).$$

Now consider $u \in A_\rho := \big\{ u \in X_\lambda^{q,r} \mid \|u\|_\lambda \le \rho \big\}$ for a radius $\rho > 0$ that will be chosen below, and set $v := \mathcal{P}u$, $w := \mathcal{P}_\perp u$. Then we have

$$\mathcal{P}\mathcal{N}(u) = -v \cdot \nabla v - \mathcal{P}(w \cdot \nabla w) - v \cdot \nabla V - V \cdot \nabla v - V \cdot \nabla V + \Delta V - \lambda \partial_1 V,$$
$$\mathcal{P}_\perp \mathcal{N}(u) = -v \cdot \nabla w - w \cdot \nabla v - \mathcal{P}_\perp(w \cdot \nabla w) - w \cdot \nabla V - V \cdot \nabla w,$$

and an application of estimates (4.9) and (4.1) together with Lemma 4.1 leads to

$$
\begin{aligned}
\|\mathcal{F}(u)\|_{X_\lambda^{q,r}} &\le C\big(\|f + \mathcal{N}(u)\|_q + \lambda^{-\frac{M}{n+1}} |\mathcal{P}(f + \mathcal{N}(u))|_{-1,r}\big) \\
&\le C\Big(\|f\|_q + \lambda^{-\frac{M}{n+1}} |f|_{-1,r} + (1 + \lambda^{-\frac{M}{n+1}})(\lambda + \lambda^2) \\
&\quad + \big(1 + \lambda^{-\frac{\theta}{n+1}} + \lambda^{-\frac{\zeta}{n+1}} + \lambda^{-\frac{M+\eta}{n+1}}\big)\big(\|u\|_{X_\lambda^{q,r}} + \|V\|_{X_\lambda^{q,r}}\big)^2\Big) \\
&\le C\big(\lambda^{-\frac{M}{n+1}}(\varepsilon + \lambda) + \big(\lambda^{-\frac{\theta}{n+1}} + \lambda^{-\frac{\zeta}{n+1}} + \lambda^{-\frac{M+\eta}{n+1}}\big)(\rho + \lambda)^2\big).
\end{aligned}
$$

Similarly, for $u_1, u_2 \in A_\rho$ we obtain

$$
\begin{aligned}
\|\mathcal{F}(u_1) - \mathcal{F}(u_2)\|_{X_\lambda^{q,r}} &\le C\big(\|\mathcal{N}(u_1) - \mathcal{N}(u_2)\|_q + \lambda^{-\frac{M}{n+1}} |\mathcal{P}(\mathcal{N}(u_1) - \mathcal{N}(u_2))|_{-1,r}\big) \\
&\le C\big(1 + \lambda^{-\frac{\theta}{n+1}} + \lambda^{-\frac{\zeta}{n+1}} + \lambda^{-\frac{M+\eta}{n+1}}\big)(\|u_1\|_\lambda + \|u_2\|_{X_\lambda^{q,r}} + \|V\|_{X_\lambda^{q,r}})\|u_1 - u_2\|_{X_\lambda^{q,r}} \\
&\le C\big(\lambda^{-\frac{\theta}{n+1}} + \lambda^{-\frac{\zeta}{n+1}} + \lambda^{-\frac{M+\eta}{n+1}}\big)(\rho + \lambda)\|u_1 - u_2\|_{X_\lambda^{q,r}}.
\end{aligned}
$$

Note that the assumptions imply $\max\{\theta, \zeta, M+\eta\} < n+1-M$, so that we can consider $\gamma \in \mathbb{R}$ with

$$1 \le \frac{n+1}{n+1-M} < \gamma < \frac{n+1}{\max\{\theta, \zeta, M+\eta\}}.$$

Now we choose $\lambda = \varepsilon = \rho^\gamma$ and $\rho > 0$ so small that

$$C\big(\rho^{\gamma - \frac{\gamma M}{n+1}} + \rho^{2 - \gamma \frac{\theta}{n+1}} + \rho^{2 - \gamma \frac{\zeta}{n+1}} + \rho^{2 - \gamma \frac{M+\eta}{n+1}}\big) \le \rho, \quad C\big(\rho^{1 - \gamma \frac{\theta}{n+1}} + \rho^{1 - \gamma \frac{\zeta}{n+1}} + \rho^{1 - \gamma \frac{M+\eta}{n+1}}\big) \le \frac{1}{2}.$$

This ensures that $\mathcal{F}\colon A_\rho \to A_\rho$ is a contractive self-mapping, and the contraction mapping principle finally yields the existence of a fixed point of $\mathcal{F}$. This completes the proof. $\qquad\square$

Proof of Theorem 2.5. We may proceed in a similar way as in the previous proof. Here we introduce the function space

$$Z_\lambda^{q,r} := \big\{ u \in D^{2,q}(\Omega)^n \cap D^{1,r}(\Omega)^n \cap L^s(\Omega)^n \mid \operatorname{div} u = 0,\ u\big|_{\partial\Omega} = 0 \big\},$$
$$\|u\|_{Z_\lambda^{q,r}} := \|u\|_\lambda := |u|_{2,q} + |u|_{1,r} + \lambda^{\frac{1}{n+1}} \|u\|_s, \qquad s := \frac{(n+1)r}{n+1-r},$$

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and the solution operator $\mathcal{S}_\lambda\colon \left(\mathrm{L}^q(\Omega)^n\cap\mathrm{D}_0^{-1,r}(\Omega)^n\right)\to\mathrm{Z}_\lambda^{q,r}$, $f\mapsto u$, where u is the unique velocity field of a solution $(u,\mathfrak{p})$ to (1.1) that exists due to Theorem 2.1. Then (v,p) is a solution to (2.10) if and only if $(u,\mathfrak{p}):=(v-V,p)$ is a fixed-point of the mapping

$$\mathcal{F}\colon \mathrm{Z}_\lambda^{q,r}\to\mathrm{Z}_\lambda^{q,r},\quad u\mapsto\mathcal{S}_\lambda(f+\mathcal{N}(u)),$$

where V and $\mathcal{N}(u)$ are given as before. The existence of such a fixed point can then be shown as in the proof of Theorem 2.7 by making use of estimates (4.3) and (4.4). $\qquad\square$

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Fachbereich Mathematik
Technische Universität Darmstadt
Schlossgartenstr. 7, 64289 Darmstadt, Germany
Email: `eiter@mathematik.tu-darmstadt.de`

Department of Mechanical Engineering and Materials Science
University of Pittsburgh
Pittsburgh, PA 15261, USA
Email: `galdi@pitt.edu`