

Local energy decay estimate for the hyperbolic type Stokes equations

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1 Introduction

This article is a survey of the work [6], which is a joint work with Takayuki Kobayashi, Professor of Osaka University and Takayuki Kubo, Professor of University of Tsukuba.

We consider the local energy decay properties of solutions to the initial-boundary value problem of the hyperbolic type Stokes equations:

$$\begin{cases} \tau \partial_t^2 u - \Delta u + \partial_t u + (1 + \tau \partial_t) \nabla \pi = 0 & \text{in } \Omega \times (0, \infty), \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, \infty), \\ u|_{\partial\Omega} = 0, \quad (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases} \quad (\text{HS})$$

with unknown velocity field $u = (u_1(x, t), \dots, u_n(x, t))$, unknown pressure $\pi = \pi(x, t)$ and given vector function (u_0, u_1) . Here, $\tau < 1$ is a positive constant describing the relaxation parameter and Ω is a domain of \mathbb{R}^n ($n \geq 2$) with smooth boundary. We consider the following cases:

- (i) Ω is an exterior domain of the n -dimensional Euclidean space \mathbb{R}^n , that is, there exists a number $r > 0$ such that $\Omega \setminus B_r = \mathbb{R}^n \setminus B_r$, where $B_r = \{x \in \mathbb{R}^n \mid |x| < r\}$.
- (ii) Ω is a perturbed half-space, that is, there exists a number $r > 0$ such that $\Omega \setminus B_r = \mathbb{R}_+^n \setminus B_r$, where $\mathbb{R}_+^n = \{(x', x_n) \mid x' \in \mathbb{R}^{n-1}, x_n > 0\}$.

This model arises from a time delayed version for the deformation tensor in the parabolic type Stokes equations (see [9]).

The investigation of the local energy decay properties are essential step to prove the global-in-time unique existence theorem corresponding to the nonlinear problem in an exterior domain and a perturbed half-space.

In the case of the parabolic type Stokes equations, when Ω is an exterior domain of \mathbb{R}^n ($n \geq 3$), Iwashita [5] investigated the local energy decay properties based on the resolvent expansion near the origin. Later on Iwashita's work, Dan, Kobayashi and Shibata [3] and Dan and Shibata [2] also proved the local energy decay estimate. They improved Iwashita's results and extended to 2D case. Here, the decay rate of the local energy decay estimate is $t^{-1}(\log t)^{-2}$ ($n = 2$), $t^{-n/2}$ ($n \geq 3$) as $t \rightarrow \infty$.

When Ω is a perturbed half-space, Kubo and Shibata [8] proved the local energy decay estimate based on the resolvent expansion near the origin obtained in [7]. Here, the decay rate of the local energy decay estimate is $t^{-(n+1)/2}$ ($n \geq 2$) as $t \rightarrow \infty$.

In the case of the hyperbolic type Stokes equations (HS), when Ω is an exterior domain or a perturbed half-space, as far as we know, there are no results. The idea of our proof is based on Dan and Shibata [1]. They proved in [1] the local energy decay estimate to the dissipative wave equations in 2D exterior domain by use of the resolvent expansion for the Laplace operator. Therefore, we prove the local energy decay estimate to (HS) by use of the resolvent expansion for the Stokes operator.

2 Main result

To state our results more precisely, we outline our notation. Let r_0 be a fixed constant satisfying (i) or (ii) in Section 1. We set $\Omega_r = \Omega \cap B_r$ for $r > r_0$ and

$$\begin{aligned} C_{0,\sigma}^\infty(\Omega) &= \{u \in C_0^\infty(\Omega) \mid \nabla \cdot u = 0 \text{ in } \Omega\}, \\ L_\sigma^2(\Omega) &= \text{the completion of } C_{0,\sigma}^\infty(\Omega) \text{ with respect to } \|\cdot\|_{L^2(\Omega)}, \\ H_{0,\sigma}^1(\Omega) &= \text{the completion of } C_{0,\sigma}^\infty(\Omega) \text{ with respect to } \|\cdot\|_{H^1(\Omega)}, \\ \hat{H}_{0,\sigma}^1(\Omega) &= \text{the completion of } C_{0,\sigma}^\infty(\Omega) \text{ with respect to } \|\nabla \cdot\|_{L^2(\Omega)}, \\ G(\Omega) &= \{\nabla \pi \in L^2(\Omega) \mid \pi \in L_{loc}^2(\Omega)\}. \end{aligned}$$

We set $v = \partial_t u$ and $\mathbb{U} = {}^T(u, v)$ and define a Hilbert space $\mathcal{H}(\Omega)$ by

$$\mathcal{H}(\Omega) = \left\{ \mathbb{U} = {}^T(u, v) \mid u \in \hat{H}_{0,\sigma}^1(\Omega), v \in L_\sigma^2(\Omega) \right\}$$

with inner product

$$(\mathbb{U}, \mathbb{W})_{\mathcal{H}(\Omega)} = (u, w)_D + \tau(v, z),$$

where $(u, w)_D = (\nabla u, \nabla w)$. Moreover, we set $L_r^2(\Omega) = \{f \in L^2(\Omega) \mid \text{supp } f \subset \Omega_r\}$ and

$$\mathcal{H}_r(\Omega) = \left\{ \mathbb{U} = {}^T(u, v) \in \mathcal{H}(\Omega) \mid \text{supp } u \cup \text{supp } v \subset \Omega_r \right\}.$$

We treat (HS) as the semigroup theoretical framework. To do this, we use the Helmholtz decomposition: $L^2(\Omega) = L_\sigma^2(\Omega) \oplus G(\Omega)$, where \oplus denotes the direct sum. Let P be a continuous projection from $L^2(\Omega)$ to $L_\sigma^2(\Omega)$. The Stokes operator A is defined by $A = -P\Delta$ with domain $\mathcal{D}(A) = H_{0,\sigma}^1(\Omega) \cap H^2(\Omega)$. We define an operator \mathbb{L} by

$$\mathbb{L} = \frac{1}{\tau} \begin{pmatrix} 0 & -\tau \\ A & 1 \end{pmatrix}$$

with domain

$$\mathcal{D}(\mathbb{L}) = \left\{ \mathbb{U} = {}^T(u, v) \in \mathcal{H}(\Omega) \mid u \in \mathcal{D}(A), v \in \hat{H}_{0,\sigma}^1(\Omega) \right\}.$$

Applying the projection P to (HS), the problem is written in the following form:

$$\frac{d}{dt}\mathbb{U}(t) = -\mathbb{L}\mathbb{U}(t) \quad \text{for } t > 0, \quad \mathbb{U}|_{t=0} = \mathbb{U}_0,$$

where $\mathbb{U}_0 = T(u_0, u_1)$. Then, we obtain the following two theorems.

Theorem 2.1. $-\mathbb{L}$ generates a C_0 contraction semigroup $\{T(t)\}_{t \geq 0}$ on $\mathcal{H}(\Omega)$.

Theorem 2.2. Let $n \geq 2$ be an integer and let $r > r_0$. Suppose that the initial data $u_0 \in H_{0,\sigma}^1(\Omega)$ and $u_1 \in L_\sigma^2(\Omega)$ and $\text{supp } u_0 \cup \text{supp } u_1 \subset \Omega_r$. Then, the solution u of (HS) holds the following properties:

(i) When Ω is an exterior domain, it holds that

$$\|u(t)\|_{H^1(\Omega_r)} + \sqrt{\tau} \|\partial_t u(t)\|_{L^2(\Omega_r)} \leq C_{n,r} (1+t)^{-\frac{n}{2}} (\|u_0\|_{H^1(\Omega)} + \sqrt{\tau} \|u_1\|_{L^2(\Omega)})$$

for any $t \geq 0$.

(ii) When Ω is a perturbed half-space, it holds that

$$\|u(t)\|_{H^1(\Omega_r)} + \sqrt{\tau} \|\partial_t u(t)\|_{L^2(\Omega_r)} \leq C_{n,r} (1+t)^{-\frac{n+1}{2}} (\|u_0\|_{H^1(\Omega)} + \sqrt{\tau} \|u_1\|_{L^2(\Omega)})$$

for any $t \geq 0$.

Remark 2.1. The decay rate of (ii) in Theorem 2.2 is one half better compared with the case (i) because the order of asymptotic behavior of the Stokes resolvent near the origin in Proposition 3.1 is one half better compared with the exterior domain case due to the reflection principle on the boundary in the half-space unlike the whole space.

3 Key lemma

We consider the resolvent problem:

$$\lambda u + Au = f \quad \text{in } \Omega, \tag{3.1}$$

where $\lambda \in \Sigma_{\ell,\epsilon} = \{\lambda \in \mathbf{C} \setminus \{0\} \mid |\lambda| < \ell, |\arg \lambda| < \pi - \epsilon\}$, $0 < \ell < 1$, $0 < \epsilon < \pi/2$ and A is the Stokes operator. Let $S(\lambda)f$ be defined as a solution to (3.1), $W_s^{m,2}(\Omega)$ be a weighted Sobolev space defined by

$$W_s^{m,2}(\Omega) = \left\{ f \mid (1 + |\cdot|^2)^{\frac{s}{2}} \partial_j^k f \in L^2(\Omega), k \leq m \right\}$$

for any non-negative integer m and real number s . Moreover, for $s > n/2$ and $s' < -n/2$, we set

$$\begin{aligned} \mathcal{B}_1 &= \mathcal{L} \left(L_\sigma^2(\Omega) \cap W_s^{0,2}(\Omega)^n, L_\sigma^2(\Omega) \cap W_{s'}^{2,2}(\Omega)^n \right), \\ \mathcal{B}_2 &= \mathcal{L} \left(L_\sigma^2(\Omega) \cap L_r^2(\Omega), L_\sigma^2(\Omega) \cap H^2(\Omega_r) \right). \end{aligned}$$

Then, by [2, Proposition 3.6], [5, Theorem 3.1, Corollary 3.2] and [8, Theorem 3.1], it holds that the following proposition.

Proposition 3.1. *There exist an $\ell > 0$ and an $S(\lambda) \in \text{Hol}(\Sigma_{\ell, \epsilon}, \mathcal{B}_1)$ which has the following expansion formula:*

(i) *When Ω is an exterior domain,*

$$S(\lambda) = \begin{cases} G_1 \lambda^{\frac{n}{2}-1} \log \lambda + G_2(\lambda) + G_3(\lambda) \lambda^{\frac{n}{2}-1} & \text{where } n \text{ is even,} \\ G_1 \lambda^{\frac{n}{2}-1} + G_2(\lambda) + G_3(\lambda) \lambda^{\frac{n}{2}-1} & \text{where } n \text{ is odd,} \end{cases}$$

where $G_1 \in \mathcal{B}_1$, $G_2(\lambda)$ is a polynomial of λ of $\deg G_2(\lambda) \leq [n/2] - 1$ and $G_3(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$. Particularly in case of $n = 2$, the following holds:

$$S(\lambda) = V_1 + V_2(\log \lambda)^{-1} + O((\log \lambda)^{-2}),$$

where $V_1, V_2 \in \mathcal{B}_2$.

(ii) *When Ω is a perturbed half-space,*

$$S(\lambda) = \begin{cases} H_1(\lambda) \lambda^{\frac{n-1}{2}} + H_2(\lambda) \lambda^{\frac{n}{2}} \log \lambda + H_3(\lambda) & \text{where } n \text{ is even,} \\ H_1(\lambda) \lambda^{\frac{n}{2}} + H_2(\lambda) \lambda^{\frac{n-1}{2}} \log \lambda + H_3(\lambda) & \text{where } n \text{ is odd,} \end{cases}$$

where $H_1, H_2 \in \text{Hol}(\Sigma_{\ell, \epsilon}, \mathcal{B}_2)$ and $H_3 \in \text{Hol}(\Sigma_{\ell, \epsilon} \cup \{0\}, \mathcal{B}_2)$.

In Section 4, we investigate the stability of $(\lambda \mathbb{I} + \mathbb{L})^{-1}$ near the origin. To do this, we use the class \mathcal{C}^k defined as follows and the properties of the class.

Definition 3.2 ([1]). Let X be a Banach space with norm $|\cdot|_X$. Let $N \geq 0$ be an integer and $k = N + \sigma$ with $0 < \sigma \leq 1$. Set

$$\mathcal{C}^k(\mathbf{R}, X) = \left\{ f \in C^\infty(\mathbf{R} \setminus \{0\}, X) \mid \langle\langle f \rangle\rangle_{k, X} < \infty \right\},$$

where

$$\begin{aligned} \langle\langle f \rangle\rangle_{k, X} &= \sum_{j=0}^N \int_{-\infty}^{\infty} \left| \left(\frac{d}{ds} \right)^j f(s) \right|_X ds + \sup_{h \neq 0} \frac{1}{|h|^\sigma} \int_{-\infty}^{\infty} \left| \Delta_h \left(\frac{d}{ds} \right)^N f(s) \right|_X ds \quad (0 < \sigma < 1), \\ \langle\langle f \rangle\rangle_{k, X} &= \sum_{j=0}^N \int_{-\infty}^{\infty} \left| \left(\frac{d}{ds} \right)^j f(s) \right|_X ds + \sup_{h \neq 0} \frac{1}{|h|} \int_{-\infty}^{\infty} \left| \Delta_h^2 \left(\frac{d}{ds} \right)^N f(s) \right|_X ds \quad (\sigma = 1). \end{aligned}$$

Here, we have set

$$\Delta_h f(s) = f(s+h) - f(s), \quad \Delta_h^2 f(s) = f(s+h) - 2f(s) + f(s-h).$$

Proposition 3.3 ([10]). *Let N be a positive integer and X be a Banach space with norm $|\cdot|_X$. Assume that $f \in C^\infty(\mathbf{R} \setminus \{0\}, X)$, $f(s) = 0$ if $|s| \geq 2$ and set $I = (-2, 2)$.*

(i) *Let $k = N + \sigma$ with $0 < \sigma < 1$ and f satisfy the following condition (a).*

(a) For any $s \in I \setminus \{0\}$,

$$\left| \left(\frac{d}{ds} \right)^j f(s) \right|_X \leq C_f \quad \text{for any integer } j \in [0, N-1],$$

$$\left| \left(\frac{d}{ds} \right)^N f(s) \right|_X \leq C_f |s|^{\sigma-1}, \quad \left| \left(\frac{d}{ds} \right)^{N+1} f(s) \right|_X \leq C_f |s|^{\sigma-2}.$$

Then, $f \in \mathcal{C}^k(\mathbf{R}, X)$ satisfies

$$\langle\langle f \rangle\rangle_{k,X} \leq C_{\sigma,N} C_f.$$

(ii) Let $k = N + 1$ and f satisfy the following conditions (a) and (b).

(a) There exist $f_0 \in X$ and a X -valued function $f_1(s)$ defined on I such that

$$\left(\frac{d}{ds} \right)^N f(s) = f_0 \log |s| + f_1(s) \quad \text{for } s \in I \setminus \{0\}.$$

(b) For any $s \in I \setminus \{0\}$,

$$\left| \left(\frac{d}{ds} \right)^j f(s) \right|_X \leq C_f \quad \text{for any integer } j \in [0, N-1],$$

$$|f_0|_X \leq C_f, \quad |f_1(s)|_X \leq C_f, \quad \left| \left(\frac{d}{ds} \right)^{N+1} f(s) \right|_X \leq C_f |s|^{-1},$$

$$\left| \left(\frac{d}{ds} \right)^{N+2} f(s) \right|_X \leq C_f |s|^{-2}.$$

Then, $f \in \mathcal{C}^k(\mathbf{R}, X)$ satisfies

$$\langle\langle f \rangle\rangle_{k,X} \leq C_{\sigma,N} C_f.$$

Proposition 3.4 ([1]). Let X be a Banach space with norm $|\cdot|_X$. Let $f(s) \in \mathcal{C}^2(\mathbf{R} \setminus \{0\}, X)$. If $\left| \left(\frac{d}{ds} \right)^j f(s) \right|_X \leq C_f |s|^{-j}$ for any $s \in \mathbf{R} \setminus \{0\}$ and $j = 0, 1, 2$. Then, it holds that

$$\frac{1}{|h|} \int_{-\infty}^{\infty} |\Delta_h^2 f(s)|_X ds \leq C_f.$$

Proposition 3.5 ([10]). Let X be a Banach space with norm $|\cdot|_X$. Let $N \geq 0$ be an integer and $0 < \sigma \leq 1$. Assume that $f \in \mathcal{C}^{N+\sigma}(\mathbf{R}, X)$. Set

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) e^{its} ds.$$

Then, the following estimate holds:

$$|F(t)|_X \leq C(1 + |t|)^{-(N+\sigma)} \langle\langle f \rangle\rangle_{N+\sigma,X}.$$

4 Outline of the proof

Theorem 2.1 follows from the Lumer-Phillips theorem. Hereafter, We concentrate the proof of Theorem 2.2.

Lemma 4.1. *Set*

$$b(a) = \frac{a}{2a\sqrt{\tau} + 2(3a\tau + 1)\sqrt{\tau + 1}}.$$

Then for any $a > 0$, there exists an $M_a > 0$ such that

$$\|(\lambda\mathbb{I} + \mathbb{L})^{-1}\|_{\mathcal{L}(\mathcal{H}(\Omega))} \leq M_a$$

for $\lambda \in D_{a,b(a)} = \{\lambda \in \mathbf{C} \mid |\operatorname{Re} \lambda| \leq b(a), |\operatorname{Im} \lambda| \geq a\} \cup \{\lambda \in \mathbf{C} \mid \operatorname{Re} \lambda \geq b(a)\}$.

In what follows, $\ell > 0$ denotes the same positive number in Proposition 3.1. Moreover, let φ_r be a function in $C_0^\infty(\mathbf{R}^n)$ such that $\varphi_r(x) = 1$ if $|x| \leq r$ and $\varphi_r(x) = 0$ if $|x| \geq r + 1$ and let ρ_d be a function in $C_0^\infty(\mathbf{R})$ such that $\rho_d(s) = 1$ if $|s| < d/2$ and $\rho_d(s) = 0$ if $|s| > d$.

Lemma 4.2. *Let $Q_d = \{\lambda \in \mathbf{C} \mid 0 < \operatorname{Re} \lambda < d, |\operatorname{Im} \lambda| < d\}$. Then the following assertions hold.*

(i) *There exist a $d > 0$ and an $\mathbb{R}(\lambda) \in \operatorname{Hol}(Q_d, \mathcal{L}(\mathcal{H}_r(\Omega), \mathcal{H}(\Omega_r)))$ such that*

$$\mathbb{R}(\lambda)\mathbb{X} = (\lambda\mathbb{I} + \mathbb{L})^{-1}\mathbb{X} \quad \text{for } \mathbb{X} \in \mathcal{H}_r(\Omega) \quad \text{and } \lambda \in Q_d,$$

where we have set

$$\mathcal{H}(\Omega_r) = \{^T(f, g) \mid f \in H^1(\Omega_r) \cap L_\sigma^2(\Omega), g \in L^2(\Omega_r) \cap L_\sigma^2(\Omega)\}.$$

(ii) *For any $\mathbb{X} \in \mathcal{H}_r(\Omega)$, $\mathbb{Y} \in \mathcal{H}(\Omega)$ and $\alpha < d$, there exists a positive constant $C = C_{n,r,\rho_d,\varphi_r}$ such that the following assertions hold.*

(a) *When Ω is an exterior domain, the following estimate holds.*

$$\langle\langle \rho_d(\cdot)(\varphi_r \mathbb{R}(\alpha + i \cdot)\mathbb{X}, \mathbb{Y})_{\mathcal{H}(\Omega)} \rangle\rangle_{\frac{n}{2}, \mathbf{R}} \leq C \|\mathbb{X}\|_{\mathcal{H}(\Omega)} \|\mathbb{Y}\|_{\mathcal{H}(\Omega)}.$$

(b) *When Ω is a perturbed half-space, the following estimate holds.*

$$\langle\langle \rho_d(\cdot)(\varphi_r \mathbb{R}(\alpha + i \cdot)\mathbb{X}, \mathbb{Y})_{\mathcal{H}(\Omega)} \rangle\rangle_{\frac{n+1}{2}, \mathbf{R}} \leq C \|\mathbb{X}\|_{\mathcal{H}(\Omega)} \|\mathbb{Y}\|_{\mathcal{H}(\Omega)}.$$

Proof. (i) In terms of $S(\lambda)$, we shall represent $(\lambda\mathbb{I} + \mathbb{L})^{-1}$. If we set $\mathbb{X} = ^T(f, g)$ and

$$(\lambda\mathbb{I} + \mathbb{L})\mathbb{U} = \mathbb{X} \quad \text{for } \mathbb{U} \in \mathcal{D}(\mathbb{L}),$$

we have

$$v = \lambda u - f \quad \text{and} \quad \{\lambda(\tau\lambda + 1) + A\}u = (\tau\lambda + 1)f + \tau g.$$

We take $\ell' < \ell$ so small that there exists an $\epsilon' < \pi/2$ such that $\lambda(\tau\lambda+1) \in \Sigma_{\ell, \epsilon}$ if $\lambda \in \Sigma_{\ell', \epsilon'}$. If we set

$$\mathbb{R}(\lambda) = \begin{bmatrix} (\tau\lambda+1)S(\lambda(\tau\lambda+1)) & \tau S(\lambda(\tau\lambda+1)) \\ \lambda(\tau\lambda+1)S(\lambda(\tau\lambda+1)) - 1 & \tau\lambda S(\lambda(\tau\lambda+1)) \end{bmatrix}, \quad (4.1)$$

we obtain

$$\mathbb{R}(\lambda)\mathbb{X} = (\lambda\mathbb{I} + \mathbb{L})^{-1}\mathbb{X} \quad \text{for } \mathbb{X} \in \mathcal{H}_r(\Omega) \quad \text{and } \lambda \in \Sigma_{\ell', \epsilon'},$$

because $\mathbb{R}(\lambda)\mathbb{X} \in \mathcal{D}(\mathbb{L})$ as it follows from the fact that $S(\lambda(\tau\lambda+1)) \in \mathcal{L}(L^2(\Omega), H^2(\Omega))$. Therefore, $\mathbb{R}(\lambda)$ satisfies the property of (i) with $d = 2\ell'/3$.

(ii) follows from Proposition 3.1, 3.3 and 3.4. \square

By Theorem 2.1, the following estimate holds:

$$\|T(t)\|_{\mathcal{L}(\mathcal{H}(\Omega))} \leq 1, \quad \forall t \geq 0. \quad (4.2)$$

Then, by a lemma due to Huang [4, Lemma 1], we have the following lemma.

Lemma 4.3. *For any $\alpha > 0$ and $\mathbb{X} \in \mathcal{H}(\Omega)$, set*

$$g(\omega) = \|((\alpha + i\omega)\mathbb{I} + \mathbb{L})^{-1}\mathbb{X}\|_{\mathcal{H}(\Omega)}.$$

Then, $g(\omega) \in L^2(\mathbf{R})$ and

$$\begin{aligned} \lim_{|\omega| \rightarrow \infty} g(\omega) &= 0, \\ \int_{-\infty}^{\infty} g(\omega)^2 d\omega &\leq \frac{\pi}{\alpha} \|\mathbb{X}\|_{\mathcal{H}(\Omega)}^2. \end{aligned}$$

Now, we shall give a proof of Theorem 2.2. Since the proofs of an exterior domain case and a perturbed half-space case are essentially the same, we only prove an exterior domain case. To do this, it is sufficient to prove the following proposition.

Proposition 4.4. *Let Ω be an exterior domain, φ_r be the same in Lemma 4.2. Then, it holds that*

$$\|\varphi_r T(t)\mathbb{X}\|_{\mathcal{H}(\Omega)} \leq C(1+t)^{-\frac{n}{2}} \|\mathbb{X}\|_{\mathcal{H}(\Omega)} \quad (4.3)$$

for any $t \geq 0$ and $\mathbb{X} \in \mathcal{H}_r(\Omega)$, where $C = C_{M_a, n, \varphi_r}$, and M_a denotes the constant arising from Lemma 4.1.

Proof. Since (4.2) holds, we have the following expression formula:

$$T(t)\mathbb{X} = \lim_{\omega \rightarrow \infty} \frac{1}{2\pi i} \int_{\alpha - i\omega}^{\alpha + i\omega} e^{\lambda t} (\lambda\mathbb{I} + \mathbb{L})^{-1} \mathbb{X} d\lambda, \quad \alpha > 0.$$

Hereafter, we set $\rho(s) = \rho_d(s)$. Let us take $\alpha < d$, $\mathbb{X} \in \mathcal{H}_r(\Omega)$ and $\mathbb{Y} \in \mathcal{H}(\Omega)$. Then, we see

$$(\varphi_r T(t)\mathbb{X}, \mathbb{Y})_{\mathcal{H}(\Omega)} = J_0(t) + J_\infty(t),$$

where

$$J_0(t) = \frac{1}{2\pi} e^{\alpha t} \int_{-\infty}^{\infty} e^{ist} \rho(s) (\varphi_r((\alpha + is)\mathbb{I} + \mathbb{L})^{-1} \mathbb{X}, \mathbb{Y})_{\mathcal{H}(\Omega)} ds,$$

$$J_\infty(t) = \frac{1}{2\pi} e^{\alpha t} \lim_{\omega \rightarrow \infty} \int_{-\omega}^{\omega} e^{ist} (1 - \rho(s)) (\varphi_r((\alpha + is)\mathbb{I} + \mathbb{L})^{-1} \mathbb{X}, \mathbb{Y})_{\mathcal{H}(\Omega)} ds.$$

By Lemma 4.2 and Proposition 3.5, we obtain

$$|J_0(t)| \leq C e^{\alpha t} (1 + t)^{-\frac{n}{2}} \|\mathbb{X}\|_{\mathcal{H}(\Omega)} \|\mathbb{Y}\|_{\mathcal{H}(\Omega)}. \quad (4.4)$$

We set

$$J_\infty(t) = \frac{1}{2\pi} e^{\alpha t} \lim_{\omega \rightarrow \infty} L_\omega(t),$$

where

$$L_\omega(t) = \int_{-\omega}^{\omega} e^{ist} (1 - \rho(s)) (\varphi_r((\alpha + is)\mathbb{I} + \mathbb{L})^{-1} \mathbb{X}, \mathbb{Y})_{\mathcal{H}(\Omega)} ds.$$

By the relation $(it)^{-1} de^{ist}/ds = e^{ist}$ and integration by parts, we have

$$L_\omega(t) = \sum_{k=1}^{\ell} \frac{(-1)^{k-1}}{(it)^k} L_\omega^k(t) + \frac{(-1)^\ell}{(it)^\ell} M_\omega^\ell(t),$$

where

$$L_\omega^k(t) = \left[e^{ist} \frac{d^{k-1}}{ds^{k-1}} \left\{ (1 - \rho(s)) (\varphi_r((\alpha + is)\mathbb{I} + \mathbb{L})^{-1} \mathbb{X}, \mathbb{Y})_{\mathcal{H}(\Omega)} \right\} \right]_{s=-\omega}^{s=\omega},$$

$$M_\omega^\ell(t) = \int_{-\omega}^{\omega} e^{ist} \frac{d^\ell}{ds^\ell} \left\{ (1 - \rho(s)) (\varphi_r((\alpha + is)\mathbb{I} + \mathbb{L})^{-1} \mathbb{X}, \mathbb{Y})_{\mathcal{H}(\Omega)} \right\} ds.$$

Since we have by Lemma 4.1

$$\left\| \frac{d^j}{ds^j} ((\alpha + is)\mathbb{I} + \mathbb{L})^{-1} \right\|_{\mathcal{L}(\mathcal{H}(\Omega))} \leq j! M_a^j \left\| ((\alpha + is)\mathbb{I} + \mathbb{L})^{-1} \right\|_{\mathcal{L}(\mathcal{H}(\Omega))} \quad \text{for } |s| \geq a,$$

it follows from Lemma 4.3 that

$$|L_\omega^k(t)| \rightarrow 0, \quad \omega \rightarrow \infty. \quad (4.5)$$

Using the Leibniz rule and the adjoint operator \mathbb{L}^* of \mathbb{L} , we obtain

$$\begin{aligned} |M_\omega^\ell(t)| &\leq \ell! \int_{\frac{d}{2} \leq |s| \leq \omega} (1 - \rho(s)) \left| \left(((\alpha + is)\mathbb{I} + \mathbb{L})^{-\ell} \mathbb{X}, ((\alpha - is)\mathbb{I} + \mathbb{L}^*)^{-1} \varphi_r \mathbb{Y} \right)_{\mathcal{H}(\Omega)} \right| ds \\ &\quad + \sum_{k=0}^{\ell-1} \binom{\ell}{k} k! \int_{\frac{d}{2} \leq |s| \leq d} \left| \frac{d^{\ell-k}}{ds^{\ell-k}} \rho(s) \right| \left| \left(\varphi_r((\alpha + is)\mathbb{I} + \mathbb{L})^{-k-1} \mathbb{X}, \mathbb{Y} \right)_{\mathcal{H}(\Omega)} \right| ds \\ &= K_1 + K_2. \end{aligned}$$

If we take $a < d/2$, by Lemma 4.1 and Lemma 4.3, we have

$$\begin{aligned} K_1 &\leq C l! M_a^{l-1} \left(\int_{\frac{d}{2} \leq |s|} \|((\alpha + is)\mathbb{I} + \mathbb{L})^{-1} \mathbb{X}\|_{\mathcal{H}(\Omega)}^2 ds \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{\frac{d}{2} \leq |s|} \|((\alpha - is)\mathbb{I} + \mathbb{L}^*)^{-1} \varphi_r \mathbb{Y}\|_{\mathcal{H}(\Omega)}^2 ds \right)^{\frac{1}{2}} \\ &\leq C_{l, M_a} \|\mathbb{X}\|_{\mathcal{H}(\Omega)} \|\mathbb{Y}\|_{\mathcal{H}(\Omega)}. \end{aligned} \quad (4.6)$$

Moreover, by Lemma 4.1, we see

$$K_2 \leq C_{l, M_a} \|\mathbb{X}\|_{\mathcal{H}(\Omega)} \|\mathbb{Y}\|_{\mathcal{H}(\Omega)}. \quad (4.7)$$

Combining (4.5), (4.6) and (4.7), we obtain

$$|J_\infty(t)| \leq \frac{e^{\alpha t}}{2\pi} t^{-l} C_{l, M_a} \|\mathbb{X}\|_{\mathcal{H}(\Omega)} \|\mathbb{Y}\|_{\mathcal{H}(\Omega)} \quad (4.8)$$

for any $l \geq 1$. Letting $\alpha \rightarrow 0$ in (4.4) and (4.8), we obtain (4.3) for any $\mathbb{X} \in \mathcal{H}_r(\Omega)$. \square

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