On the blow-up criteria for the strong solutions to inhomogeneous fluids of Korteweg type

Huanyuan Li

Graduate School of Mathematical Sciences, The University of Tokyo

1 Introduction

In this paper, the motion of the viscous fluids with capillary effect are discussed, which is governed by the inhomogeneous incompressible Navier-Stokes-Korteweg equations.

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu(\rho)\mathcal{D}(u)) + \nabla\Pi + \mathrm{div}(\kappa(\rho)\nabla\rho \otimes \nabla\rho) = 0, \\ \mathrm{div} u = 0. \end{cases} \tag{1}$$

Assume the fluid is occupied by a bounded smooth domain $\Omega \subset \mathbb{R}^d, d = 2, 3$. $x \in \Omega$ is the spatial coordinate, $t \geq 0$ is the time. And $\rho = \rho(x, t), u = (u^1, \cdots, u^d)(x, t)$ and $\Pi = \Pi(x, t)$ denote the density, velocity field and pressure of the fluid, respectively.

$$\mathcal{D}(u) = \frac{1}{2}\Big[\nabla u + (\nabla u)^T\Big]$$

represents the viscous deformation tensor, where ∇u is the gradient matrix $(\partial u_i/\partial x_j)$ and $(\nabla u)^T$ is its transpose. $\kappa = \kappa(\rho)$ and $\mu = \mu(\rho)$ stand for the capillary and viscosity coefficients of the fluid respectively, and are both functions of density ρ. which are assumed to satisfy

$$\kappa, \mu \in C^1[0, \infty), \quad \text{and} \quad \kappa \geq 0, \mu \geq \underline{\mu} > 0 \quad \text{on } [0, \infty) \tag{2}$$

for some positive constant $\underline{\mu}$.

The Navier-Stokes-Korteweg system (1)-(2) is complemented with the following initial and boundary conditions:

$$(\rho, u)(x, t = 0) = (\rho_0, u_0) \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega \times [0, T). \tag{3}$$

The study on the compressible Navier-Stokes-Korteweg is very popular. However, there is little literatures on the inhomogeneous case. To illustrate the reasonability of our model

(1), we give more interpretation on the Koeteweg type fluids. In general, the capillary tensor is written as

$$\mathrm{div}K = \nabla\left(\rho\kappa(\rho)\triangle\rho + \frac{1}{2}\left(\kappa(\rho)+\rho\kappa'(\rho)\right)|\nabla\rho|^2\right) - \mathrm{div}(\kappa(\rho)\nabla\rho\otimes\nabla\rho). \tag{4}$$

In the case of the inhomogeneous incompressible Korteweg system, the first term in the capillary tensor (4) can be absorbed by the pressure term due to the incompressibility condition, thus we directly write the capillary term as a general divergence term (see the Remark 1.1 in [1], it is exactly the equations (1). See more physical background and mathematical modelling in [15, 18].

If we take $\kappa\equiv 0$, the system (1) reduces the inhomogeneous incompressible Navier-Stokes equations with density-dependent viscosity, which has been studied extensively since the last century by Russian mathematicians. Here we just mention some important works when the vacuum is admitted. In the begining of 21st century, Korean researchers Cho and Kim [3] established the local well-posedness of strong solutions for all initial data satisfying a natural compatibility condition. And later the local strong solutions have been extended to a global one when the initial gradient of the velocity is suitably small in L^2 space by Huang and Wang [13]. For the related results, the authors can see the references [11]-[13] and therein.

Let us come back to the fluids of Korteweg type, that is, $\kappa(\rho)$ is not a zero function. Because of physical importance and mathematical complexity, the Navier-Stokes-Korteweg equations have been studied widely by many people, especially a great of efforts have been devoted to compressible capillary fluids, see the references [5, 9, 10] and therein. While there are few literatures on the system (1). When the vacumm is admitted, the first local existence of unique strong solution was obtained by Tan and Wang [19] when the capillary coefficients κ is a positive constant. And very recently, their result was extended to the case when $\kappa(\rho)$ is a C^1 function of the density by Wang [20].

In this paper we aim to bulid two blow-up criteria for the strong solutions to the initial and boundary value problem (1)-(3) in dimension three ($d=3$) and dimension two ($d=2$), respectively.

In the case when the initial density may vanish in an open subset of Ω, that is, the initial vacuum is allowed, the local well-posedness of strong solutions to (1)-(3) was obtained by Wang [20] in a three dimensional bounded smooth domain. Because we also discuss the 2D case, the local well-posedness of strong solutions in a two-dimensional bounded domain is necessary. In fact, this can be achived in the same manner as Wang [20] and Cho and Kim [3], also see the remark 2 in Tan and Wang [19]. Now as a preparation work, we introduce the follwoing local well-posedness of strong solutions over a bounded smooth domain $\Omega\subset\mathbb{R}^d, d=2,3$ (two and three dimensional versions).

Theorem 1. *Assume that the initial data* (ρ_0, u_0) *satisfies the regularity condition*

$$0 \le \rho_0 \in W^{2,q}, \quad d < q < \frac{2d}{d-2}, \quad u_0 \in H^1_{0,\sigma} \cap H^2, \tag{5}$$

and the compatibility condition

$$-\mathrm{div}(\mu(\rho_0)(\nabla u_0 + (\nabla u_0)^T)) + \nabla \Pi_0 + \mathrm{div}(\kappa(\rho_0)\nabla\rho_0 \otimes \nabla\rho_0) = \rho_0^{1/2} g, \tag{6}$$

for some $(\Pi_0, g) \in H^1(\Omega) \times L^2(\Omega)$. *Then there exist a small time* T *and a unique strong solution* (ρ, u, P) *to the initial boundary value problem* (1)-(3) *satisfying*

$$\begin{cases} \rho \in C([0,T]; W^{2,q_0}), \quad u \in C([0,T]; H_0^1 \cap H^2) \cap L^2(0,T; W^{2,q_0}), \\ \rho_t \in C([0,T]; W^{1,q_0}), \quad \nabla P \in C([0,T]; L^2) \cap L^2(0,T; L^{q_0}), \quad u_t \in L^2(0,T; H_0^1), \end{cases} \tag{7}$$

At the above local well-posedness of strong solutions in Theorem 1 at hand, we will bulid the blow-up criteria of possible breakdown of local strong solutions to the initial and boundary value problem in 3D and 2D, respectively. First, we state the blow-up criterion of 3D case.

Theorem 2. *Suppose the dimension* $d = 3$, *and all the assumptions in Theorem 1 are satisfied. Let* (ρ, u, P) *be a strong solution of the problem* (1)-(3) *when* $d = 3$. *If* $0 < T^* < \infty$ *is the maximal time of existence, then*

$$\lim_{T \to T*} (\|\nabla\rho\|_{L^\infty(0,T;W^{1,q})} + \|u\|_{L^s(0,T;L^r_\omega)}) = \infty. \tag{8}$$

for any r *and* s *satisfying*

$$\frac{2}{s} + \frac{3}{r} \le 1, \quad 3 < r \le \infty, \tag{9}$$

where L^r_ω *denotes the weak* L^r *space.*

Remark 1. *When* $\rho \equiv 1$, *the Navier-Stokes-Korteweg equations reduce to the classical incompressible Navier-Stokes equations, therefore our blow-up criterion indicates the generalization of Serrin's criterion using weak Lesbegue spaces for incompressible Navier-Stokes equations, see the work of H. Sohr (2001), S. Bosia et. al. (2014).*

Next, we bulid a better blow-up criterion than (8) in the 2D case. We remark from the basic energy estimate that

$$\sup_{0<T<T^*} (\|\sqrt{\rho} u\|_{L^\infty(0,T;L^2)} + \|\nabla u\|_{L^2(0,T;L^2)}) \le C.$$

When there is no vacuum, that is, $\inf \rho_0 > \exists \underline{\rho} > 0$, we can deduce

$$\int_0^T \|u\|_{L^4}^4 dt \le C \int_0^T \|u\|_{L^2}^2 \cdot \|\nabla u\|_{L^2}^2 dt \le C \sup \|u\|_{L^2}^2 \cdot \int_0^T \|\nabla u\|_{L^2}^2 dt,$$

which implies that $u \in L^4(0,T;L^4)$. Then the Serrin norm in 2D is finite when $r = s = 4$ since of $2/r + 2/s \le 1$. Thus, we can remove the second Serrin's type term when we deduce the blow-up criterion for 2D flow without vacuum. A natural question is that if we can obtain a blow-up criterion just in terms of the density when the vacuum is allowed.

We can overcome the difficulty of vacuum in view of a recent work of Huang and Wang [12], in which a new type blow-up criterion for the 2D inhomogeneous incompressible Navier-Stokes flow only involving the density is built. More precisely, our second result can be stated as follows.

Theorem 3. *Suppose the dimension $d = 2$, and all the assumptions in Theorem 1 are satisfied. Let (ρ, u, P) be a strong solution of the problem (1)-(3) when $d = 2$. If $0 < T^* < \infty$ is the maximal time of existence, then*

$$\lim_{T \to T*} \|\nabla \rho\|_{L^\infty(0,T;W^{1,q})} = \infty. \tag{10}$$

Remark 2. *As said in the Remark 1, when $\rho \equiv 1$, the Navier-Stokes-Korteweg equations reduce to the classical incompressible Navier-Stokes equations, therefore our blow-up criterion indicates the global well-posedness for Navier-Stokes equations in 2D due to Leray in 1934.*

The proofs of Theorem 2 and Theorem 3 are based on the contradiction argument. In view of the local existence result, to prove Theorem 2 and Theorem 3, it suffices to verify that (ρ, u) satisfy (5) and (6) at the time T^* under the assumption of the left hand sides of (8) and (10) are finite, respectively.

The remainder of this paper is arranged as follows. In Sec. 2, we give some auxiliary lemmas which is useful in our later analysis. The outline of proofs of Theorem 2 and Theorem 3 will be done by combining the contradiction argument with the estimates derived in Sec. 3 and Sec. 4 respectively.

2 Preliminaries

Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^d, d = 2, 3$. For notations simplicity below, we omit the integration domain Ω. And the Lesbegue and Sobolev spaces are defined in a standard way.

Denote the Lorentz space and its norm by $L^{p,q}$ and $\|\cdot\|_{L^{p,q}}$, respectively, where $1 < p < \infty$ and $1 \le q \le \infty$. And we recall the weak-L^p space L^p_ω which is defined as follows:

$$L^p_\omega := \{f \in L^1_{loc} : \|f\|_{L^p_\omega} = \sup_{\lambda > 0} \lambda |\{|f(x)| > \lambda\}|^{\frac{1}{p}} < \infty\}.$$

And it should be noted that

$$L^p \subsetneqq L^p_\omega, \quad L^\infty_\omega = L^\infty, \quad L^p_\omega = L^{p,\infty}.$$

For the details of Lorentz space, we refer to the first chapter in Grafakos [8]. The following lemma involving the weak Lesbegue spaces has been proved in Kim [14], Xu and Zhang [21], which will play an important role in the subsequent analysis.

Lemma 1. *Assume $g \in H^1$, and $f \in L^r_\omega$ with $r \in (3, \infty]$, then $f \cdot g \in L^2$. Furthermore, for any $\epsilon > 0$, we have*

$$\|f \cdot g\|^2_{L^2} \le \epsilon \|g\|^2_{H^1} + C(\epsilon)(\|f\|^s_{L^r_\omega} + 1)\|g\|^2_{L^2}, \tag{11}$$

where C is a positive constant depending only on ϵ, r and the domain Ω.

However, to deal with a inhomogeneous problem with vacuum in 2D, we should make good use of degenerate weight like $\sqrt{\rho}$ with velocity .We look for a similar estimate for $\sqrt{\rho}u$ as in Gagliardo-Nirenberg inequality. Desjardin [4] provides a useful estimate for us, which is stated as follows.

Lemma 2. *Assume that $0 \le \rho \le \bar{\rho}, u \in H^1_0$; then*

$$\|\sqrt{\rho}u\|^2_{L^4} \le C(1 + \|\rho u\|_{L^2})\|\nabla u\|_{L^2}\sqrt{\log(2 + \|\nabla u\|^2_{L^2})} \tag{12}$$

where C is a positive constant depending only on $\bar{\rho}$ and the domain Ω.

2.1 Regularity estimates for the stationary Stokes equations

High-order a priori estimates of velocity field u rely on the following regularity results for the stationary density-dependent Stokes equations.

Lemma 3. *Suppose the dimension $d = 2, 3$, and assume that $\rho \in W^{2,q}, d < q < \infty$, and $0 \le \rho \le \bar{\rho}$. Let $(u, \Pi) \in H^1_{0,\sigma} \times L^2$ be the unique weak solution to the boundary value problem*

$$-\mathrm{div}(2\mu(\rho)\mathcal{D}(u)) + \nabla\Pi = F, \quad \mathrm{div} u = 0 \text{ in } \Omega, \quad \text{and} \int \Pi dx = 0, \tag{13}$$

where $\mathcal{D}(u) = \frac{1}{2}[\nabla u + (\nabla u)^T]$ and

$$\mu \in C^1[0, \infty), \quad \underline{\mu} \le \mu(\rho) \le \bar{\mu} \text{ on } [0, \bar{\rho}].$$

Then we have the following regularity results:

(1) If $F \in L^2$, then $(u, \Pi) \in H^2 \times H^1$ and

$$\|u\|_{H^2} + \|\Pi\|_{H^1} \le C(1 + \|\nabla\rho\|_{L^\infty})\|F\|_{L^2}, \tag{14}$$

(2) If $F \in L^r$ for some $r \in (2, \infty)$, then $(u, \Pi) \in W^{2,r} \times W^{1,r}$ and

$$\|u\|_{W^{2,r}} + \|\Pi\|_{W^{1,r}} \leq C(1 + \|\nabla\rho\|_{L^\infty})\|F\|_{L^r}. \tag{15}$$

The proof of Lemma 3 has been given by Wang [20]. And refer to Lemma 2.1 in his paper.

3 Outline of the proof of Theorem 2

Suppose the dimension $d = 3$, and let (ρ, u, Π) be a strong solution to the initial and boundary value problem (1)-(3) as derived in Theorem 1 when $d = 3$. In this section, we give a sketch of the proof of the blow-up criterion in 3D case.

As mentioned in the Section 1, the theorem 2 will be proved by using a contradiction argument. Denote $0 < T^* < \infty$ the maximal existence time for the strong solution to the initial and boundary value problem (1)-(3) when $d = 3$. Suppose that (8) were false, that is

$$M_0 := \lim_{T \to T^*} (\|\nabla\rho\|_{L^\infty(0,T;W^{1,q})} + \|u\|_{L^s(0,T;L^r_\omega)}) < \infty. \tag{16}$$

Under the condition (16), one will extend the existence time of the strong solutions to (1)-(3) beyond T^*, which contradicts the definition of maximum of T^*. To extend the solutions, we will show that there exists a constant $C > 0$, depending only on M_0, ρ_0, u_0 and T^* such that

$$\begin{aligned} &\sup_{0 \leq t \leq T^*} \{(\|\rho\|_{W^{2,q}} + \|\rho_t\|_{W^{1,q}}) + (\|\nabla u\|_{H^1} + \|\sqrt{\rho}u_t\|_{L^2})\} \\ &+ \int_0^{T^*} (\|\nabla u_t\|_{L^2}^2 + \|\nabla u\|_{W^{1,q}}^2) dt \leq C. \end{aligned}$$

The $W^{1,q}$-estimate of the density ρ has been shown when we assume the left hand side of (16) is finite. Then the first key step is to derive the L^2-norm of the first order spatial derivatives of u under the assumptions of initial data and (16). Here we define the material derivative $\dot{u} := u_t + u \cdot \nabla u$.

Step 1: Estimate of $\sup \|\nabla u\|_{L^2}$

Under the condition (16), it holds that for any $0 < T < T^*$,

$$\sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 + \int_0^T \|\sqrt{\rho}\dot{u}\|_{L^2}^2 dt \leq C. \tag{17}$$

Proof. Multiplying the momentum equations $(1)_2$ by u_t, and integrating the resulting equations over Ω, we have

$$\begin{aligned}
&\int \rho|\dot{u}|^2dx + \frac{d}{dt}\int \mu(\rho)|\mathcal{D}(u)|^2dx \\
=&-\int \rho\dot{u}\cdot(u\cdot\nabla u)dx - \int \mu'(\rho)u\cdot\nabla\rho|\mathcal{D}(u)|^2dx + \int \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u_t dx \\
=&\frac{d}{dt}\int \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx + \int \kappa'(\rho)(u\cdot\nabla\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx \\
&+2\int \kappa(\rho)\nabla(u\cdot\nabla\rho)\otimes\nabla\rho : \nabla u dx - \int \rho\dot{u}\cdot(u\cdot\nabla u)dx - \int \mu'(\rho)u\cdot\nabla\rho|\mathcal{D}(u)|^2dx \\
=&\frac{d}{dt}\int \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx + \sum_{k=1}^{4} I_k.
\end{aligned} \tag{18}$$

After a direct estimate under the assumption (16), the terms $I_1 - I_4$ can de dominated as

$$\sum_{k=1}^{4} I_k \le \epsilon\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + C\|u\cdot\nabla u\|_{L^2}^2 + C(1+\|\nabla u\|_{L^2}^2). \tag{19}$$

To close the estimate, the Lemma 3 will play an important role in estimating $\|\nabla u\|_{H^1}$, take $F = -\rho\dot{u} - \mathrm{div}(\kappa(\rho)\nabla\rho\otimes\nabla\rho)$, we derive that

$$\begin{aligned}
\|\nabla u\|_{H^1} + \|P\|_{H^1} &\le C(1+\|\nabla\rho\|_{L^\infty})\|F\|_{L^2} \\
&\le C(1+\|\nabla\rho\|_{L^\infty})\|\rho\dot{u} + \mathrm{div}(\kappa(\rho)\nabla\rho\otimes\nabla\rho)\|_{L^2} \\
&\le C_*\|\sqrt{\rho}\dot{u}\|_{L^2} + C\|\nabla\rho\|_{L^6}^3 + C\|\nabla\rho\|_{L^\infty}\|\nabla^2\rho\|_{L^2} \\
&\le C_*\|\sqrt{\rho}\dot{u}\|_{L^2} + C,
\end{aligned} \tag{20}$$

where C_* is a positive number.

Now we substitute (33)-(20) into (18), deduces

$$\begin{aligned}
&\int \rho|\dot{u}|^2dx + \frac{d}{dt}\int \mu(\rho)|\mathcal{D}(u)|^2dx \\
\le&\frac{d}{dt}\int \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx + \epsilon\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + C(1+\|\nabla u\|_{L^2}^2) \\
&+ C(\epsilon)\|u\cdot\nabla u\|_{L^2}^2 \\
\le&\frac{d}{dt}\int \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx + \epsilon\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + C(1+\|\nabla u\|_{L^2}^2) \\
&+ \delta\|\nabla u\|_{H^1}^2 + C(\epsilon,\delta)(\|u\|_{L^r_\omega}^s + 1)\|\nabla u\|_{L^2}^2 \\
\le&\frac{d}{dt}\int \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx + \epsilon\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + C(1+\|\nabla u\|_{L^2}^2) \\
&+ C_*\delta\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + C(\epsilon,\delta)(\|u\|_{L^r_\omega}^s + 1)\|\nabla u\|_{L^2}^2,
\end{aligned} \tag{21}$$

where we used Lemma 1 in the second inequality, and (20) was used to get the third one. Then choosing ϵ, δ small enough, we get

$$\begin{aligned} &\int \rho|\dot{u}|^2dx + \frac{d}{dt}\int \mu(\rho)|\mathcal{D}(u)|^2dx \\ \le &\frac{d}{dt}\int \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx + C(1+\|\nabla u\|_{L^2}^2)(\|u\|_{L^r_\omega}^s + 1). \end{aligned} \tag{22}$$

By the assumption (16) and Cauchy-Schwarz inequality, it is easily seen that

$$C\int |\kappa(\rho)||\nabla\rho\otimes\nabla\rho : \nabla u|dx \le \frac{\mu}{4}\|\nabla u\|_{L^2}^2 + C. \tag{23}$$

Taking this into account, we can conclude from (35) and the Gronwall inequality that (17) holds for all $0 \le T < T^*$. Therefore the first step is completed. □

Step 2: estimate of $\sup \|\sqrt{\rho}u_t\|_{L^2}$

By a standard procedure, differentiating the momentum equations with respect to time t, multiplying the resulting equations by u_t, we can obtain the necesssary estimate by use of the compatibility condition and key assume (16).

Under the condition (16)*, it holds that for any* $0 < T < T^*$*,*

$$\sup_{0\le t\le T}\|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^T \|\nabla u_t\|_{L^2}^2 dt \le C. \tag{24}$$

Step 3: Higher estimate of the density

At the results step 1 and step 2 at hand, now we can derive the necessary higher $D^{2,q}$-estimate on the density.

Under the condition (16)*, it holds that for any* $0 < T < T^*$*,*

$$\sup_{0\le t\le T}(\|\rho_t\|_{W^{1,q}} + \|u\|_{H^2} + \|P\|_{H^1}) + \int_0^T (\|u\|_{W^{2,q}}^2 + \|P\|_{W^{1,q}}^2)dt \le C. \tag{25}$$

Proof. Applying (15) in Lemma 3 with $F = -\rho u_t - \rho u\cdot\nabla u - \mathrm{div}(\kappa(\rho)\nabla\rho\otimes\nabla\rho)$, we get

$$\begin{aligned} \|\nabla u\|_{W^{1,q}} + \|P\|_{W^{1,q}} &\le C(\|\rho u_t\|_{L^q} + \|\rho u\cdot\nabla u\|_{L^q} + \|\kappa(\rho)|\nabla^2\rho||\nabla\rho|\|_{L^q} + \|\kappa'(\rho)|\nabla\rho|^3\|_{L^q}) \\ &\le C(\|\rho u_t\|_{L^q} + \|\rho u\cdot\nabla u\|_{L^q} + 1) \\ &\le C(\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\|\sqrt{\rho}u_t\|_{L^6}^{\frac{3q-6}{2q}} + \|\nabla u\|_{L^2}^{\frac{6(q-1)}{5q-6}}\|\nabla u\|_{W^{1,q}}^{\frac{4q-6}{5q-6}} + 1). \end{aligned} \tag{26}$$

Applying Young's inequality and Sobolev embedding inequality,

$$\begin{aligned} \|\nabla u\|_{W^{1,q}}^2 + \|P\|_{W^{1,q}}^2 &\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{q}}\|\nabla u_t\|_{L^2}^{\frac{3(q-2)}{q}} + C\|\nabla u\|_{L^2}^{\frac{12(q-1)}{q}} + C \\ &\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{q}}\|\nabla u_t\|_{L^2}^{\frac{3(q-2)}{q}} + C. \end{aligned} \tag{27}$$

Hence

$$\begin{aligned}\int_0^T (\|\nabla u\|_{W^{1,q}}^2 + \|P\|_{W^{1,q}}^2) dt &\le C\int_0^T \|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{q}} \|\nabla u_t\|_{L^2}^{\frac{3(q-2)}{q}} dt + C \\ &\le C(\sup_{0\le t\le T} \|\sqrt{\rho}u_t\|_{L^2}^2)^{\frac{6-q}{2q}} \int_0^T \|\nabla u_t\|_{L^2}^2 dt + C \\ &\le C.\end{aligned} \tag{28}$$

Therefore we complete the proof of step 3. □

Proof of Theorem 2. In fact, in view of (16) and (25), it is easy to see that the functions $(\rho, u)(x, t = T^*) = lim_{t\to T^*}(\rho, u)$ have the same regularities imposed on the initial data (5) at the time $t = T^*$. Furthermore,

$$\begin{aligned}&- \operatorname{div}(2\mu(\rho)\mathcal{D}(u)) + \nabla P + \operatorname{div}(\kappa(\rho)\nabla\rho \otimes \nabla\rho)|_{t=T^*} \\ &= \lim_{t\to T^*} \rho^{\frac12}(\rho^{\frac12}u_t + \rho^{\frac12}u\cdot\nabla u) := \rho^{\frac12} g|_{t=T^*}\end{aligned}$$

with $g = (\rho^{\frac12}u_t + \rho^{\frac12}u\cdot\nabla u)|_{t=T^*} \in L^2$ due to (37). Thus the functions $(\rho, u)|_{t=T^*}$ satisfy the compatibility condition (6) at time T^*. Therefore we can take $(\rho, u)|_{t=T^*}$ as the initial data and apply the local existence theorem (Theorem 1) to extend the local strong solution beyond T^*. This contradicts the definition of maximal existence time T^*, and thus, the proof of Theorem 2 is completed. □

4 Outline of the proof of Theorem 3

Suppose the dimension $d = 2$, let (ρ, u, Π) be a strong solution to the initial and boundary value problem (1)-(3) as derived in Theorem 1 when $d = 2$. Similar to the proof of blow-up criterion in the dimension three in Sec. 3, Theorem 3 will be also proved by using a contradiction argument. Denote $0 < T^* < \infty$ the maximal existence time for the strong solution to the initial and boundary value problem (1)-(3). Suppose that (10) were false, that is

$$M_0 := \lim_{T\to T^*} \|\nabla\rho\|_{L^\infty(0,T;W^{1,q})} < \infty. \tag{29}$$

Under the condition (29), one will extend the existence time of the strong solutions to (1)-(3) beyond T^*, which contradicts the definition of maximum of T^*.

To continue the main steps, we give the following estimate which is used later.

Step 0: H^1-estimate of the velocity

Under the assumption (29)*, it holds for all* $0 < T < T^*$*,*

$$\|\nabla u\|_{H^1} \le C\|\rho u_t\|_{L^2} + C\|\rho u\|_{L^4}^2 \|\nabla u\|_{L^2} + C, \tag{30}$$

and consequently by Sobolev embedding,

$$\|\nabla u\|_{H^1} \le C\|\rho u_t\|_{L^2} + C\|\nabla u\|_{L^2}^3 + C. \tag{31}$$

Proof. According to the Lemma 3 and the Gagliardo-Nirenberg inequality,

$$\begin{aligned}\|\nabla u\|_{H^1} &\le C(1+\|\nabla\rho\|_{L^\infty})(\|\rho u_t\|_{L^2} + \|\rho u\cdot\nabla u\|_{L^2} + 1)\\ &\le C\|\rho u_t\|_{L^2} + C\|\rho u\|_{L^4}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}} + C\\ &\le C\|\rho u_t\|_{L^2} + C\|\rho u\|_{L^4}^2\|\nabla u\|_{L^2} + C + \frac{1}{2}\|\nabla u\|_{H^1},\end{aligned}$$

which complete the proof of (30). □

Step 1: key estimate of $\sup\|\nabla u\|_{L^2}$

Under the condition (29)*, it holds that for any* $0 < T < T^*$,

$$\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2 + \int_0^T \|\sqrt{\rho}u_t\|_{L^2}^2 dt \le C. \tag{32}$$

Proof. Multiplying the momentum equations $(1)_2$ by u_t, and integrating the resulting equations over Ω, we have

$$\begin{aligned}&\int \rho|u_t|^2dx + \frac{d}{dt}\int \mu(\rho)|\mathcal{D}(u)|^2dx\\ =&-\int(\rho u\cdot\nabla u)\cdot u_t dx - \int u\cdot\nabla\mu(\rho)|\mathcal{D}(u)|^2dx + \int\kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u_t dx\\ =&\frac{d}{dt}\int\kappa(\rho)\nabla\rho\otimes\nabla\rho:\nabla u dx + \int\kappa'(\rho)(u\cdot\nabla\rho)\nabla\rho\otimes\nabla\rho:\nabla u dx\\ &+2\int\kappa(\rho)\nabla(u\cdot\nabla\rho)\otimes\nabla\rho:\nabla u dx - \int(\rho u\cdot\nabla u)\cdot u_t dx - \int u\cdot\nabla\mu(\rho)|\mathcal{D}(u)|^2dx\\ =&\frac{d}{dt}\int\kappa(\rho)\nabla\rho\otimes\nabla\rho:\nabla u dx + \sum_{k=1}^4 Q_k.\end{aligned} \tag{33}$$

The terms I_1 to I_4 can be estimated in a similar way, that is,

$$\sum_{k=1}^4 Q_k \le \frac{1}{2}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\rho u\|_{L^4}^4\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 + C\|\nabla u\|_{L^2}^2. \tag{34}$$

In view of Lemma 2, we get

$$\|\sqrt{\rho}u\|_{L^4}^4 \le C\|\nabla u\|_{L^2}^2\cdot\log(2+\|\nabla u\|_{L^2}^2). \tag{35}$$

Therefore

$$\begin{aligned}&\frac{1}{2}\int \rho|u_t|^2dx+\frac{d}{dt}\int(\mu(\rho)|\mathcal{D}(u)|^2-\kappa(\rho)\nabla\rho\otimes\nabla\rho:\nabla u)dx\\&\le C\|\nabla u\|_{L^2}^2(1+\|\nabla u\|_{L^2}^2)(1+\log(2+\|\nabla u\|_{L^2}^2))\end{aligned}\tag{36}$$

and we know that

$$\frac{3}{4}\underline{\mu}\|\nabla u\|_{L^2}^2-C_0\le\int(\mu(\rho)|\mathcal{D}(u)|^2-\kappa(\rho)\nabla\rho\otimes\nabla\rho:\nabla u)dx\le\frac{5}{4}\underline{\mu}\|\nabla u\|_{L^2}^2+C_0,\tag{37}$$

owing to the following estimate

$$\int\kappa(\rho)\nabla\rho\otimes\nabla\rho:\nabla u dx\le\frac{1}{4}\underline{\mu}\|\nabla u\|_{L^2}^2+C_0.$$

We can conclude from (36) and the logarithmic type Gronwall inequality that (32) holds for all $0\le T<T^*$. □

Before we prove the boundedness of $\|\sqrt{\rho}u_t\|_{L^2}$, we insert the following estimate on the L^∞-norm of u.

Under the condition (29)*, it holds that for any* $0<T<T^*$,

$$\sup_{0\le t\le T}(\|u\|_{L^2(0,T;L^\infty)}+\|u\|_{L^4(0,T;L^\infty)})\le C.\tag{38}$$

Proof. By the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\int_0^T\|u\|_{L^\infty}^4dt&\le C\int_0^T\|u\|_{L^2}^2\|\nabla u\|_{H^1}^2dt\\&\le C\int_0^T(\|\nabla u\|_{L^2}^2\|\rho u_t\|_{L^2}^2+\|\nabla u\|_{L^2}^8+\|\nabla u\|_{L^2}^2)dt,\end{aligned}\tag{39}$$

which completes the proof of (38). □

Step 2: Higher estimate of the density

The estimates of $\|\sqrt{\rho}u_t\|_{L^2}$ and higher $D^{2,q}$-estimates of density will be derived by performing similar calculations, thus we omit here.

Proof of Theorem 3. Therefore, having all the estimates at hand, as explained in the end of Sec. 3, it is easy to extend the strong solutions beyond time T^*. Thus, the proof of Theorem 3 is completed. □

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Graduate School of Mathematical Sciences
The University of Tokyo
Tokyo 153-8914
JAPAN
E-mail address: lhytu@ms.u-tokyo.ac.jp