On the ill-posedness of the stationary Navier-Stokes equations in scaling invariant Besov spaces

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1 Introduction

We consider the stationary Navier-Stokes equations in $\mathbb{R}^n$, $n \geq 3$;

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla \pi = f & \text{in } x \in \mathbb{R}^n, \\ \nabla \cdot u = 0 & \text{in } x \in \mathbb{R}^n, \end{cases} \tag{SNS}$$

where $u = u(x) = (u_1(x), u_2(x), \ldots, u_n(x))$ and $\pi = \pi(x)$ denote the unknown velocity vector and the unknown pressure of the fluid at the point $x \in \mathbb{R}^n$, respectively, while $f = f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ is the given external force. For this stationary problem, there have been various studies on existence, uniqueness, and regularity of weak and strong solutions to (SNS). For example, Leray[8] and Ladyzhenskaya[7] proved the existence of solutions to (SNS), and later on, Heywood[5] constructed the solution of (SNS) as a limit of solutions of the non-stationary Navier-Stokes equations. Then Secchi[10] investigated existence and regularity of solutions to (SNS) in $L^n \cap L^p$, $p > n$.

In this study, we focus on the well-posedness problem on (SNS). Let us first define the notions of well-posedness and ill-posedness of (SNS):

Definition 1.1. *Let $(D, \|\cdot\|_D)$ and $(S, \|\cdot\|_S)$ be two Banach spaces (Here D and S indirectly denote the spaces of data (external forces) and of solutions, respectively). We say that* (SNS) *is well-posed from D to S if there exists $\varepsilon > 0$ such that*

(i) *For any $f \in B_D(\varepsilon)$, there exist a solution $u \in S$ of* (SNS),

(ii) *There exists $\delta = \delta(\varepsilon) > 0$ such that if a solution given by* (i) *belongs to $B_S(\delta)$, then it is a unique one,*

(iii) *The map $f \in (B_D(\varepsilon), \|\cdot\|_D) \mapsto u \in (B_S(\delta), \|\cdot\|_S)$, which is well-defined by* (i) *and* (ii), *is continuous,* *where* $B_D(\varepsilon) \equiv \{f \in D;\ \|f\|_D < \varepsilon\}$ *and* $B_S(\eta) \equiv \{u \in S;\ \|u\|_S < \delta\}$*. In addition,* (SNS) *is ill-posed from* D *to* S *if* (SNS) *is not well-posed from* D *to* S*.*

It seems to be an important problem to find more general spaces D and S where (SNS) is well-posed from D to S.

As for this problem, Cunanan-Okabe-Tsutsui[4] and Kaneko-Kozono-Shimizu[6] recently showed that (SNS) is well-posed from $D = \dot{B}^{-3+\frac{n}{p}}_{p,q}$ to $S = P\dot{B}^{-1+\frac{n}{p}}_{p,q}$ for all $1 \le p < n$ and $1 \le q \le \infty$, where P is the Leray projection. Moreover, in the case $p = n$, we can easily show that (SNS) is also well-posed from $D = \dot{B}^{-2}_{n,q}$ to $S = PL^n$ when $1 \le q \le 2$. These spaces D and S are scaling invariant for the external force f and the velocity u in (SNS) respectively. Hence, it seems to be important to investigate whether or not (SNS) is well-posed from $D = \dot{B}^{-3+\frac{n}{p}}_{p,q}$ to $S = P\dot{B}^{-1+\frac{n}{p}}_{p,q}$ even when $n < p \le \infty$ and $1 \le q \le \infty$, and when $p = n$ and $2 < q \le \infty$.

Our purpose is to prove that if $n < p \le \infty$, $1 \le q \le \infty$, and if $p = n$, $2 < q \le \infty$, then (SNS) is *ill-posed* from $D = \dot{B}^{-3+\frac{n}{p}}_{p,q}$ to $S = P\dot{B}^{-1+\frac{n}{p}}_{p,q}$ in the sense that the solution map $f \in D \mapsto u \in S$ is, even if it exists, *not* continuous. More precisely, under such a condition, there exists a sequence $\{f_N\}_{N\in\mathbb{N}}$ of external forces with $f_N \to 0$ in D such that there exists a unique solution $u_N \in PL^n$ of (SNS) for each f_N, which never converges to zero in S (actually, even in $\dot{B}^{-1}_{\infty,\infty}$). Our result makes it clear that the well-posedness and ill-posedness can be divided between the case $(p, q) \in [1, n) \times [1, \infty]$ and the case $(p, q) \in \{n\} \times (2, \infty] \cup (n, \infty] \times [1, \infty]$, respectively.

For the proof of our theorem, we construct such a sequence of external forces according to that of initial data proposed by Bourgain-Pavlović[3] and Yoneda[11] with some modifications. In fact, we make use of some properties of trigonometric functions, and show the norm inflation of the second approximation of solutions in $\dot{B}^{-1}_{\infty,\infty}$. Based on the method of Bejenaru-Tao[1] showing the ill-posedness of the quadratic non-linear Schrödinger equation, we prove the uniquely existence of a sequence $\{u_N\}_{N\in\mathbb{N}} \subset PL^n$ of solutions corresponding to $\{f_N\}_{N\in\mathbb{N}}$, which never converges to zero in $\dot{B}^{-1}_{\infty,\infty}$.

2 Result

First of all, we should prepare definitions and properties of the Littlewood-Paley decomposition and the homogeneous Besov spaces.

We denote by $\mathcal{S}$ the space of rapidly decreasing functions, and $\mathcal{S}'$ denotes the dual space of $\mathcal{S}$, which is called the space of tempered distributions. We define $\mathcal{S}_0$ to be the space of all $\varphi \in \mathcal{S}$ such that

$$\int_{\mathbb{R}^n} x^\alpha \varphi(x) dx = 0$$

for any multi-indices α, and define $\mathcal{S}_0'$ as the dual space of $\mathcal{S}_0$. It is known that $\mathcal{S}_0$ is a closed subspace of $\mathcal{S}$, and that there holds the topological isomorphism

$$\mathcal{S}_0' \cong \mathcal{S}'/\mathcal{P},$$

where $\mathcal{S}'/\mathcal{P}$ denotes the quotient space with the polynomials space $\mathcal{P}$.

Let us introduce the Littlewood-Paley decomposition of functions. First, we take $\phi \in \mathcal{S}$ such that

$$0 \le \phi \le 1, \quad \mathrm{supp}(\phi) = \left\{ \xi \in \mathbb{R}^n;\ \frac{1}{2} \le |\xi| \le 2 \right\}, \quad \sum_{j\in\mathbb{Z}} \phi(2^{-j}\xi) = 1\ (\xi \neq 0). \tag{1}$$

Then, we define a family $\{\varphi_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}$ of functions as

$$\hat{\varphi}_j(\xi) = \phi(2^{-j}\xi), \quad j \in \mathbb{Z}, \tag{2}$$

where $\hat{f} = \mathcal{F}f$ denotes the Fourier transform of f defined by $\hat{f}(\xi) \equiv \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx$. We should note here that

$$\mathrm{supp}(\hat{\varphi}_j) = \left\{ \xi \in \mathbb{R}^n; 2^{j-1} \le |\xi| \le 2^{j+1} \right\}. \tag{3}$$

Associated with $\{\varphi_j\}_{j\in\mathbb{Z}}$ above, we define the homogeneous Besov spaces $\dot{B}^s_{p,q}$ by

$$\dot{B}^s_{p,q} \equiv \left\{ f \in \mathcal{S}'/\mathcal{P};\ \|f\|_{\dot{B}^s_{p,q}} < \infty \right\}$$

for $s \in \mathbb{R}$, $1 \le p, q \le \infty$ with the norms

$$\|f\|_{\dot{B}^s_{p,q}} \equiv \begin{cases} \left(\sum_{j\in\mathbb{Z}} (2^{js} \|\varphi_j * f\|_p)^q \right)^{\frac{1}{q}}, & q < \infty, \\ \sup_{j\in\mathbb{Z}} (2^{js} \|\varphi_j * f\|_p), & q = \infty. \end{cases}$$

It is known that this definition is independent of choice of a function ϕ satisfying (1). As for the homogeneous Besov spaces, there hold some embedding properties as below:

$$\dot{B}^s_{p,q_1} \hookrightarrow \dot{B}^s_{p,q_2}, \quad s \in \mathbb{R},\ 1 \le p \le \infty,\ 1 \le q_1 \le q_2 \le \infty, \tag{4}$$

$$\dot{B}^{s_1}_{p_1,q} \hookrightarrow \dot{B}^{s_2}_{p_2,q}, \quad 1 \le q \le \infty,\ -\infty < s_2 \le s_1 < \infty,\ 1 \le p_1 \le p_2 \le \infty, \quad \text{with } s_1 - n/p_1 = s_2 - n/p_2. \tag{5}$$

In addition, the Riesz potential $(-\Delta)^{\frac{\alpha}{2}} f \equiv \mathcal{F}^{-1}(|\xi|^\alpha \hat{f}(\xi))$ $(\alpha \in \mathbb{R})$ gives an isomorphism from $\dot{B}^{s+\alpha}_{p,q}$ onto $\dot{B}^s_{p,q}$ for any $s \in \mathbb{R}$, $1 \le p, q \le \infty$, which implies that

$$\|(-\Delta)^{\frac{\alpha}{2}} f\|_{\dot{B}^s_{p,q}} \cong \|f\|_{\dot{B}^{s+\alpha}_{p,q}}. \tag{6}$$

The above properties (4), (5) and (6) will be often used implicitly in what follows.

Now let us return to the problem on (SNS). First, we rewrite (SNS) to the generalized form so that we can treat it easily. We introduce the Leray projection $P : L^p \to L^p_\sigma \equiv \overline{\{f \in C_0^\infty;\ \nabla \cdot f = 0\}}^{\|\cdot\|_{L^p}}$. We should note here that in $\mathbb{R}^n$, P is defined as a matrix-valued operator $P = (P_{jk})_{1\le j,k\le n}$ with $P_{jk} \equiv \delta_{jk} + R_j R_k$, where $R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}$, $j = 1, 2, \ldots, n$, denotes the Riesz transform. By applying P to (SNS), we obtain

$$-\Delta u + P(u \cdot \nabla u) = Pf,$$

implied by $P(\nabla\pi) = 0$ and $Pu = u$, since $\nabla \cdot u = 0$. Hence, the solution u of (SNS) can be expressed as

$$u = Lf + B(u, u), \tag{rSNS}$$

where $Lf \equiv (-\Delta)^{-1}Pf$ and $B(u, v) \equiv -(-\Delta)^{-1}P(u \cdot \nabla v)$.

As for the well-posedness of (rSNS) (or (SNS)) in homogeneous Besov spaces, the following previous result is well-known:

Proposition 2.1. (Cunanan-Okabe-Tsutsui[4], Kaneko-Kozono-Shimizu[6]) *Let $n \ge 3$. Suppose that $1 \le p < n$ and $1 \le q \le \infty$. Then* (rSNS) *is well-posed from $\dot{B}^{-3+\frac{n}{p}}_{p,q}$ to $P\dot{B}^{-1+\frac{n}{p}}_{p,q}$.*

Remark 2.2. We should note here that the space $\dot{B}^{-3+\frac{n}{p}}_{p,q}$ $(1 \le p, q \le \infty)$ for the external force f and the space $\dot{B}^{-1+\frac{n}{p}}_{p,q}$ for the solution u are both scaling invariant with respect to (SNS), respectively. Moreover, it is seen from the embedding (5) that if $p_1 \le p_2$, then we see $\dot{B}^{-3+\frac{n}{p_1}}_{p_1,q} \hookrightarrow \dot{B}^{-3+\frac{n}{p_2}}_{p_2,q}$ and $\dot{B}^{-1+\frac{n}{p_1}}_{p_1,q} \hookrightarrow \dot{B}^{-1+\frac{n}{p_2}}_{p_2,q}$.

For the case $p = n$ and $1 \le q \le 2$, it is still unknown whether or not (rSNS) is well-posed from $\dot{B}^{-2}_{n,q}$ to $P\dot{B}^{0}_{n,q}$. However, by extending the solution space to PL^n, we can show the following:

Proposition 2.3. (Well-posedness when $p = n$ and $1 \le q \le 2$) *Let $n \ge 3$. Then* (rSNS) *is well-posed from $\dot{B}^{-2}_{n,q}$ to PL^n if $1 \le q \le 2$.*

Indeed, we can prove Proposition 2.1 and Proposition 2.3 by showing the *quantitatively well-posedness* of (rSNS) defined as follows, which is one of sufficient conditions of the well-posedness defined in Definition 1.1:

Definition 2.4. *Let $(D, \|\cdot\|_D)$ and $(S, \|\cdot\|_S)$ be two Banach spaces. We call that* (rSNS) *is quantitatively well-posed from* D *to* S *if there hold two estimates as follows*:

$$\|Lf\|_S \le C\|f\|_D, \quad \forall f \in D,$$
$$\|B(u,v)\|_S \le C\|u\|_S\|v\|_S, \quad \forall u, v \in S.$$

Actually, we can see the boundedness of L by that of P (or Riesz transforms) in homogeneous Besov spaces. We can also see the boundedness of B by using the Hölder type estimate of function products deduced by the Bony's paraproduct formula (for Proposition 2.3, it suffices to use the Hölder inequality). For the detail, see [6].

Our main result now reads:

Theorem 2.5. (Main theorem) *Let* $n \ge 3$. *Suppose that* D *and* $\tilde{D}$ *are two spaces with* $D \hookrightarrow \tilde{D}$ *as either* (1) *or* (2):

(1) $D = \dot{B}^{-2}_{n,1}$, $\tilde{D} = \dot{B}^{-3+\frac{n}{p}}_{p,q}$ *with* $n < p \le \infty$ *and* $1 \le q \le \infty$,

(2) $D = \dot{B}^{-2}_{n,2}$, $\tilde{D} = \dot{B}^{-2}_{n,q}$ *with* $2 < q \le \infty$.

Let $\varepsilon, \delta > 0$ *be constants appeared in Definition 1.1 which guarantee the well-posedness of* (rSNS) *from* D *to* PL^n, *and take* $0 < \eta < \varepsilon$ *arbitrarily. Then the solution map*

$$f \in (B_D(\eta), \|\cdot\|_{\tilde{D}}) \mapsto u \in (B_{PL^n}(\delta), \|\cdot\|_{\dot{B}^{-1}_{\infty,\infty}})$$

is discontinuous, where $(B_D(\eta), \|\cdot\|_{\tilde{D}})$ *and* $(B_{PL^n}(\delta), \|\cdot\|_{\dot{B}^{-1}_{\infty,\infty}})$ *denote the ball* $B_D(\eta)$ *equipped with the* $\tilde{D}$ *topology and* $B_{PL^n}(\delta)$ *with the* $\dot{B}^{-1}_{\infty,\infty}$ *topology, respectively. In other words,* (rSNS) *is ill-posed from* $\tilde{D}$ *to* $P\dot{B}^{-1}_{\infty,\infty}$.

Remark 2.6. Suppose that D and $\tilde{D}$ are as the above theorem. We now arbitrarily choose a sequence $\{g_N\}_{N\in\mathbb{N}}$ such that $\sup_{N\in\mathbb{N}} \|g_N\|_D < \varepsilon$. Then by Proposition 2.3, there exists a unique solution $v_N \in PL^n$ for each g_N. In addition, if $g_N \to 0$ in D, then we see $v_N \to 0$ in PL^n by the well-posedness (continuity of the solution map). Theorem 2.5 means, however, that the weaker convergence $g_N \to 0$ in $\tilde{D}$ cannot sufficiently guarantee $v_N \to 0$ even in the weakest scaling invariant norm $\dot{B}^{-1}_{\infty,\infty}$.

3 Proof of the main theorem

In order to prove Theorem 2.5, we make use of the well-posed theory proposed by Bejenaru-Tao[1] as below:

Proposition 3.1. (Bejenaru-Tao[1]) *Suppose that* (rSNS) *is quantitatively well-posed from D to S. We now define the nonlinear maps $A_n : D \to S$ for $n \in \mathbb{N}$ by*

$$\begin{cases} A_1 f \equiv Lf, \\ A_n f \equiv \sum_{k,l\geq 1, k+l=n} B(A_k f, A_l f), \quad n \geq 2. \end{cases}$$

(1) *Each $A_n f$ belongs to S and there exists a constant $C > 0$ such that*

$$\|A_n f\|_S \leq C^n \|f\|_D^n, \quad \forall n \in \mathbb{N}.$$

Moreover, there exists a constant $\varepsilon > 0$ such that if $f \in B_D(\varepsilon)$, then there exists a unique solution $u \in S$ of (rSNS)*, which is expressed as $u = \sum_{n=1}^{\infty} A_n f$.*

(2) *Suppose that D and S are given other norms $\|\cdot\|_{\tilde{D}}$ and $\|\cdot\|_{\tilde{S}}$, respectively, which are weaker than D and S in the sense that*

$$\|f\|_{\tilde{D}} \leq C\|f\|_D, \quad \|u\|_{\tilde{S}} \leq C\|u\|_S.$$

Assume that the solution map $f \mapsto u$ is continuous from $(B_D(\varepsilon), \|\cdot\|_{\tilde{D}})$ to $(B_S(\delta), \|\cdot\|_{\tilde{S}})$. Then for each n, $A_n : D \to S$ is also continuous from $(B_D(\varepsilon), \|\cdot\|_{\tilde{D}})$ to $(B_S(\delta), \|\cdot\|_{\tilde{S}})$.

Proposition 3.1 means that if at least one of A_n is discontinuous, then (rSNS) is ill-posed from $\tilde{D}$ to $\tilde{S}$. By this proposition and Proposition 2.3, it suffices to show the following lemma in order to prove Theorem 2.5:

Lemma 3.2. *Let $n \geq 3$. Suppose that D and $\tilde{D}$ are two spaces with $D \hookrightarrow \tilde{D}$ as either* (1) *or* (2) *of Theorem 2.5, and $\eta > 0$ is a constant given in that theorem. Then there exists a sequence $\{f_N\}_{N\in\mathbb{N}}$ of external forces and a constant $C = C(\eta) > 0$ satisfying the following* (i)*,* (ii) *and* (iii)*:*

(i) $\sup_{N\in\mathbb{N}} \|f_N\|_D < \eta$,

(ii) $\|f_N\|_{\tilde{D}} \to 0$ *as* $N \to \infty$,

(iii) $\inf_{N\in\mathbb{N}} \|A_2(f_N)\|_{\dot{B}^{-1}_{\infty,\infty}} = \inf_{N\in\mathbb{N}} \|B(Lf_N, Lf_N)\|_{\dot{B}^{-1}_{\infty,\infty}} > C$.

Proof of Lemma 3.2. We first take $\psi \in \mathcal{S}$ as

$$\mathrm{supp}(\hat{\psi}) = \{\xi \in \mathbb{R}^n; |\xi| \leq 1\}, \quad \hat{\psi}(\xi) > 0 \text{ in } \{\xi \in \mathbb{R}^n; |\xi| < 1\},$$

and we define

$$\Psi_m^{(j)} \equiv (-\Delta)\left\{\psi_{x_j}\cos(mx_1)\right\}, \quad j = 2, 3, \; m \in \mathbb{N},$$

where $\psi_{x_j} \equiv \frac{\partial \psi}{\partial x_j}$. Using this function, we construct $\{f_N\}_{N\in\mathbb{N}}$ differently in the case (1) and (2) of Theorem 2.5.

Step 1. The case (1) : $D = \dot{B}^{-2}_{n,1}$, $\tilde{D} = \dot{B}^{-3+\frac{n}{p}}_{p,q}$ *with* $n < p \le \infty$ *and* $1 \le q \le \infty$.

We define a parametrized vector-valued function as

$$g_{\lambda,M} \equiv \lambda\{e_2\Psi^{(3)}_M - e_3\Psi^{(2)}_M\}, \quad \lambda > 0,\ M \ge 100,$$

where $e_2 \equiv (0,1,0,\dots 0)$ and $e_3 \equiv (0,0,1,\dots,0)$ are unit vectors along x_2 and x_3, respectively. This function is inspired by a initial data sequence proposed by Bourgain-Pavlović[3]. It is clearly seen that $\nabla \cdot g_{\lambda,M} = 0$ and hence $Pg_{\lambda,M} = g_{\lambda,M}$. Therefore, we have

$$Lg_{\lambda,M} = (-\Delta)^{-1}g_{\lambda,M} = \lambda\cos(Mx_1)\{e_2\psi_{x_3}(x) - e_3\psi_{x_2}(x)\}.$$

Now let us consider the estimate of $g_{\lambda,M}$. We recall $\{\varphi_j\}_{j\in\mathbb{Z}}$ in the definition of Besov spaces (see (1)-(3)). Since

$$\mathcal{F}[\psi_{x_j}\cos(Mx_1)](\xi) = -\frac{1}{2}i\xi_j\{\hat{\psi}(\xi - Me_1) + \hat{\psi}(\xi + Me_1)\}, \quad j = 2,3,$$

we see that there exist at most three indices $j \in \mathbb{Z}$ such that $\varphi_j * Lg_{\lambda,M} \not\equiv 0$. Indeed, such indices must satisfy

$$\{\xi \in \mathbb{R}^n; 2^{j-1} \le |\xi| \le 2^{j+1}\} \cap \{\xi \in \mathbb{R}^n; M-1 \le |\xi| \le M+1\} \ne \emptyset,$$

i.e., $(M-1)/2 \le 2^j \le 2(M+1)$. Therefore, we obtain the estimates

$$\begin{aligned} \|g_{\lambda,M}\|_D = \|g_{\lambda,M}\|_{\dot{B}^{-2}_{n,1}} &= \|(-\Delta)^{-1}g_{\lambda,M}\|_{\dot{B}^0_{n,1}} \\ &= \sum_{j\in\mathbb{Z}} \|\varphi_j * Lg_{\lambda,M}\|_{L^n} \\ &\le C\lambda. \end{aligned} \tag{7}$$

and

$$\begin{aligned} \|g_{\lambda,M}\|_{\tilde{D}} = \|g_{\lambda,M}\|_{\dot{B}^{-3+\frac{n}{p}}_{p,q}} &\le \|(-\Delta)^{-1}g_{\lambda,M}\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}} \\ &= \sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p})}\|\varphi_j * Lg_{\lambda,M}\|_{L^p} \\ &\le C\lambda M^{-1+\frac{n}{p}} \to 0 \ \ as\ M \to \infty. \end{aligned} \tag{8}$$

for any $M \ge 100$, implied by $-1 + n/p < 0$. Here we have used the Young inequality, the equality

$$\|\varphi_j\|_{L^1} = \|2^{nj}\varphi_0(2^j\cdot)\|_{L^1} = \|\varphi_0\|_{L^1}, \quad \forall j \in \mathbb{Z}, \tag{9}$$

and the estimate

$$\|Lg_{\lambda,M}\|_{L^p} \le C\|\nabla\psi\|_{L^p}, \quad \forall\lambda > 0,\ \forall M \ge 100,\ 1 \le \forall p \le \infty. \tag{10}$$

We next calculate $B(Lg_{\lambda,M}, Lg_{\lambda,M})$. It is seen that

$$\begin{aligned}(Lg_{\lambda,M})\cdot\nabla(Lg_{\lambda,M}) &= \frac{1}{2}\lambda^2(e_2\Phi_1+e_3\Phi_2)+\frac{1}{2}\lambda^2(e_2\Phi_1\cos(2Mx_1)+e_3\Phi_2\cos(2Mx_1))\\ &\equiv I_1+I_2,\end{aligned}$$

where $\psi_{x_2^\alpha x_3^\beta}\equiv \frac{\partial^{(\alpha+\beta)}}{\partial x_2^\alpha x_3^\beta}\psi$ and

$$\Phi_1\equiv \psi_{x_3}\psi_{x_2x_3}-\psi_{x_2}\psi_{x_3^2},\quad \Phi_2\equiv -\psi_{x_3}\psi_{x_2^2}+\psi_{x_2}\psi_{x_2x_3}. \tag{11}$$

Since

$$\operatorname{supp}(\hat{I}_1)\subset\operatorname{supp}(\hat{\psi}*\hat{\psi})\subset\{\xi\in\mathbb{R}^n;|\xi|\le 2\},$$

we see that

$$\|(-\Delta)^{-1}PI_1\|_{\dot{B}^{-1}_{\infty,\infty}}=\sup_{j\in\mathbb{Z},j\le 2}2^{-j}\|\varphi_j*(-\Delta)^{-1}PI_1\|_{L^\infty}\ge C\lambda^2>0$$

for some constant $C>0$. On the other hand, it is seen that

$$\operatorname{supp}(\hat{I}_2)\subset\operatorname{supp}((\hat{\psi}*\hat{\psi})(\cdot\pm 2Me_1))\subset\{\xi\in\mathbb{R}^n;2M-2\le|\xi|\le 2M+2\},$$

which yields $\varphi_j*((-\Delta)^{-1}PI_2)\equiv 0$ for any $j\le 2$. Therefore, we obtain the estimate that

$$\begin{aligned}\|B(Lg_{\lambda,M},Lg_{\lambda,M})\|_{B^{-1}_{\infty,\infty}} &= \sup_{j\in\mathbb{Z}}2^{-j}\|\varphi_j*(-\Delta)^{-1}P(I_1+I_2)\|_{L^\infty}\\ &\ge \sup_{j\in\mathbb{Z},j\le 2}2^{-j}\|\varphi_j*(-\Delta)^{-1}PI_1\|_{L^\infty}\\ &\ge C\lambda^2\end{aligned}\tag{12}$$

for any $M\ge 100$.

Now for given $\eta>0$, we can fix $\lambda=\lambda_0$ so that $\sup_{M\ge 100}\|g_{\lambda_0,M}\|_D<\eta$ from (7). In addition, from (8) and (12), we see that a sequence $\{f_N\}_{N\in\mathbb{N}}$ defined by

$$f_N\equiv g_{\lambda_0,N+100},\quad N=1,2,3,\dots$$

satisfies (i), (ii), and (iii) of Lemma 3.2. This proves Lemma 3.2 in the case (1) of Theorem 2.5.

Step 2. The case (2) : $D=\dot{B}^{-2}_{n,2}$, $\tilde{D}=\dot{B}^{-2}_{n,q}$ *with* $2<q\le\infty$.

We define another parametrized vector-valued function as

$$h_{\lambda,M}\equiv\frac{\lambda}{\sqrt{\Gamma(M)}}\sum_{k=10}^{M}k^{-\frac{1}{2}}\{e_2\Psi^{(3)}_{2^{k^2}}-e_3\Psi^{(2)}_{2^{k^2}}\},\quad \lambda>0,\ M\ge 100,$$

where $\Gamma(M) \equiv \sum_{k=10}^{M} k^{-1}$. This function is inspired by a initial data sequence proposed by Yoneda[11]. As similar to $g_{\lambda,M}$, we see that $\nabla \cdot h_{\lambda,M} = 0$ and

$$Lh_{\lambda,M} = (-\Delta)^{-1} h_{\lambda,M} = \frac{\lambda}{\sqrt{\Gamma(M)}} \sum_{k=10}^{M} k^{-\frac{1}{2}} \cos(2^{k^2} x_1)\{e_2 \psi_{x_3}(x) - e_3 \psi_{x_2}(x)\}$$

Let us consider the estimate of $h_{\lambda,M}$. By a similar way to Step 1, we see that for each k, there exist at most three indices $j \in \mathbb{Z}$ such that $\varphi_j * (\psi_{x_l} \cos(2^{k^2} x_1)) \not\equiv 0$ $(l = 2, 3)$, which must satisfy $(2^{k^2} - 1)/2 \le 2^j \le 2(2^{k^2} + 1)$. Moreover, the set $\{2^{k^2}\}_{k \ge 10}$ is so discrete that we see

$$\{j \in \mathbb{Z}; \varphi_j * (\psi_{x_l} \cos(2^{k_1^2} x_1)) \not\equiv 0\} \cap \{j \in \mathbb{Z}; \varphi_j * (\psi_{x_l} \cos(2^{k_2^2} x_1)) \not\equiv 0\} = \emptyset$$

for any $k_1, k_2 \ge 10$ with $k_1 \ne k_2$. Hence we obtain the estimate

$$\begin{aligned}
\|h_{\lambda,M}\|_{\dot{B}^{-2}_{n,q}} &= \|(-\Delta)^{-1} h_{\lambda,M}\|_{\dot{B}^{0}_{n,q}} \\
&= \left\{ \sum_{j \in \mathbb{Z}} \|\varphi_j * (-\Delta)^{-1} h_{\lambda,M}\|_{L^n}^q \right\}^{\frac{1}{q}} \\
&\le \frac{C\lambda}{\sqrt{\Gamma(M)}} \left\{ \sum_{k=10}^{M} k^{-\frac{q}{2}} \right\}^{\frac{1}{q}} \\
&\le \begin{cases} C\lambda, & q = 2, \\ \frac{C\lambda}{\sqrt{\Gamma(M)}}, & 2 < q \le \infty. \end{cases}
\end{aligned} \tag{13}$$

Here we have used the Young inequality, (9), and (10). Since $\Gamma(M) \to \infty$ as $M \to \infty$, we see from (13) that

$$\|h_{\lambda,M}\|_{\dot{B}^{-2}_{n,q}} \to 0 \quad \text{as } M \to \infty, \quad \text{if } 2 < q \le \infty. \tag{14}$$

We next calculate $B(Lh_{\lambda,M}, Lh_{\lambda,M})$. It is seen that

$$\begin{aligned}
&(Lh_{\lambda,M}) \cdot \nabla (Lh_{\lambda,M}) \\
&= \frac{\lambda^2}{2}(e_2\Phi_1 + e_3\Phi_2) + \frac{\lambda^2}{2\Gamma(M)}(e_2\Phi_1 + e_3\Phi_2) \sum_{k=10}^{M} k^{-1} \cos(2^{k^2+1} x_1) \\
&\quad + \frac{\lambda^2}{2\Gamma(M)}(e_2\Phi_1 + e_3\Phi_2) \left\{ \sum_{\substack{10 \le k,l \le M \\ k \ne l}} k^{-\frac{1}{2}} l^{-\frac{1}{2}} \cos((2^{k^2} + 2^{l^2}) x_1) + \cos((2^{k^2} - 2^{l^2}) x_1) \right\} \\
&\equiv J_1 + J_2 + J_3,
\end{aligned}$$

where Φ_1 and Φ_2 are as (11). Since the above coefficients 2^{k^2+1}, $2^{k^2}+2^{l^2}$ and $|2^{k^2}-2^{l^2}|$ are large enough, we see

$$\varphi_j * (-\Delta)^{-1}P(J_1+J_2) \equiv \varphi_j * (-\Delta)^{-1}PJ_1, \quad \forall j \le 2.$$

Hence, by a similar way to the argument on I_1 and I_2 in Step 1, we obtain

$$\|B(Lh_{\lambda,M}, Lh_{\lambda,M})\|_{\dot{B}^{-1}_{\infty,\infty}} \ge \|(-\Delta)^{-1}PJ_1\|_{\dot{B}^{-1}_{\infty,\infty}} \ge C\lambda^2 > 0. \tag{15}$$

Now for given $\eta > 0$, we can fix $\lambda = \lambda_0$ so that $\sup_{M\ge 100}\|h_{\lambda_0,M}\|_{\dot{B}^0_{n,2}} < \eta$ from (13). In addition, from (14) and (15), we see that a sequence $\{f_N\}_{N\in\mathbb{N}}$ defined by

$$f_N \equiv h_{\lambda_0,N+100}, \quad N = 1,2,3,\dots$$

satisfies

$$\sup_{N\in\mathbb{N}}\|f_N\|_{\dot{B}^0_{n,2}} < \eta, \quad \lim_{N\to\infty}\|f_N\|_{\dot{B}^0_{n,q}} = 0 \text{ if } 2 < q \le \infty, \quad \inf_{N\in\mathbb{N}}\|B(Lf_N, Lf_N)\|_{\dot{B}^{-1}_{\infty,\infty}} \ge C\lambda_0^2.$$

This proves Lemma 3.2 in the case (2) of Theorem 2.5, and hence the proof of Lemma 3.2 is completed.

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