Existence of weak solutions for a diffuse interface model for two-phase flows of incompressible fluids with different densities and nonlocal free energies

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1 Introduction

This contribution is a summary of the results in A. and T. [AT18]. We consider here a two-phase flow for incompressible fluids of different densities and different viscosities. The two fluids are assumed to be macroscopically immiscible and to be miscible in a thin interface region, i.e., we consider a diffuse interface model (also called phase field model) for the two-phase flow. In contrast to sharp interface models, where the interface between the two fluids is a sufficiently smooth hypersurface, diffuse interface model can describe topological changes due to pinch off and droplet collision.

There are several diffuse interface models for such two-phase flows. Firstly, in the case of matched densities, i.e., the densities of both fluids are assumed to be identical, there is a well-known model H, cf. Hohenberg and Halperin or Gurtin et al. [HH77, GPV96]. In the case that the fluid densities do not coincide there are different models. On one hand Lowengrub and Truskinovsky [LT98] derived a quasi-incompressible model, where the mean velocity field of the mixture is in general not divergence free. On the other hand, Ding et al. [DSS07] proposed a model with a divergence free mean fluid velocities. But this model is not known to be thermodynamically consistent. In A., Garcke and Grün [AGG11] a thermodynamically consistent diffuse interface model for two-phase flow with different densities and a divergence free mean velocity field was derived, which we call AGG model for short. The existence of weak solutions of the AGG model was shown in [ADG13]. For analytic result in the case of matched densities, i.e., the model H, we refer to [Abe09b] and the reference given there. Existence of weak and strong solutions for the model by Lowengrub and Truskinovsky was proven in [Abe09a, Abe11].

Concerning the Cahn-Hilliard equation, Giacomin and Lebowitz [GL97, GL98] observed that a physically more rigorous derivation leads to a nonlocal equation, which we call a nonlocal Cahn-Hilliard equation. There are two types of nonlocal Cahn-Hilliard equations. One is the equation where the second order differential operator in the equation for the chemical potential is replaced by a convolution operator with a sufficiently smooth even function. We call it a nonlocal Cahn-Hilliard equation with a regular kernel in the following. The other is one where the second order differential operator is replaced by a regional fractional Laplacian. We call it a nonlocal Cahn-Hilliard equation with a singular kernel, since the regional fractional Laplacian is defined by using singular kernel. The nonlocal Cahn-Hilliard equation with a regular kernel was analyzed in [GZ03, G14, GL98, LP11a, LP11b]. On the other hand, the nonlocal Cahn-Hilliard equation with a singular kernel was first analyzed in A., Bosia and Grasselli [ABG15], where they proved the existence and uniqueness of a weak solution of the nonlocal Cahn-Hilliard equation, its regularity properties and the existence of a (connected) global attractor. Concerning corresponding fractional Allen-Cahn equation, there is an earlier study by [NNG08].

Concerning the nonlocal model H with a regular kernel, where the convective Cahn-Hilliard equation is replaced by the convective nonlocal Cahn-Hilliard equation with a regular kernel, first studies were done by [CFG12, FG12a, FG12b]. More recently, the nonlocal AGG model with a regular kernel, where the convective Cahn-Hilliard equation is replaced by the convective nonlocal Cahn-Hilliard equation with a regular kernel, was studied by Frigeri [F15] and he showed the existence of a weak solution for that model. The method of the proof in [F15] is based on the Faedo-Galerkin method of a suitably mollified system and the method of passing to the limit with two parameters tending to zero. The method is different from [ADG13] which is based on implicit time discretization and a Leray-Schauder fixed point argument.

In this contribution, we consider a nonlocal AGG model with a singular kernel, where a convective Cahn-Hilliard equation in the AGG model is replaced by a convective nonlocal Cahn-Hilliard equation with a singular kernel. Our aim is to prove the existence of a weak solution of such a system.

We consider the existence of weak solutions of the following system, which couples a nonhomogeneous Navier-Stokes equation system with a nonlocal Cahn-Hilliard equation:

$$\partial_t(\rho\mathbf{v}) + \operatorname{div}(\mathbf{v}\otimes(\rho\mathbf{v}+\widetilde{\mathbf{J}})) - \operatorname{div}(2\eta(\varphi)D\mathbf{v}) + \nabla p = \mu\nabla\varphi \qquad \text{in } Q, \tag{1.1}$$

$$\operatorname{div}\mathbf{v} = 0 \qquad \text{in } Q, \tag{1.2}$$

$$\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi = \operatorname{div}(m(\varphi)\nabla\mu) \qquad \text{in } Q, \tag{1.3}$$

$$\mu = \Psi'(\varphi) + \mathcal{L}\varphi \qquad \text{in } Q, \tag{1.4}$$

where $\rho = \rho(\varphi) := \frac{\tilde\rho_1+\tilde\rho_2}{2} + \frac{\tilde\rho_2-\tilde\rho_1}{2}\varphi$, $\widetilde{\mathbf{J}} = -\frac{\tilde\rho_2-\tilde\rho_1}{2}m(\varphi)\nabla\mu$, $Q = \Omega\times(0,\infty)$. We assume that $\Omega\subset\mathbb{R}^d$, $d=2,3$, is a bounded domain with C^2-boundary. Here and in the following $\mathbf{v}$, p, and ρ are the (mean) velocity, the pressure and the density of the mixture of the two fluids, respectively. Furthermore $\tilde\rho_j$, $j=1,2$, are the specific densities of the unmixed fluids, φ is the difference of the volume fractions of the two fluids, and μ is the chemical potential related to φ. Moreover, $D\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v}+\nabla\mathbf{v}^T)$, $\eta(\varphi) > 0$ is the viscosity of the fluid mixture, and $m(\varphi) > 0$ is a mobility coefficient.

Finally, $\mathcal{L}$ is defined as

$$\begin{aligned}\mathcal{L}u(x) &= \text{p.v.} \int_\Omega (u(x) - u(y))k(x, y, x - y)dy \\ &= \lim_{\varepsilon\to 0} \int_{\Omega\setminus B_\varepsilon(x)} (u(x) - u(y))k(x, y, x - y)dy \qquad \text{for } x \in \Omega\end{aligned} \tag{1.5}$$

for suitable $u \colon \Omega \to \mathbb{R}$. Here the kernel $k \colon \mathbb{R}^d \times \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to \mathbb{R}$ is assumed to be $(d + 2)$-times continuously differentiable and to satisfy the conditions

$$k(x, y, z) = k(y, x, -z)\,, \tag{1.6}$$

$$|\partial_x^\beta \partial_y^\gamma \partial_z^\delta k(x, y, z)| \leqslant C_{\beta,\gamma,\delta}|z|^{-d-\alpha-|\delta|}\,, \tag{1.7}$$

$$c_0|z|^{-d-\alpha} \leqslant k(x, y, z) \leqslant C_0|z|^{-d-\alpha}\,. \tag{1.8}$$

for all $x, y, z \in \mathbb{R}^d$, $z \neq 0$ and $\beta, \gamma, \delta \in \mathbb{N}_0^d$ with $|\beta| + |\gamma| + |\delta| \leqslant d + 2$ and some constants $C_{\beta,\gamma,\delta}, c_0, C_0 > 0$. Here α is the order of the operator, cf. [AK07]). We restrict ourselves to the case $\alpha \in (1, 2)$. If $\omega \in C_b^{d+2}(\mathbb{R}^d \times \mathbb{R}^d)$ is symmetric, then $k(x, y, z) = \omega(x, y)|z|^{-d-\alpha}$ is an example of a kernel satisfying the previous assumptions.

We add to our system the boundary and initial conditions

$$\mathbf{v}|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times (0, \infty), \tag{1.9}$$

$$\partial_{\mathbf{n}}\mu|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times (0, \infty), \tag{1.10}$$

$$(\mathbf{v}, \varphi)\,|_{t=0} = (\mathbf{v}_0, \varphi_0) \qquad \text{in } \Omega. \tag{1.11}$$

Here $\partial_{\mathbf{n}} = \mathbf{n} \cdot \nabla$ and $\mathbf{n}$ denotes the exterior normal at $\partial\Omega$. We note that (1.9) is the usual no-slip boundary condition for the velocity field and $\partial_{\mathbf{n}}\mu|_{\partial\Omega} = 0$ describes that there is no mass flux of the fluid components through the boundary. Furthermore we complete the system above by an additional boundary condition for φ, which will be part of the weak formulation, cf. Definition 3.2 below. If φ is smooth enough (e.g. $\varphi(t) \in C^{1,\beta}(\overline{\Omega})$ for every $t \geq 0$) and k fulfills suitable assumptions, then

$$\mathbf{n}_{x_0} \cdot \nabla c(x_0) = 0 \qquad \text{for all } x_0 \in \partial\Omega$$

where $\mathbf{n}_{x_0}$ depends on the interaction kernel k, cf. [ABG15, Theorem 6.1], and $x_0 \in \partial\Omega$.

The total energy of the system at time $t \geq 0$ is given by

$$E_{\text{tot}}(\varphi, \mathbf{v}) = E_{\text{kin}}(\varphi, \mathbf{v}) + E_{\text{free}}(\varphi) \tag{1.12}$$

where

$$E_{\text{kin}}(\varphi, \mathbf{v}) = \int_\Omega \rho \frac{|\mathbf{v}|^2}{2}\, dx, \qquad E_{\text{free}}(\varphi) = \int_\Omega \Psi(\varphi)\, dx + \frac{1}{2}\mathcal{E}(\varphi, \varphi)$$

are the kinetic energy and the free energy of the mixture, respectively, and

$$\mathcal{E}(u, v) = \int_\Omega \int_\Omega (u(x) - u(y))(v(x) - v(y))k(x, y, x - y)\, dx\, dy \tag{1.13}$$

for all $u, v \in H^{\frac{\alpha}{2}}(\Omega)$ is the natural bilinear form associated to $\mathcal{L}$, which will also be used to formulate the natural boundary condition for φ weakly. Every sufficiently smooth solution of the system above satisfies the energy identity

$$\frac{d}{dt}E_{\mathrm{tot}}(\varphi, \mathbf{v}) = -\int_\Omega 2\eta(\varphi)|D\mathbf{v}|^2\,dx - \int_\Omega m(\varphi)|\nabla\mu|^2\,dx$$

for all $t \geq 0$. This can be shown by testing (1.1) with $\mathbf{v}$, (1.3) with μ and (1.4) with $\partial_t\varphi$, where the product of $\mathcal{L}\varphi$ and $\partial_t\varphi$ coincides with

$$\mathcal{E}(\varphi(t), \partial_t\varphi(t))$$

under the natural boundary condition for $\varphi(t)$.

In order to guarantee that $\varphi(x,t) \in [-1,1]$ almost everywhere, we consider a class of singular free energies, which will be specified below and which includes the homogeneous free energy of the so-called regular solution models used by Cahn and Hilliard [CH58]:

$$\Psi(\varphi) = \frac{\theta}{2}\left((1+\varphi)\ln(1+\varphi) + (1-\varphi)\ln(1-\varphi)\right) - \frac{\theta_c}{2}\varphi^2, \quad \varphi \in [-1,1] \tag{1.14}$$

where $0 < \theta < \theta_c$. In order to deal with these terms we apply techniques, which were developed in Abels and Wilke [AW07] and extended to the present nonlocal Cahn-Hilliard equation in [ABG15].

Our proof of existence of a weak solution of (1.1)-(1.4) together with a suitable initial and boundary condition follows closely the proof of the main result of [ADG13]. The following are the main differences and difficulties of our paper compared with [ADG13]. Since we do not expect H^1-regularity in space for the volume fraction φ of a weak solution of our system, we should eliminate $\nabla\varphi$ from our weak formulation taking into account the incompressibility of $\mathbf{v}$. Implicit time discretization has to be constructed carefully, using a suitable mollification of φ and an addition of a small Laplacian term to the chemical potential equation taking into account of the lack of H^1-regularity in space of φ. While the arguments for the weak convergence of temporal interpolants of weak solutions of the time-discrete problem are similar to [ADG13], the function space used for the order parameter has less regularity in space since the nonlocal operator of order less than 2 is involved in the equation for the chemical potential. For the convergence of the singular term $\Psi'(\varphi)$, we employ the argument in [ABG15]. The only difference is that we work in space-time domains directly. For the validity of the energy inequality, additional arguments using the equation of chemical potential and the fact that weak convergence together with norm convergence in uniformly convex Banach spaces imply strong convergence are needed.

The structure of this contribution is as follows: In Section 2 we summarize preliminaries, we fix notations and collect the needed results on nonlocal operator. In Section 3, we define weak solutions of our system and state our main result concerning the existence of weak solutions. In Section 4, we define an implicit time discretization of our system and state an existence result of its weak solutions. In Section 5, we obtain compactness in time of temporal interpolants of the weak solutions of time-discrete problem and obtain weak solutions of our system as weak limits of a suitable subsequence.

2 Preliminaries

For $a, b \in \mathbb{R}^d$ we denote $a \otimes b = (a_i b_j)_{i,j=1}^d$ and $A_{\mathrm{sym}} = \frac{1}{2}(A + A^T)$ for $A \in \mathbb{R}^{d\times d}$. The duality product of a Banach space X and its dual X' is denoted by

$$\langle f, g\rangle \equiv \langle f, g\rangle_{X',X} = f(g), \qquad f \in X', g \in X.$$

We write $X \hookrightarrow\hookrightarrow Y$ if X is compactly embedded into Y. Moreover, if H is a Hilbert space, $(\cdot\,,\cdot)_H$ denotes its inner product. Furthermore, we use the abbreviation $(.\,,.)_M = (.\,,.)_{L^2(M)}$.

Lebesgue and Sobolev spaces: Let $M \subseteq \mathbb{R}^d$ be measurable. As usual $L^q(M)$, $1 \leq q \leq \infty$, denotes the Lebesgue space, $\|.\|_q$ its norm and $L^q(M; X)$ denotes the set of all strongly measurable q-integrable functions/essentially bounded functions, where X is a Banach space. If $M = (a, b)$, we denote these spaces for simplicity by $L^q(a, b; X)$ and $L^q(a, b)$. Recall that $f \in L^q_{\mathrm{loc}}([0,\infty); X)$ if and only if $f \in L^q(0, T; X)$ for every $T > 0$. Furthermore, $L^q_{\mathrm{uloc}}([0,\infty); X)$ is the *uniformly local* variant of $L^q(0,\infty; X)$ consisting of all strongly measurable $f\colon [0,\infty) \to X$ such that

$$\|f\|_{L^q_{\mathrm{uloc}}([0,\infty);X)} = \sup_{t\geq 0} \|f\|_{L^q(t,t+1;X)} < \infty.$$

If $T < \infty$, we define $L^q_{\mathrm{uloc}}([0, T); X) := L^q(0, T; X)$.

In the following let $\Omega \subset \mathbb{R}^d$ be a domain. Then $W^m_q(\Omega)$, $m \in \mathbb{N}_0$, $1 \leq q \leq \infty$, denotes the usual L^q-Sobolev space, $W^m_{q,0}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^m_q(\Omega)$, $W^{-m}_q(\Omega) = (W^m_{q',0}(\Omega))'$, and $W^{-m}_{q,0}(\Omega) = (W^m_{q'}(\Omega))'$. The L^2-Bessel potential spaces are denoted by $H^s(\Omega)$, $s \geq 0$.

Let $f_\Omega = \frac{1}{|\Omega|}\int_\Omega f(x)\,dx$ denote the mean value of $f \in L^1(\Omega)$. For $m \in \mathbb{R}$ we define

$$L^q_{(m)}(\Omega) := \{f \in L^q(\Omega) : f_\Omega = m\}, \qquad 1 \leq q \leq \infty.$$

Then the orthogonal projection onto $L^2_{(0)}(\Omega)$ is given by

$$P_0 f := f - f_\Omega = f - \frac{1}{|\Omega|}\int_\Omega f(x)\,dx \qquad \text{for all } f \in L^2(\Omega).$$

For the following we denote

$$H^1_{(0)} \equiv H^1_{(0)}(\Omega) = H^1(\Omega) \cap L^2_{(0)}(\Omega), \qquad (c, d)_{H^1_{(0)}(\Omega)} := (\nabla c, \nabla d)_{L^2(\Omega)}.$$

Note that $H^1_{(0)}(\Omega)$ is a Hilbert space because of Poincaré's inequality. More generally, we define for $s \geq 0$

$$H^s_{(0)} \equiv H^s_{(0)}(\Omega) = H^s(\Omega) \cap L^2_{(0)}(\Omega), \quad H^{-s}_{(0)}(\Omega) = (H^s_{(0)}(\Omega))',$$
$$H^{-s}_0(\Omega) = (H^{-s}(\Omega))', \qquad H^{-s}(\Omega) = (H^s_0(\Omega))'.$$

Finally, $f \in H^s_{\mathrm{loc}}(\Omega)$ if and only if $f|_{\Omega'} \in H^s(\Omega')$ for every open and bounded subset Ω' with $\overline{\Omega'} \subset \Omega$.

We denote by $L^2_\sigma(\Omega)$ is the closure of $C^\infty_{0,\sigma}(\Omega)$ in $L^2(\Omega)^d$, where $C^\infty_{0,\sigma}(\Omega)$ is the set of all divergence free vector fields in $C^\infty_0(\Omega)^d$. The corresponding Helmholtz projection, i.e., the L^2-orthogonal projection onto $L^2_\sigma(\Omega)$, is denoted by P_σ, cf. e.g. Sohr [Soh01].

Spaces of continuous vector-fields: For the following we assume that $I = [0, T]$ with $0 < T < \infty$ or $I = [0, \infty)$ if $T = \infty$ and that X is a Banach space. The Banach space of all bounded and continuous $f\colon I \to X$ is denoted by $BC(I; X)$. It is equipped with the supremum norm. Moreover, $BUC(I; X)$ is defined as the subspace of all bounded and uniformly continuous functions. Furthermore, we define $BC_w(I; X)$ as the topological vector space of all bounded and weakly continuous functions $f\colon I \to X$. $C^\infty_0(0, T; X)$ denotes the vector space of all smooth functions $f\colon (0, T) \to X$ with $\operatorname{supp} f \subset\subset (0, T)$. By definition $f \in W^1_p(0, T; X)$, $1 \le p < \infty$, if and only if $f, \frac{df}{dt} \in L^p(0, T; X)$, where $\frac{df}{dt}$ denotes the vector-valued distributional derivative of f. Furthermore, $W^1_{p,\mathrm{uloc}}([0, \infty); X)$ is defined by replacing $L^p(0, T; X)$ by $L^p_{\mathrm{uloc}}([0, \infty); X)$ and we set $H^1(0, T; X) = W^1_2(0, T; X)$ and $H^1_{\mathrm{uloc}}([0, \infty); X) := W^1_{2,\mathrm{uloc}}([0, \infty); X)$. Finally, we note:

Lemma 2.1. *Let X, Y be two Banach spaces such that $Y \hookrightarrow X$ and $X' \hookrightarrow Y'$ densely. Then $L^\infty(I; Y) \cap BUC(I; X) \hookrightarrow BC_w(I; Y)$.*

For a proof see e.g. Abels [Abe09a].

2.1 Results on the Nonlocal Operator $\mathcal{L}$

In the following let $\mathcal{E}$ be defined as in (1.13). Assumptions (1.6)–(1.8) yield that there are positive constants c and C such that

$$c\|u\|^2_{H^{\frac{\alpha}{2}}(\Omega)} \leqslant |m(u)|^2 + \mathcal{E}(u, u) \leqslant C\|u\|^2_{H^{\frac{\alpha}{2}}(\Omega)} \qquad \text{for all}\, u \in H^{\frac{\alpha}{2}}(\Omega).$$

This implies that the following norm equivalences hold:

$$\mathcal{E}(u, u) \sim \|u\|^2_{H^{\frac{\alpha}{2}}(\Omega)} \qquad \text{for all}\, u \in H^{\frac{\alpha}{2}}_{(0)}(\Omega), \tag{2.15}$$

$$\mathcal{E}(u, u) + |m(u)|^2 \sim \|u\|^2_{H^{\frac{\alpha}{2}}(\Omega)} \qquad \text{for all}\, u \in H^{\frac{\alpha}{2}}(\Omega), \tag{2.16}$$

cf. [ABG15, Lemma 2.4 and Corollary 2.5].

In the following we will use a variational extension of the nonlocal linear operator $\mathcal{L}$ (see (1.5)) by defining $\mathcal{L}\colon H^{\frac{\alpha}{2}}(\Omega) \to H_0^{-\frac{\alpha}{2}}(\Omega)$ as

$$\langle \mathcal{L}u, \varphi\rangle_{H_0^{-\frac{\alpha}{2}}, H^{\frac{\alpha}{2}}} = \mathcal{E}(u, \varphi) \quad \text{for all } \varphi \in H^{\frac{\alpha}{2}}(\Omega).$$

In particular we have

$$\langle \mathcal{L}u, 1\rangle = \mathcal{E}(u, 1) = 0$$

by definition. We note that $\mathcal{L}$ agrees with (1.5) as soon as $u \in H^\alpha_{\mathrm{loc}}(\Omega) \cap H^{\frac{\alpha}{2}}(\Omega)$ and $\varphi \in C^\infty_0(\Omega)$, cf. [AK07, Lemma 4.2]. But this weak formulation also includes a natural boundary condition for u, cf. [ABG15, Theorem 6.1] for a discussion.

We will also need the following regularity result, which essentially states that the operator $\mathcal{L}$ is of lower order with respect to the usual Laplace operator. This result is from [ABG15, Lemma 2.6].

Lemma 2.2. *Let $g \in L^2_{(0)}(\Omega)$ and $\theta > 0$. Then the unique solution $u \in H^1_{(0)}(\Omega)$ for the problem*

$$-\theta \int_\Omega \nabla u \cdot \nabla \varphi + \mathcal{E}(u, \varphi) = (g, \varphi)_{L^2} \qquad \text{for all } \varphi \in H^1_{(0)}(\Omega) \tag{2.17}$$

belongs to $H^2_{loc}(\Omega)$ and satisfies the estimate

$$\theta \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{H^{\alpha/2}(\Omega)} \leqslant C\|g\|^2_{L^2(\Omega)},$$

where C is independent of $\theta > 0$ and g.

For the following let $\phi\colon [a, b] \to \mathbb{R}$ be a continuous function and set $\phi(x) = +\infty$ for $x \notin [a, b]$. As in [ABG15, Section 3] we fix $\theta \geqslant 0$ and consider the functional

$$F_\theta(c) = \frac{\theta}{2} \int_\Omega |\nabla c|^2 \, dx + \frac{1}{2}\mathcal{E}(c, c) + \int_\Omega \phi(c(x)) \, dx \tag{2.18}$$

where

$$\begin{aligned} \operatorname{dom} F_0 &= \left\{ c \in H^{\alpha/2}(\Omega) \cap L^2_{(m)}(\Omega) : \phi(c) \in L^1(\Omega) \right\}, \\ \operatorname{dom} F_\theta &= H^1(\Omega) \cap \operatorname{dom} F_0 \qquad \text{if } \theta > 0 \end{aligned}$$

for a fixed $m \in (a, b)$. Moreover, we define

$$\mathcal{E}_\theta(u, v) = \theta \int_\Omega \nabla u \cdot \nabla v \, dx + \mathcal{E}(u, v)$$

for all $u, v \in H^1(\Omega)$ if $\theta > 0$ and $u, v \in H^{\alpha/2}(\Omega)$ if $\theta = 0$.

We denote by $\partial F_\theta(c)\colon L^2_{(m)}(\Omega) \to \mathcal{P}(L^2_{(0)}(\Omega))$ the subgradient of F_θ at $c \in \operatorname{dom} F$ in the sense that $w \in \partial F_\theta(c)$ if and only if

$$(w, c' - c)_{L^2} \leqslant F_\theta(c') - F_\theta(c) \qquad \text{for all } c' \in L^2_{(m)}(\Omega).$$

The following characterization of $\partial F_\theta(c)$ is an important tool for the existence proof.

Theorem 2.3. *Let $\phi\colon [a, b] \to \mathbb{R}$ be a convex function that is twice continuously differentiable in (a, b) and satisfies $\lim_{x\to a} \phi'(x) = -\infty$, $\lim_{x\to b} \phi'(x) = +\infty$. Moreover, we set $\phi(x) = +\infty$ for $x \notin (a, b)$ and let F_θ be defined as in* (2.18)*. Then $\partial F_\theta\colon \mathcal{D}(\partial F_\theta) \subseteq L^2_{(m)}(\Omega) \to L^2_{(0)}(\Omega)$ is a single valued, maximal monotone operator with*

$$\begin{aligned} \mathcal{D}(\partial F_0) = \Big\{ & c \in H^\alpha_{loc}(\Omega) \cap H^{\alpha/2}(\Omega) \cap L^2_{(m)}(\Omega) : \phi'(c) \in L^2(\Omega), \exists f \in L^2(\Omega) : \\ & \mathcal{E}(c, \varphi) + \int_\Omega \phi'(c)\varphi \, dx = \int_\Omega f\varphi \, dx \quad \forall \varphi \in H^{\alpha/2}(\Omega) \Big\} \end{aligned}$$

if $\theta = 0$ and

$$\begin{aligned} \mathcal{D}(\partial F_\theta) = \Big\{ & c \in H^2_{loc}(\Omega) \cap H^1(\Omega) \cap L^2_{(m)}(\Omega) : \phi'(c) \in L^2(\Omega), \exists f \in L^2(\Omega) : \\ & \mathcal{E}_\theta(c, \varphi) + \int_\Omega \phi'(c)\varphi \, dx = \int_\Omega f\varphi \, dx \quad \forall \varphi \in H^1(\Omega) \Big\} \end{aligned}$$

if $\theta > 0$ *as well as*

$$\partial F_\theta(c) = -\theta\Delta c + \mathcal{L}c + P_0\phi'(c) \quad \text{in } \mathcal{D}'(\Omega) \qquad \text{for } \theta \geqslant 0.$$

Moreover, the following estimates hold

$$\theta\|c\|_{H^1}^2 + \|c\|_{H^{\alpha/2}}^2 + \|\phi'(c)\|_2^2 \leqslant C\left(\|\partial F_\theta(c)\|_2^2 + \|c\|_2^2 + 1\right) \tag{2.19}$$

$$\int_\Omega\int_\Omega (\phi'(c(x)) - \phi'(c(y)))(c(x) - c(y))k(x, y, x-y)\,dx\,dy \leqslant C\left(\|\partial F_\theta(c)\|_2^2 + \|c\|_2^2 + 1\right)$$

$$\theta\int_\Omega \phi''(c)|\nabla c|^2\,dx \leqslant C\left(\|\partial F_\theta(c)\|_2^2 + \|c\|_2^2 + 1\right)$$

for some constant $C > 0$ *independent of* $c \in \mathcal{D}(\partial F_\theta)$ *and* $\theta \geqslant 0$.

The result follows from [ABG15, Corollary 3.2 and Theorem 3.3].

3 Existence of Weak Solutions

In this section we define weak solutions for the system (1.1)-(1.4), (1.9)-(1.11) together with a natural boundary condition for φ given by the bilinear form $\mathcal{E}$, summarize the assumptions and state the main result.

Assumption 3.1. *Let* $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, *be a bounded domain with* C^2*-boundary. The following conditions hold true:*

(i) $\rho(\varphi) = \frac{1}{2}(\tilde\rho_1 + \tilde\rho_2) + \frac{1}{2}(\tilde\rho_2 - \tilde\rho_1)\varphi$ *for all* $\varphi \in [-1, 1]$.

(ii) $m \in C^1(\mathbb{R})$, $\eta \in C^0(\mathbb{R})$ *and there are constants* $m_0, K > 0$ *such that* $0 < m_0 \leq a(s), m(s), \eta(s) \leq K$ *for all* $s \in \mathbb{R}$.

(iii) $\Psi \in C([-1, 1]) \cap C^2((-1, 1))$ *and*

$$\lim_{s\to\pm 1}\Psi'(s) = \pm\infty\,, \quad \Psi''(s) \geq -\kappa \ \text{ for some } \kappa \in \mathbb{R}\,. \tag{3.20}$$

A standard example for a homogeneous free energy density Ψ satisfying the previous conditions is given by (1.14). Since for solutions we will have $\varphi(x, t) \in [-1, 1]$ almost everywhere, we only need the functions a, m, η on this interval. But for simplicity we assume a, m, η to be defined on $\mathbb{R}$.

Definition 3.2. *Let* $\mathbf{v}_0 \in L^2_\sigma(\Omega)$ *and* $\varphi_0 \in H^{\alpha/2}(\Omega)$ *with* $|\varphi_0| \leq 1$ *almost everywhere in* Ω *and let Assumption 3.1 be satisfied. Then* $(\mathbf{v}, \varphi, \mu)$ *such that*

$$\mathbf{v} \in BC_w([0,\infty); L^2_\sigma(\Omega)) \cap L^2(0,\infty; H^1_0(\Omega)^d)\,,$$
$$\varphi \in BC_w([0,\infty); H^{\alpha/2}(\Omega)) \cap L^2_{uloc}([0,\infty); H^\alpha_{loc}(\Omega))\,, \quad \Psi'(\varphi) \in L^2_{uloc}([0,\infty); L^2(\Omega))\,,$$
$$\mu \in L^2_{uloc}([0,\infty); H^1(\Omega)) \ \text{ with } \ \nabla\mu \in L^2(0,\infty; L^2(\Omega))$$

is called a weak solution of (1.1)-(1.4), (1.4)-(1.9) *if the following conditions hold true:*

$$
\begin{aligned}
-(\rho\mathbf{v}, \partial_t\boldsymbol{\psi})_Q + (\operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v}), \boldsymbol{\psi})_Q + (2\eta(\varphi)D\mathbf{v}, D\boldsymbol{\psi})_Q - \Big((\mathbf{v}\otimes\widetilde{\mathbf{J}}), \nabla\boldsymbol{\psi}\Big)_Q \\
= -(\varphi\nabla\mu, \boldsymbol{\psi})_Q
\end{aligned}
\tag{3.21}
$$

for all $\boldsymbol{\psi} \in C_0^\infty(\Omega\times(0,\infty))^d$ *with* $\operatorname{div}\boldsymbol{\psi} = 0$,

$$
-(\varphi, \partial_t\psi)_Q + (\mathbf{v}\cdot\nabla\varphi, \psi)_Q = -(m(\varphi)\nabla\mu, \nabla\psi)_Q \tag{3.22}
$$

$$
\int_0^\infty \int_\Omega \mu\psi\,dx\,dt = \int_0^\infty \int_\Omega \Psi'(\varphi)\psi\,dx\,dt + \int_0^\infty \mathcal{E}(\varphi(t), \psi(t))\,dt \tag{3.23}
$$

for all $\psi \in C_0^\infty((0,\infty); C^1(\overline{\Omega}))$ *and*

$$
(\mathbf{v}, \varphi)|_{t=0} = (\mathbf{v}_0, \varphi_0)\ . \tag{3.24}
$$

Finally, the energy inequality

$$
\begin{aligned}
E_{tot}(\varphi(t), \mathbf{v}(t)) + \int_s^t \int_\Omega 2\eta(\varphi)\,|D\mathbf{v}|^2\,dx\,d\tau + \int_s^t \int_\Omega m(\varphi)|\nabla\mu|^2\,dx\,d\tau \\
\leq E_{tot}(\varphi(s), \mathbf{v}(s))
\end{aligned}
\tag{3.25}
$$

holds true for all $t \in [s,\infty)$ *and almost all* $s \in [0,\infty)$ *(including* $s = 0$*). Here* E_{tot} *is as in* (1.12).

The main result of [AT18] is:

Theorem 3.3 (Existence of Weak Solutions [AT18])**.**
Let Assumption 3.1 hold. Then for every $\mathbf{v}_0 \in L^2_\sigma(\Omega)$ *and* $\varphi_0 \in H^{\alpha/2}(\Omega)$ *such that* $|\varphi_0| \leq 1$ *almost everywhere and* $\fint_\Omega \varphi_0\,dx \in (-1,1)$ *there exists a weak solution* $(\mathbf{v}, \varphi, \mu)$ *of* (1.1)-(1.4), (1.9)-(1.11).

4 Approximation by an Implicit Time Discretization

Let Ψ be as in Assumption 3.1. We define $\Psi_0\colon [-1,1] \to \mathbb{R}$ by $\Psi_0(s) = \Psi(s) + \kappa\frac{s^2}{2}$ for all $s \in [a,b]$. Then $\Psi_0\colon [-1,1] \to \mathbb{R}$ is convex and $\lim_{s\to\pm1}\Psi_0'(s) = \pm\infty$. A basic idea for the following is to use this decomposition to split the free energy E_{free} into a singular convex part E and a quadratic perturbation. In the equations this yields a decomposition into a singular monotone operator and a linear remainder. To this end we define an energy $E\colon L^2(\Omega) \to \mathbb{R}\cup\{+\infty\}$ with domain

$$
\operatorname{dom} E = \{\varphi \in H^{\alpha/2}(\Omega) \mid -1 \leq \varphi \leq 1 \ \text{ a.e.}\}
$$

given by

$$
E(\varphi) = \begin{cases} \frac{1}{2}\mathcal{E}(\varphi,\varphi) + \int_\Omega \Psi_0(\varphi)\,dx & \text{for } \varphi \in \operatorname{dom} E\,, \\ +\infty & \text{else}\,. \end{cases} \tag{4.26}
$$

This yields the decomposition

$$E_{\text{free}}(\varphi) = E(\varphi) - \frac{\kappa}{2}\|\varphi\|_{L^2}^2 \qquad \text{for all } \varphi \in \text{dom } E.$$

Moreover, E is convex and $E = F_0$ if one chooses $\phi = \Psi_0$ and F_0 is as in Subsection 2.1. This is a key relation for the following analysis in order to make use of Theorem 2.3, which in particular implies that $\partial E = \partial F_0$ is a maximal monotone operator.

In order to prove Theorem 3.3, we use an implicit time discretization. To this end, let $h = \frac{1}{N}$ for $N \in \mathbb{N}$ and $\mathbf{v}_k \in L^2_\sigma(\Omega)$, $\varphi_k \in H^1(\Omega)$ with $\varphi_k(x) \in [-1, 1,]$ almost everywhere and $\rho_k = \frac{1}{2}(\tilde\rho_1 + \tilde\rho_2) + \frac{1}{2}(\tilde\rho_2 - \tilde\rho_1)\varphi_k$ be given. Then $\Psi(\varphi_k) \in L^1(\Omega)$. We also define a smoothing operator P_h on $L^2(\Omega)$ as follows. We choose u as the solution of the following heat equation

$$\begin{cases} \partial_t u - \Delta u &= 0 \quad \text{in } \Omega \times (0,T)\,, \\ u|_{t=0} &= \varphi' \quad \text{on } \Omega\,, \\ \partial_\nu u|_{\partial\Omega} &= 0 \quad \text{on } \partial\Omega \times (0,T), \end{cases}$$

where $\varphi' \in L^2(\Omega)$, and set $P_h\varphi' := u|_{t=h}$. Then $P_h\varphi' \in H^2(\Omega)$ and $P_h\varphi' \to \varphi'$ in $L^2(\Omega)$ as $h \to 0$ for all $\varphi' \in L^2(\Omega)$. Moreover, we have $|P_h\varphi'| \le 1$ in Ω if $|\varphi'(x)| \le 1$ almost everywhere and $P_h\varphi' \to_{h\to 0} \varphi'$ in $H^{\frac{\alpha}{2}}(\Omega)$ as $h \to 0$ for all $\varphi' \in H^{\frac{\alpha}{2}}(\Omega)$.

Now we determine $(\mathbf{v}, \varphi, \mu) = (\mathbf{v}_{k+1}, \varphi_{k+1}, \mu_{k+1})$, $k \in \mathbb{N}$, successively as solution of the following problem: Find $\mathbf{v} \in H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$, $\varphi \in \mathcal{D}(\partial E)$ and

$$\mu \in H^2_n(\Omega) = \{u \in H^2(\Omega) \mid \partial_{\mathbf{n}} u|_{\partial\Omega} = 0 \text{ on } \partial\Omega\},$$

such that

$$\begin{aligned} \left(\frac{\rho\mathbf{v} - \rho_k\mathbf{v}_k}{h}, \boldsymbol{\psi}\right)_\Omega &+ (\operatorname{div}(\rho(P_h\varphi_k)\mathbf{v}\otimes\mathbf{v}), \boldsymbol{\psi})_\Omega + (2\eta(\varphi_k)D\mathbf{v}, D\boldsymbol{\psi})_\Omega + \left(\operatorname{div}(\mathbf{v}\otimes\widetilde{\mathbf{J}}), \boldsymbol{\psi}\right)_\Omega \\ &= -\left((P_h\varphi_k)\nabla\mu, \boldsymbol{\psi}\right)_\Omega \end{aligned} \tag{4.27}$$

for all $\boldsymbol{\psi} \in C^\infty_{0,\sigma}(\Omega)$,

$$\frac{\varphi - \varphi_k}{h} + \mathbf{v}\cdot\nabla P_h\varphi_k = \operatorname{div}\left(m(P_h\varphi_k)\nabla\mu\right) \quad \text{almost everywhere in } \Omega\,, \tag{4.28}$$

and

$$\int_\Omega \left(\mu + \kappa\,\frac{\varphi + \varphi_k}{2}\right)\psi\,dx = \mathcal{E}(\varphi,\psi) + \int_\Omega \Psi_0'(\varphi)\psi\,dx + h\int_\Omega \nabla\varphi\cdot\nabla\psi\,dx \tag{4.29}$$

for all $\psi \in H^{\alpha/2}(\Omega)$, where

$$\widetilde{\mathbf{J}} \equiv \widetilde{\mathbf{J}}_{k+1} := -\tfrac{\tilde\rho_2 - \tilde\rho_1}{2} m(P_h\varphi_k)\nabla\mu_{k+1} = -\tfrac{\tilde\rho_2 - \tilde\rho_1}{2} m(P_h\varphi_k)\nabla\mu\,.$$

For the following let

$$E_{\text{tot},h}(\varphi,\mathbf{v}) = \int_\Omega \rho\frac{|\mathbf{v}|^2}{2}\,dx + \int_\Omega \Psi(\varphi)\,dx + \frac{1}{2}\mathcal{E}(\varphi,\varphi) + \frac{h}{2}\int_\Omega |\nabla\varphi|^2\,dx. \tag{4.30}$$

denote the total energy of the system (4.27)-(4.29).

Remark 4.1. *(i) As in [ADG13] we obtain the important relation*

$$-\frac{\rho-\rho_k}{h}-\mathbf{v}\cdot\nabla\rho(P_h\varphi_k)=\operatorname{div}\widetilde{\mathbf{J}}\,,$$

by multiplication of (4.28) *with* $-\frac{\tilde\rho_2-\tilde\rho_1}{2}=\frac{\partial\rho(\varphi)}{\partial\varphi}$*. Because of* $\operatorname{div}(\mathbf{v}\otimes\widetilde{\mathbf{J}})=(\operatorname{div}\widetilde{\mathbf{J}})\mathbf{v}+\left(\widetilde{\mathbf{J}}\cdot\nabla\right)\mathbf{v}$ *this yields that*

$$\begin{aligned}&\left(\frac{\rho\mathbf{v}-\rho_k\mathbf{v}_k}{h},\boldsymbol{\psi}\right)_\Omega+(\operatorname{div}(\rho(P_h\varphi_k)\mathbf{v}\otimes\mathbf{v}),\boldsymbol{\psi})_\Omega+(2\eta(\varphi_k)D\mathbf{v},D\boldsymbol{\psi})_\Omega\\&+\left(\left(\operatorname{div}\widetilde{\mathbf{J}}-\frac{\rho-\rho_k}{h}-\mathbf{v}\cdot\nabla\rho(P_h\varphi_k)\right)\frac{\mathbf{v}}{2},\boldsymbol{\psi}\right)_\Omega+\left(\left(\widetilde{\mathbf{J}}\cdot\nabla\right)\mathbf{v},\boldsymbol{\psi}\right)_\Omega=-\left((P_h\varphi_k)\nabla\mu,\boldsymbol{\psi}\right)_\Omega\end{aligned}\tag{4.31}$$

for all $\boldsymbol{\psi}\in C^\infty_{0,\sigma}(\Omega)$ *to* (4.27)*, which will be used to derive suitable a-priori estimates.*

(ii) Integrating (4.28) *in space one obtains* $\int_\Omega\varphi\,dx=\int_\Omega\varphi_k\,dx$ *because of* $\operatorname{div}\mathbf{v}=0$ *and the boundary conditions.*

The following lemma is important to control the derivative of the singular free energy density $\Psi'(\varphi)$. For its proof, we refer to [AT18].

Lemma 4.2 ([AT18])**.** *Let* $\varphi\in\mathcal{D}(\partial F_h)$ *and* $\mu\in H^1(\Omega)$ *be a solution of* (4.29) *for given* $\varphi_k\in H^1(\Omega)$ *with* $|\varphi_k(x)|\le1$ *almost everywhere in* Ω *such that*

$$\tfrac{1}{|\Omega|}\int_\Omega\varphi\,dx=\tfrac{1}{|\Omega|}\int_\Omega\varphi_k\,dx\in(-1,1)\,.$$

Then there is a constant $C=C(\int_\Omega\varphi_k,\Omega)>0$*, independent of* φ,μ,φ_k*, such that*

$$\|\Psi_0'(\varphi)\|_{L^2(\Omega)}+\left|\int_\Omega\mu\,dx\right|\le C(\|\nabla\mu\|_{L^2}+\|\nabla\varphi\|^2_{L^2}+1)\text{ and}$$
$$\|\partial F_h(\varphi)\|_{L^2(\Omega)}\le C\left(\|\mu\|_{L^2}+1\right)\,.$$

The following lemma is about the existence of solutions to the time-discrete system. For its proof, we could follow the line of the corresponding arguments in [ADG13]. The main tools are Theorem 2.3 and Leray-Schauder principle. For details, we refer to [AT18]. As before we denote

$$H^2_n(\Omega):=\{u\in H^2(\Omega):\mathbf{n}\cdot\nabla u|_{\partial\Omega}=0\}.$$

Lemma 4.3 ([AT18])**.** *For every* $\mathbf{v}_k\in L^2_\sigma(\Omega)$*,* $\varphi_k\in H^1(\Omega)$ *with* $|\varphi_k(x)|\le1$ *almost everywhere, and* $\rho_k=\frac12(\tilde\rho_1+\tilde\rho_2)+\frac12(\tilde\rho_2-\tilde\rho_1)\varphi_k$ *there is some solution* $(\mathbf{v},\varphi,\mu)\in\left(H^1_0(\Omega)^d\cap L^2_\sigma(\Omega)\right)\times\mathcal{D}(\partial F_h)\times H^2_n(\Omega)$ *of the system* (4.28)-(4.29) *and* (4.31)*. Moreover, the solution satisfies the discrete energy estimate*

$$\begin{aligned}E_{tot,h}(\varphi,\mathbf{v})&+\int_\Omega\rho_k\frac{|\mathbf{v}-\mathbf{v}_k|^2}{2}\,dx+\int_\Omega\frac{|\nabla\varphi-\nabla\varphi_k|^2}{2}\,dx+\frac12\mathcal{E}(\varphi-\varphi_k,\varphi-\varphi_k)\\&+h\int_\Omega2\eta(\varphi_k)|D\mathbf{v}|^2\,dx+h\int_\Omega m(\varphi_k)|\nabla\mu|^2\,dx\le E_{tot,h}(\varphi_k,\mathbf{v}_k)\,.\end{aligned}\tag{4.32}$$

5 Proof of the Main Result

5.1 Compactness in Time

In order to prove our main result Theorem 3.3 we will pass to the limit $h \to 0$ resp. $N \to \infty$ for the approximate solution, which are obtain by suitable interpolations of our time-discrete solutions. To this end let $N \in \mathbb{N}$ be given and let $(\mathbf{v}_{k+1}, \varphi_{k+1}, \mu_{k+1})$, $k \in \mathbb{N}$, be chosen successively as a solution of (4.27)-(4.29) with $h = \frac{1}{N}$ and $(\mathbf{v}_0, \varphi_0^N)$ as initial value.

As in [ADG13] we define $f^N(t)$ for $t \in [-h, \infty)$ by the relation $f^N(t) = f_k$ for $t \in [(k-1)h, kh)$, where $k \in \mathbb{N}_0$ and $f \in \{\mathbf{v}, \varphi, \mu\}$. Moreover, let $\rho^N = \frac{1}{2}(\tilde{\rho}_1 + \tilde{\rho}_2) + \frac{1}{2}(\tilde{\rho}_2 - \tilde{\rho}_1)\varphi^N$. Furthermore we introduce the notation

$$\left(\Delta_h^+ f\right)(t) := f(t+h) - f(t)\,, \qquad \left(\Delta_h^- f\right)(t) := f(t) - f(t-h)\,,$$
$$\partial_{t,h}^{\pm} f(t) := \frac{1}{h}\left(\Delta_h^{\pm} f\right)(t)\,, \qquad f_h := \left(\tau_h^* f\right)(t) = f(t-h)\,.$$

In order to derive the weak formulation in the limit let $\boldsymbol{\psi} \in \left(C_0^\infty(\Omega \times (0,\infty))\right)^d$ with $\operatorname{div} \boldsymbol{\psi} = 0$ be arbitrary and choose $\widetilde{\boldsymbol{\psi}} := \int_{kh}^{(k+1)h} \boldsymbol{\psi}\, dt$ as test function in (4.27). By summation with respect to $k \in \mathbb{N}_0$ this yields

$$\int_0^\infty\!\!\int_\Omega \partial_{t,h}^-(\rho^N \mathbf{v}^N) \cdot \boldsymbol{\psi}\, dx\, dt + \int_0^\infty\!\!\int_\Omega \operatorname{div}\left(\rho_h^N \mathbf{v}^N \otimes \mathbf{v}^N\right) \cdot \boldsymbol{\psi}\, dx\, dt + \int_0^\infty\!\!\int_\Omega 2\eta(\varphi_h^N) D\mathbf{v}^N : D\boldsymbol{\psi}\, dx\, dt$$
$$- \int_0^\infty\!\!\int_\Omega \left(\mathbf{v}^N \otimes \widetilde{\mathbf{J}}^N\right) : D\boldsymbol{\psi}\, dx\, dt = -\int_0^\infty\!\!\int_\Omega \nabla\mu^N \varphi_h^N \cdot \boldsymbol{\psi}\, dx\, dt \tag{5.33}$$

for all $\boldsymbol{\psi} \in \left(C_0^\infty(\Omega \times (0,\infty))\right)^d$ with $\operatorname{div} \boldsymbol{\psi} = 0$. Using a simple change of variable, one sees

$$\int_0^\infty\!\!\int_\Omega \partial_{t,h}^-(\rho^N \mathbf{v}^N) \cdot \boldsymbol{\psi}\, dx\, dt = -\int_0^\infty\!\!\int_\Omega (\rho^N \mathbf{v}^N) \cdot \partial_{t,h}^+ \boldsymbol{\psi}\, dx\, dt$$

for sufficiently small $h > 0$. In the same way one derives

$$\int_0^\infty\!\!\int_\Omega \partial_{t,h}^- \varphi^N\, \zeta\, dx\, dt + \int_0^\infty\!\!\int_\Omega \mathbf{v}^N \varphi_h^N \cdot \nabla\zeta\, dx\, dt = \int_0^\infty\!\!\int_\Omega m(\varphi_h^N) \nabla\mu^N \cdot \nabla\zeta\, dx\, dt \tag{5.34}$$

for all $\zeta \in C_0^\infty((0,\infty); C^1(\overline{\Omega}))$ as well as

$$\int_0^\infty \int_\Omega (\mu^N + \kappa\, \frac{\varphi^N + \varphi_h^N}{2})\psi\, dx\, dt = \int_0^\infty \mathcal{E}(\varphi^N, \psi)\, dt + \int_0^\infty \int_\Omega \Psi_0'(\varphi^N)\psi\, dx\, dt$$
$$+ h \int_0^\infty \int_\Omega \nabla\varphi^N \cdot \nabla\psi\, dx\, dt \tag{5.35}$$

for all $\psi \in C_0^\infty((0,\infty); C^1(\overline{\Omega}))$.

Let $E^N(t)$ be defined as

$$E^N(t) = \frac{(k+1)h - t}{h} E_{\text{tot}}(\varphi_k, \mathbf{v}_k) + \frac{t - kh}{h} E_{\text{tot}}(\varphi_{k+1}, \mathbf{v}_{k+1}) \quad \text{for} \quad t \in [kh, (k+1)h)$$

and define

$$D^N(t) := \int_\Omega 2\eta(\varphi_k)|D\mathbf{v}_{k+1}|^2\,dx + \int_\Omega m(\varphi_k)|\nabla\mu_{k+1}|^2\,dx$$

for all $t \in (t_k, t_{k+1})$, $k \in \mathbb{N}_0$. Then the discrete energy estimate (4.32) yields

$$-\frac{d}{dt}E^N(t) = \frac{E_{\text{tot}}(\varphi_k, \mathbf{v}_k) - E_{\text{tot}}(\varphi_{k+1}, \mathbf{v}_{k+1})}{h} \geq D^N(t) \tag{5.36}$$

for all $t \in (t_k, t_{k+1})$, $k \in \mathbb{N}_0$. Integration implies

$$\begin{aligned} E_{\text{tot}}(\varphi^N(t), \mathbf{v}^N(t)) + \int_s^t \int_\Omega \left(2\eta(\varphi_h^N)|D\mathbf{v}^N|^2 + m(\varphi_h^N)|\nabla\mu^N|^2\right) dx\,d\tau \\ \leq E_{\text{tot}}(\varphi^N(s), \mathbf{v}^N(s)) \end{aligned} \tag{5.37}$$

for all $0 \leq s \leq t < \infty$ with $s, t \in h\mathbb{N}_0$.

Because of Lemma 4.2 and since $E_{\text{tot}}(\varphi_0^N, \mathbf{v}_0)$ is bounded, we conclude that

$$\begin{aligned} &(\mathbf{v}^N)_{N\in\mathbb{N}} \subseteq L^2(0,\infty; H^1(\Omega)^d) \cap L^\infty(0,\infty; L^2(\Omega)^d)\,, \\ &(\nabla\mu^N)_{N\in\mathbb{N}} \subseteq L^2(0,\infty; L^2(\Omega)^d)\,, \\ &(\varphi^N)_{N\in\mathbb{N}} \subseteq L^\infty(0,\infty; H^{\frac{\alpha}{2}}(\Omega))\,, \text{ and} \\ &(h^{\frac{1}{2}}\nabla\varphi^N)_{N\in\mathbb{N}} \subseteq L^\infty(0,\infty; L^2(\Omega)) \end{aligned} \tag{5.38}$$

are bounded. Moreover, there is a nondecreasing $C\colon (0,\infty) \to (0,\infty)$ such that

$$\int_0^T \left|\int_\Omega \mu^N\,dx\right| dt \leq C(T) \text{ for all } 0 < T < \infty\,.$$

Therefore there are subsequences (denoted again by the index $N \in \mathbb{N}$, $h > 0$, respectively) such that

$$\begin{aligned} \mathbf{v}^N &\rightharpoonup \mathbf{v} \text{ in } L^2(0,\infty; H^1(\Omega)^d)\,, \\ \mathbf{v}^N &\rightharpoonup^* \mathbf{v} \text{ in } L^\infty(0,\infty; L^2(\Omega)^d)\,, \\ \varphi^N &\rightharpoonup^* \varphi \text{ in } L^\infty(0,\infty; H^{\frac{\alpha}{2}}(\Omega))\,, \\ \mu^N &\rightharpoonup \mu \text{ in } L^2(0,T; H^1(\Omega)) \text{ for all } 0 < T < \infty\,, \\ \nabla\mu^N &\rightharpoonup \nabla\mu \text{ in } L^2(0,\infty; L^2(\Omega)^d)\,, \end{aligned}$$

where $\mu \in L^2_{uloc}([0,\infty); H^1(\Omega))$.

In the following $\widetilde{\varphi}^N$ denotes the piecewise linear interpolant of $\varphi^N(t_k)$ in time, where $t_k = kh$, $k \in \mathbb{N}_0$. Then $\partial_t\widetilde{\varphi}^N = \partial_{t,h}^-\varphi^N$ and therefore

$$\|\widetilde{\varphi}^N - \varphi^N\|_{H^{-1}(\Omega)} \leq h\|\partial_t\widetilde{\varphi}^N\|_{H^{-1}(\Omega)}\,. \tag{5.39}$$

Using that $\mathbf{v}^N\varphi^N$ and $\nabla\mu^N$ are bounded in $L^2(0,\infty; L^2(\Omega)^d)$ and (5.34) we conclude that $\partial_t\widetilde{\varphi}^N \in L^2(0,\infty; H^{-1}(\Omega))$ is bounded. Since $(\varphi^N)_{N\in\mathbb{N}}$ and therefore $(\widetilde{\varphi}^N)_{N\in\mathbb{N}}$ are bounded in $L^\infty(0,\infty; H^{\frac{\alpha}{2}}(\Omega))$, the Lemma of Aubin-Lions yields

$$\widetilde{\varphi}^N \to \widetilde{\varphi} \text{ in } L^2(0,T; L^2(\Omega)) \tag{5.40}$$

for all $0 < T < \infty$ for some $\widetilde{\varphi} \in L^\infty(0,\infty;L^2(\Omega))$ (and a suitable subsequence). In particular $\widetilde{\varphi}^N(x,t) \to \widetilde{\varphi}(x,t)$ almost every $(x,t) \in \Omega \times (0,\infty)$. Because of (5.39),

$$\|\widetilde{\varphi}^N - \varphi^N\|_{L^2(-h,\infty;H^{-1}(\Omega))} \to 0 \tag{5.41}$$

and thus $\widetilde{\varphi} = \varphi$. Since $\widetilde{\varphi}^N \in H^1_{\mathrm{uloc}}([0,\infty);H^{-1}(\Omega)) \cap L^\infty([0,\infty);H^{\frac{\alpha}{2}}(\Omega)) \hookrightarrow BUC([0,\infty);L^2(\Omega))$ and $\widetilde{\varphi}^N \in L^\infty(0,\infty;H^{\frac{\alpha}{2}}(\Omega))$ are bounded, Lemma 2.1 implies $\varphi \in BC_w([0,\infty);H^{\frac{\alpha}{2}}(\Omega))$. Moreover, $(\widetilde{\varphi}^N - \varphi^N)_{N\in\mathbb{N}} \subseteq L^\infty(-h,\infty;H^{\frac{\alpha}{2}}(\Omega))$ is bounded since $(\varphi^N)_{N\in\mathbb{N}}, (\widetilde{\varphi}^N)_{N\in\mathbb{N}} \subseteq L^\infty(-h,\infty;H^{\frac{\alpha}{2}}(\Omega))$ are bounded. By interpolation with (5.41) we conclude

$$\widetilde{\varphi}^N - \varphi^N \to 0 \text{ in } L^2(-h,T;L^2(\Omega)) \tag{5.42}$$

and therefore

$$\varphi^N \to \varphi \text{ in } L^2(0,T;L^2(\Omega)) \tag{5.43}$$

for all $0 < T < \infty$. Moreover, we have

$$\begin{aligned}\|\varphi_h^N - \varphi\|_{L^2(0,T;L^2(\Omega))} &\le \|\varphi_h^N - \varphi_h\|_{L^2(0,T;L^2(\Omega))} + \|\varphi_h - \varphi\|_{L^2(0,T;L^2(\Omega))}\\ &\le h^{\frac{1}{2}}\|\varphi_0^N\|_{L^2(\Omega)} + \|\varphi^N - \varphi\|_{L^2(0,T-h;L^2(\Omega))} + \|\varphi_h - \varphi\|_{L^2(0,T;L^2(\Omega))}.\end{aligned} \tag{5.44}$$

Because of $\|\varphi_h - \varphi\|_{L^2(0,T;L^2(\Omega))} \to_{h\to 0} 0$, we obtain $\|\varphi_h^N - \varphi\|_{L^2(0,T;L^2(\Omega))} \to_{h\to 0} 0$.

Finally using the bounds of $\widetilde{\varphi}^N$ in $H^1(0,T;H^{-1}(\Omega))$ and in $L^\infty(0,T;H^\alpha(\Omega))$ for all $0 < T < \infty$ as well as $\widetilde{\varphi}^N \to \varphi$ in $L^2(0,T;L^2(\Omega))$ we conclude $\widetilde{\varphi}^N(0) \to \varphi(0)$ in $L^2(\Omega)$. Since $\widetilde{\varphi}^N(0) = \varphi_0^N \to_{N\to\infty} \varphi_0$ in $L^2(\Omega)$, we derive $\varphi(0) = \varphi_0$.

Since ρ^N depends affine linearly on φ^N, the conclusions hold true for ρ^N.

To show the convergence of (5.35), we closely follow the corresponding argument in [ABG15]. The only difference is that work on the space-time domains directly, while they work on the spacial domains fixing a time variable in [ABG15]. We include the argument here for completeness. We first observe that $\Psi_0'(\varphi^N)$ is bounded in $L^2(0,T;L^2(\Omega))$ for $0 < T < \infty$. Using this bound, we can pass to a subsequence such that $\Psi_0'(\varphi^N)$ converges weakly in $L^2(0,T;L^2(\Omega))$ to χ for all $0 < T < \infty$ as N tends to infinity. Let $\psi \in C_0^\infty((0,\infty);C^1(\overline{\Omega}))$. Thanks to the convergences listed above, we can pass to the limit $N \to \infty$ in (5.35) to find

$$\int_0^\infty \int_\Omega (\mu + \kappa\varphi)\psi \, dx \, dt = \int_0^\infty \mathcal{E}(\varphi,\psi)\, dt + (\chi,\psi)_{L^2((0,\infty)\times\Omega)}.$$

To show (3.23), we only have to identify the weak limit $\chi = \lim_{N\to\infty} \Psi_0'(\varphi^N)$. Let $T > 0$. Since (5.43) holds, passing to a subsequence, we have $\varphi^N \to \varphi$ almost everywhere in $\Omega \times (0,T)$. On the other hand, thanks to Egorov's theorem, there exists a set $Q_m \subset \Omega \times (0,T)$ such that $|Q_m| \ge |\Omega \times (0,T)| - \frac{1}{2m}$ and on which $\varphi^N \to \varphi$ uniformly. We now use (uniform with respect to N) estimate on $\Psi_0'(\varphi^N)$ in $L^2(\Omega \times (0,T)$. By definition, the quantity

$$M_{\delta,N} = \left|\left\{(x,t) \in \Omega \times (0,T) \mid |\varphi^N(x,t)| > 1-\delta\right\}\right|$$

is decreasing in δ for all $n \in \mathbb{N}$. Since $\Psi_0'(y)$ is unbounded for $y \to \pm 1$, we set

$$c_\delta := \inf_{|c|\ge 1-\delta} |\Psi_0'(c)| \to_{\delta\to 0} \infty,$$

we have by the Tschebychev inequality

$$\int_{\Omega\times(0,T)} |\Psi_0'(\varphi^N)|^2\,dx\,dt \ge c_\delta^2 |M_{\delta,N}|.$$

From the uniform (with respect to N) estimate of the norm of $\Psi_0'(\varphi^N)$ in $L^2(\Omega\times(0,T))$, we obtain $M_{\delta,n}\to 0$ for $\delta\to 0$ uniformly in $n\in\mathbb{N}$. Therefore, we deduce

$$\lim_{\delta\to 0}\left|\left\{(x,t)\in\Omega\times(0,T)\mid |\varphi^N(x,t)|>1-\delta\right\}\right|=0$$

uniformly in $N\in\mathbb{N}$. Thus there exists $\delta=\delta(m)$ independent of N, such that

$$\left|\left\{(x,t)\in\Omega\times(0,T)\mid |\varphi^N(x,t)|>1-\delta\right\}\right|\le\frac{1}{2m},\qquad\forall N\in\mathbb{N}$$

Consider now $N\in\mathbb{N}$ so large that by uniform convergence we have $|\varphi^{N'}(x,t)-\varphi^N(x,t)|<\frac{\delta}{2}$ for all $N'\ge N$ and all $(x,t)\in Q_m$. Moreover, let $Q'_{mN}\subset Q_m$ be defined by

$$Q'_{mN}=Q_m\cap\left\{(x,t)\in\Omega\times(0,T)\mid |\varphi^N(x,t)|\le 1-\delta\right\}.$$

By the above construction, we immediately deduce that $|Q'_{mN}|\ge|\Omega\times(0,T)|-\frac{1}{m}$ and that $\left|\varphi^{N'}(x,t)\right|<1-\frac{\delta}{2}$ for all $N'\ge N$ and for all $(x,t)\in Q_{m,N}$. Therefore by the regularity assumptions on the potential Ψ_0', we deduce that $\Psi_0'(\varphi^N)\to\Psi_0'(\varphi)$ uniformly on Q'_{mN}. Since m is arbitrary, we have $\Psi_0'(\varphi^N)\to\Psi_0'(\varphi)$ almost everywhere in $\Omega\times(0,T)$. By a diagonal argument, passing to a subsequence, we have $\Psi_0'(\varphi^N)\to\Psi_0'(\varphi)$ almost everywhere in $\Omega\times(0,\infty)$ and $\Psi_0'(\varphi^N)\to\Psi_0'(\varphi)$ as $h\to 0$ in $L^q(Q_T)$ for every $1\le q<2$ and $0<T<\infty$. Finally, the uniqueness of weak and strong limits gives $\chi=\Psi_0'(\varphi)$ as claimed.

The next step is to show strong convergence $\mathbf{v}^N\to\mathbf{v}$ in $L^2(0,T;L^2(\Omega)^d)$ for all $0<T<\infty$ to conclude a convergence pointwise almost everywhere. As above let $\widetilde{\rho\mathbf{v}}^N$ be the piecewise linear interpolant of $\left(\rho^N\mathbf{v}^N\right)(t_k)$, where $t_k=kh$, $h\in\mathbb{N}_0$. Then it holds that $\partial_t\left(\widetilde{\rho\mathbf{v}}^N\right)=\partial_{t,h}^-\left(\rho^N\mathbf{v}^N\right)$.

Using that

$$\begin{aligned}
\rho_h^N\mathbf{v}^N\otimes\mathbf{v}^N &\text{ is bounded in } L^2(0,T;L^{\frac{3}{2}}(\Omega)),\\
D\mathbf{v}^N &\text{ is bounded in } L^2(0,T;L^2(\Omega)),\\
\mathbf{v}^N\otimes\nabla\mu^N &\text{ is bounded in } L^{\frac{8}{7}}(0,T;L^{\frac{4}{3}}(\Omega)),\\
\nabla\mu^N\varphi_h^N &\text{ is bounded in } L^2(0,T;L^{\frac{6}{6-\alpha}}(\Omega)).
\end{aligned}$$

together with (5.33), we obtain that $\partial_t\left(\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N)\right)$ is bounded in $L^{\frac{8}{7}}(0,T;(W_6^1(\Omega))')$ for all $0<T<\infty$. Here we remark that the boundedness of $\nabla\mu^N\in L^2(0,T;L^2(\Omega))$ and $\varphi_h^N\in L^\infty(0,T;L^{\frac{6}{3-\alpha}}(\Omega))$ imply that $\nabla\mu^N\varphi_h^N\in L^2(0,T;L^{\frac{6}{6-\alpha}}(\Omega))\hookrightarrow L^2(0,T;L^{\frac{6}{5}}(\Omega))$ is bounded.

Since $\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N)\in L^2(0,T;H^1(\Omega)^d)$ is bounded, the Lemma of Aubin-Lions implies

$$\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N)\to\mathbf{w}\ \text{ in }\ L^2(0,T;L^2(\Omega)^d)$$

for all $0 < T < \infty$ for some $\mathbf{w} \in L^\infty(0,\infty; L^2(\Omega)^d)$. Since the projection $\mathbb{P}_\sigma : L^2(0,T;L^2(\Omega)^d) \to L^2(0,T;L^2_\sigma(\Omega))$ is weakly continuous, we conclude from the weak convergence $\widetilde{\rho\mathbf{v}}^N \rightharpoonup \rho\mathbf{v}$ in $L^2(0,T;L^2(\Omega))$ that $\mathbf{w} = \mathbb{P}_\sigma(\rho\mathbf{v})$. This yields

$$\int_0^T \int_\Omega \rho^N |\mathbf{v}^N|^2 = \int_0^T \int_\Omega \mathbb{P}_\sigma(\rho^N \mathbf{v}^N) \cdot \mathbf{v}^N \longrightarrow \int_0^T \int_\Omega \mathbb{P}_\sigma(\rho\mathbf{v}) \cdot \mathbf{v} = \int_0^T \int_\Omega \rho\, |\mathbf{v}|^2$$

because of $\mathbb{P}_\sigma(\rho^N \mathbf{v}^N) \to_{N\to\infty} \mathbb{P}_\sigma(\rho\mathbf{v})$ in $L^2(0,T;L^2(\Omega)^d)$. Since weak convergence and convergence of the norms imply strong convergence in a Hilbert space, we conclude $(\rho^N)^{\frac{1}{2}}\mathbf{v}^N \to (\rho)^{\frac{1}{2}}\mathbf{v}$ in $L^2(0,T;L^2(\Omega)^d)$. Because of

$$\rho^N \to \rho \ \text{ almost everywhere in } \ (0,\infty)\times\Omega \ \text{ and } \ |\rho^N| \geq c > 0\,,$$

we derive

$$\mathbf{v}^N = (\rho^N)^{-\frac{1}{2}} \left((\rho^N)^{\frac{1}{2}} \mathbf{v}^N \right) \to_{N\to\infty} \mathbf{v} \ \text{ in } \ L^2(0,T;L^2(\Omega)^d)\,.$$

This yields in particular that $\mathbf{v}^N \to_{N\to\infty} \mathbf{v}$ pointwise almost everywhere in $(0,\infty)\times\Omega$ (for a subsequence).

Using these convergence results together with the fact that for all divergence free $\boldsymbol{\psi}$ the following convergence holds

$$\int_0^T \int_\Omega \nabla\mu^N P_N \varphi_h^N \cdot \boldsymbol{\psi}\, dx\, dt \to_{N\to\infty} \int_0^T \int_\Omega \nabla\mu\varphi \cdot \boldsymbol{\psi}\, dx\, dt\,,$$

we can pass to the limit in the equations (5.33), (5.34) to get (3.21), (3.22). The fact that $\mathbf{v}(0) = \mathbf{v}_0$ in $L^2(\Omega)^d$ is shown in the same way as in [ADG13]. Therefore we omit the proof.

5.2 Energy Inequality

It remains to show the energy inequality (3.25). If we show that $\varphi^N(t) \to \varphi(t)$ in $H^{\frac{\alpha}{2}}_{(m)}$ for almost every $t \in (0,\infty)$ and $\sqrt{h}\nabla\varphi^N \to 0$ in $(L^2(\Omega))^d$ for almost every $t \in (0,\infty)$, the rest of the proof is almost the same as in [ADG13] and we omit it. To this end it suffices to show $(\varphi^N, \sqrt{h}\nabla\varphi^N)$ converges strongly to $(\varphi, 0)$ in $L^2(0,T; H^{\frac{\alpha}{2}}_{(m)}(\Omega) \times (L^2(\Omega))^d)$ for every $T > 0$. If we take $\psi = \varphi^N$ in (5.35), we have

$$\int_0^\infty \int_\Omega \left(\mu^N + \kappa \frac{\varphi^N + \varphi_h^N}{2} \right) \varphi^N \, dx\, dt = \int_0^\infty \mathcal{E}(\varphi^N, \varphi^N)\, dt + \int_0^\infty \int_\Omega \Psi_0'(\varphi^N)\varphi^N \, dx\, dt$$
$$+ h \int_0^\infty \int_\Omega \nabla\varphi^N \cdot \nabla\varphi^N \, dx\, dt\,. \tag{5.45}$$

Since $\varphi^N \to \varphi$ in $L^2(Q_T)$, $\mu^N \rightharpoonup \mu$ in $L^2(Q_T)$ and $\Psi_0'(\varphi^N) \rightharpoonup \Psi_0'(\varphi^N)$ in $L^2(Q_T)$ as $N \to \infty$, we have

$$\lim_{N\to\infty} \left\{ \int_0^\infty \mathcal{E}(\varphi^N(t), \varphi^N(t))\, dt + h \int_0^\infty \int_\Omega \nabla\varphi^N \cdot \nabla\varphi^N \, dx\, dt \right\}$$
$$= \int_0^\infty \int_\Omega (\mu\varphi + \kappa\varphi^2)\, dx\, dt - \int_0^\infty \int_\Omega \Psi_0'(\varphi)\varphi \, dx\, dt = \int_0^\infty \mathcal{E}(\varphi(t), \varphi(t))\, dt \tag{5.46}$$

because of (3.23).

Next we show $\varphi^N \rightharpoonup \varphi$ in $L^2(0,T;H^{\frac{\alpha}{2}}_{(m)})$ and $\sqrt{h}\nabla\varphi^N \rightharpoonup 0$ in $L^2(0,T;L^2)$ as $N\to\infty$ for any $T>0$. Let $T>0$ be arbitrarily fixed. $(\varphi^N)_{N\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H^{\frac{\alpha}{2}}_{(m)})$, hence also in $L^2(0,T;H^{\frac{\alpha}{2}}_{(m)})$. Then there exists some $\varphi'\in L^2(0,T;H^{\frac{\alpha}{2}}_{(m)})$ such that $\varphi^N\rightharpoonup\varphi'$ in $L^2(0,T;H^{\frac{\alpha}{2}}_{(m)})$. Since $\varphi^N\to\varphi$ in $L^2(Q_T)$, $\varphi=\varphi'$. Hence $\varphi^N\rightharpoonup\varphi$ in $L^2(0,T;H^{\frac{\alpha}{2}}_{(m)})$.

For any fixed $\boldsymbol{\psi}\in C_0^\infty(Q_T)^d$,

$$\int_{Q_T}\sqrt{h}\ \nabla\varphi_N\cdot\boldsymbol{\psi}\,d(x,t) = -\int_{Q_T}\sqrt{h}\ \varphi^N\ \mathrm{div}\ \boldsymbol{\psi}\,d(x,t)$$

tends to zero as $N\to\infty$ since $\varphi^N\to\varphi$ in $L^2(Q_T)$. Since $\sup_{N\in\mathbb{N}}\|\sqrt{h}\nabla\varphi^N\|_{L^2(Q_T)^d}<\infty$ and $\overline{C_0^\infty(Q_T)^d}^{\|\cdot\|_{L^2(Q_T)^d}} = L^2(Q_T)^d$, we have $\sqrt{h}\nabla\varphi^N\rightharpoonup 0$ in $L^2(Q_T)^d$. Hence we have $(\varphi^N,\sqrt{h}\nabla\varphi^N)\rightharpoonup(\varphi,0)$ in $L^2(0,T;H^{\frac{\alpha}{2}}_{(m)}\times(L^2)^d)$.

Because of (5.46), we also have the convergence of the norms of $(\varphi^N,\sqrt{h}\nabla\varphi^N)$ to that of $(\varphi,0)$ in $L^2(0,T;H^{\frac{\alpha}{2}}_{(m)}\times(L^2)^d)$. Hence we have shown the claim.

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References

[Abe09a] Abels H., *Existence of weak solutions for a diffuse interface model for viscous, incompressible fluids with general densities*, Comm. Math. Phys., vol. 289 (2009), p.45-73.

[Abe09b] Abels H., *On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities*, Arch. Rat. Mech. Anal., vol. 194 (2009), p.463-506.

[Abe11] Abels H., *Strong well-posedness of a diffuse interface model for a viscous, quasi-incompressible two-phase flow*, SIAM J. Math. Anal., vol. 44(1) (2012), p.316-340.

[ABG15] Abels H., Bosia S., Grasselli M., *Cahn-Hilliard equation with nonlocal singular free energies*, Ann. Mat. Pura Appl. vol. 194(4) (2015), p.1071-1106.

[ADG13] Abels H., Depner D., Garcke H., *Existence of weak solutions for a diffuse interface model for two-phase flows of incompressible fluids with different densities*, J. Math. Fluid Mech. vol. 15(3) (2013), p.453-480.

[AGG11] Abels H., Garcke H., Grün G., *Thermodynamically consistent, frame indifferent diffuse interface models for incompressible two-phase flows with different densities*, Math. Models Meth. Appl. Sci., vol. 22(3) (2012) 1150013, 40 pp.

[AK07] Abels H., Kassmann M., *An analytic approach to purely nonlocal Bellman equations arising in models of stochastic control*, J. Differ. Eqs. vol. 236 (2007), p.29-56.

[AT18] Abels H., Terasawa Y., *Weak solutions for a diffuse interface model for two-phase flows of incompressible fluids with different densities and nonlocal free energies*, arXiv:1806.01030v2 (2018), 26pp.

[AW07] Abels H., Wilke M., *Convergence to equilibrium for the Cahn-Hilliard equation with a logarithmic free energy*, Nonlin. Anal., vol. 67 (2007), p.3176-3193.

[CH58] Cahn, J.W., Hilliard, J.E., *Free energy of a nonuniform system. I. Interfacial energy*, J. Chem. Phys., vol. 28(2) (1958), p.258-267.

[CFG12] Colli P., Frigeri M., Grasselli M. *Global existence of weak solutions to a nonlocal Cahn-Hilliard-Navier-Stokes system*, J. Math. Anal. Appl. vol. 386 (2012), p.428-444.

[DSS07] Ding H., Spelt P.D.M., Shu C., *Diffuse interface model for incompressible two-phase flows with large density ratios*, J. Comp. Phys., vol. 22 (2007), p.2078-2095.

[F15] Frigeri S., *Global existence of weak solutions for two-phase flows of incompressible fluids with unmatched densities*, Math. Models Methods Appl. Sci. 26 (2016), p. 1955-1993.

[FG12a] Frigeri S., Grasselli M., *Global and trajectories attractors for a nonlocal Cahn-Hilliard-Navier-Stokes-system*, J. Dynam. Differential Equations, vol. 24 (2012), p.827-856.

[FG12b] Frigeri S., Grasselli M., *Nonlocal Cahn-Hilliard-Navier-Stokes systems with singular potentials*, Dyn. Partial Differ. Equ., vol. 9 (2012), p.273-304.

[G14] Gal, C.G., Grasselli, M., *Longtime behavior of nonlocal Cahn-Hilliard equations*, Discrete Contin. Dyn. Syst. Ser. A, vol. 34 (2014), p.145-179.

[GZ03] Gajewski, H., Zacharias, K. *On a nonlocal phase separation model*, J. Math. Anal. Appl., vol. 286 (2003), p.11-31.

[GL97] Giacomin, G., Lebowitz, J.L., *Phase segregation dynamics in particle systems with long range interations. I. Macroscopic limits*, J. Stat. Phys., vol. 87 (1997), p.37-61.

[GL98] Giacomin, G., Lebowitz, J.L., *Phase segregation dynamics in particle systems with long range interations. II. Interface motion*, SIAM J. Appl. Math. vol. 58 (1998), p.1707-1729.

[GPV96] Gurtin M.E., Polignone D., Viñals J., *Two-phase binary fluids and immiscible fluids described by an order parameter*, Math. Models Meth. Appl. Sci., vol. 6(6) (1996), p.815-831.

[HH77] Hohenberg P.C., Halperin B.I., *Theory of dynamic critical phenomena*, Rev. Mod. Phys., vol. 49 (1977), p.435-479.

[LP11a] Londen, S.-O., Petzeltová, H., *Convergence of solutions of a non-local phase-field system*, Discret. Contin. Dyn. Syst. Ser. S, vol. 4 (2011), p.653-670.

[LP11b] Londen, S.-O., Petzeltová, H., *Regularity and separation from potential barriers for a non-local phase-field system*, J. Math. Anal. Appl., vol. 379 (2011), p.724-735.

[LT98] Lowengrub J., Truskinovsky L., *Quasi-incompressible Cahn-Hilliard fluids and topological transitions*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., vol. 454 (1998), p.2617-2654.

[NNG08] Nec, A.A., Nepomnyashchy, Y., Golovin, A.A., *Front-type solutions of fractional Allen-Cahn equation*, Phys. D, vol. 237 (2008), p.3237-3251.

[Soh01] Sohr H., *The Navier-Stokes Equations*, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser Verlag, Basel, 2001.

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