Partially reversible capital investment with both fixed and proportional costs under demand risk*

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Abstract

This study investigates a firm's capital expansion and reduction policy with both fixed and proportional costs when the output demand follows the geometric Brownian motion. We formulate the firm's problem as an impulse control problem and solve it by using quasi-variational inequalities. Through numerical analysis, we find that the output demand risk delays the capital expansion and reduction. Furthermore, the output demand risk decreases the magnitude of capital expansion, but it increases that of capital reduction.

Keywords: Capital expansion and reduction; quasi-variational inequalities; stochastic impulse control

1 Introduction

Irreversibility and uncertainty are two main features in capital investment decision making (Pindyck, 1991). Suppose that a firm's manager considers a capital investment to produce an output. The demand of the output is governed by a stochastic differential equation, so the firm's manager faces output demand risk. If the manager decides to invest in the capital, the investment expenditure is generally sunk. Real options analysis has revealed the effects of uncertainty on investment timing if the investment is irreversible (Caballero, 1991; Sarkar, 2000; Wong, 2007).

In this study, we relax the condition of irreversibility associated with capital investment. We consider that a firm can sell a capital at a secondary market or to another firm. Then, the investment cost is partially sunk. This relaxation generates a new problem to the firm. If the investment expenditure is totally sunk, the firm only should consider the timing of capital expansion and its size. By contrast, if the investment expenditure is partially sunk, the firm must consider the timing of capital reduction and its size, in addition to the optimization problem of capital expansion. Then, the firm faces the problem of when and how much the firm expands and reduces its capital.

Abel and Eberly (1996), Guo and Pham (2005), Merhi and Zervos (2007), De Angelis and Ferrari (2014), Federico and Pham (2014), and Tsujimura (2019) examine the capital expansion and reduction problem when changing the level of capital incurs a cost proportional to its size. In particular, the proportional cost represents the price of capital per unit. They formulated the firms problem as singular stochastic control problems. This problem is characterized by two thresholds. In case the controlled diffusion process, which represents the level of capital stock directly or indirectly, reaches a threshold, the firm increases (or reduces) the capital to ensure

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that the controlled diffusion process does not cross the thresholds. Consequently, the controlled diffusion process is a reflected diffusion process.

If changing the level of capital incurs a fixed and a proportional cost, the size of changing the level of capital is larger than that in the absence of the fixed cost to cover both fixed and proportional costs. Then, the firm's capital investment problem with these two types of cost is formulated as an impulse control problem. The firm's problem is characterized by four thresholds. The two thresholds determine the timing of changing the capital, whereas the other two thresholds determine the size of changing the capital. Once the controlled diffusion process, which represents the level of capital stock directly or indirectly, similar to that in the singular stochastic control problems, reaches one threshold between the former two thresholds, the firm increases (or reduces) the capital to ensure that the controlled diffusion process does not cross the thresholds. Thus, the controlled diffusion process jumps the other associated threshold between the later two thresholds due to the existence of the fixed cost. For more details on both stochastic controls, refer to, for example, Øksendal (1999), Bensoussan et al. (2010), and Tsujimura (2020). Bar-Ilan et al. (2002), Bensoussan and Chevalier-Roignant (2019), and Federico et al. (2019) examine the capital expansion problem when changing the capital level requires both fixed and proportional costs. This study extends these analyses, including capital reduction, with both fixed and proportional costs, and reveals the optimal timing and size of capital expansion and reduction.

We formulate the firm's problem as an impulse control problem and solve it by using quasi-variational inequalities (QVI), and then we derive the optimal capital investment policy, which is characterized by four thresholds to increase and reduce the capital. We verify that the QVI policy is an optimal capital expansion and reduction policy (if it exists). Through numerical analysis, we conduct a comparative static analysis to ensure that our analysis provides useful implications for firms' investment decision making.

The remainder of this paper is organized as follows. We describe the setup of a firm's capital expansion and reduction problem in Section 2. Section 3 defines the quasi-variational inequalities to solve the firm's problem. Section 4 solves the problem of the firm through the QVI. We numerically derive the optimal capital expansion and reduction policy and show some comparative static results in Section 5. Finally, Section 6 concludes the paper.

2 Firm's Capital Investment Problem

In this section, we formulate a firm's capital expansion and reduction problem. The firm produces an output using capital K and sells it in a competitive market. Demand of the output X is random, and its process is followed by the geometric Brownian motion:

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t, \quad X_{0^-} = x > 0, \tag{2.1}$$

where $\mu > 0$ and $\sigma > 0$ are constants. Moreover, W_t is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$. The firm controls the capital corresponding to the output demand. Let ζ_i be the ith amount of change in capital at time τ_i, $i = 0, 1, \cdots$. We assume that

$\tau_0 = 0$. The dynamics of the capital stock are governed by

$$\begin{cases} \mathrm{d}K_t = -\delta K_t \mathrm{d}t, & \tau_i \le t < \tau_{i+1}, \\ K_{\tau_i} = K_{\tau_i^-} + \zeta_i, \\ K_{0^-} = k \ (> 0), \end{cases} \tag{2.2}$$

where $\delta \in (0,1)$ is a constant depreciation rate. The firm's operating profit $\hat{\pi}$ at time t is given by

$$\hat{\pi}(K_t, X_t) = K_t^{\alpha} X_t^{\beta}, \tag{2.3}$$

where $\alpha \in (0,1)$, $\beta > 0$.

Changing the capital stock incurs a fixed $c > 0$ and a proportional cost. The firm can purchase a capital at a constant unit price $p > 0$ and sell it at a constant price $(1-\lambda)p > 0$. The purchase and sale price is the proportional cost. The parameter $\lambda \in (0,1)$ represents the degree of irreversibility of investment. The investment cost is completely sunk if λ goes to 1. The capital expansion and reduction costs are given by

$$C(\zeta_i) = \begin{cases} c + p\zeta_i, & \zeta_i > 0, \\ c, & \zeta_i = 0, \\ c + (1-\lambda)p\zeta_i, & \zeta_i < 0. \end{cases} \tag{2.4}$$

Note that, for all ζ, ζ', $C(\zeta)$ satisfies the following:

$$C(\zeta + \zeta') \le C(\zeta) + C(\zeta'). \tag{2.5}$$

Inequality (2.5) represents subadditivity with respect to ζ, implying that reasonable $\{\mathcal{F}_t\}_{t\ge 0}$-stopping times become strictly increasing sequences; that is, $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_i < \cdots < \infty$.

A capital expansion and reduction policy $\hat{v}$ is defined as the following double sequence:

$$\hat{v} := \{(\tau_i, \zeta_i)\}_{i \ge 0}. \tag{2.6}$$

The firm's expected discounted profit $\hat{J}(k, x; \hat{v})$ is given by

$$\hat{J}(k, x; \hat{v}) = \mathbb{E}\left[\int_0^{\infty} \mathrm{e}^{-rt} \hat{\pi}(K_t, X_t) \mathrm{d}t - \sum_{i=1}^{\infty} \mathrm{e}^{-r\tau_i} C(\zeta_i) \mathbb{I}_{\{\tau_i < \infty\}}\right], \tag{2.7}$$

where $r > 0$ is the discount rate, $\zeta := \{\zeta_i\}_{i \ge 0}$, and $\mathbb{I}_{\Theta}$ is the indicator function of the set Θ.

Hereinafter, for simplicity, we assume that $\beta = 1 - \alpha$, as in Abel and Eberly (1996) and change variables as $Y_t := K_t / X_t$. Then, the operating profit function $\hat{\pi}$ and the expected discounted profit $\hat{J}(k, x; \hat{v})$, respectively, can be rewritten as follows:

$$\hat{\pi}(K_t, X_t) = K_t^{\alpha} X_t^{1-\alpha} = Y_t^{\alpha} X_t = \pi(Y_t) X_t. \tag{2.8}$$

$$\hat{J}(k, x) = x \hat{J}\left(\frac{k}{x}, 1\right) = x J(y). \tag{2.9}$$

The firm's expected discounted profit is rewritten in the following form:

$$J(y;v) = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\pi(Y_t)\mathrm{d}t - \sum_{i=1}^\infty \mathrm{e}^{-r\tau_i}C(\xi_i)\mathbb{I}_{\{\tau_i<\infty\}}\right], \tag{2.10}$$

where v is the capital expansion and reduction policy is composed by the capital expansion and reduction timing and its magnitude $\xi := \zeta/x$:

$$v := \{(\tau_i, \xi_i)\}_{i\geq 0}. \tag{2.11}$$

Proposition 2.1. *The firm's expected discounted profit function $J(y;v)$ is well defined and finite if the following conditions hold:*

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}Y_t^\alpha \mathrm{d}t\right] < \infty \tag{2.12}$$

$$\lim_{t\to\infty} \mathbb{E}\left[\mathrm{e}^{-rt}Y_t\mathrm{d}t\right] = 0 \tag{2.13}$$

$$P\left[\lim_{i\to\infty} \tau_i \leq \hat{T}\right] = 0, \quad \hat{T} \in [0,\infty), \tag{2.14}$$

Proof. Condition (2.12) implies that

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\pi(Y_t)\mathrm{d}t\right] < \infty. \tag{2.15}$$

Condition (2.14) implies that the capital expansion and reduction policy will only occur finitely before a terminal time, $\hat{T}$. If condition (2.14) holds, then we obtain

$$\mathbb{E}\left[\sum_{i=1}^\infty \mathrm{e}^{-r\tau_i}\mathbb{I}_{\{\tau_i<\infty\}}\right] < \infty. \tag{2.16}$$

According to integration by parts formula, for every $0 < s \leq t < \infty$ (Rogers and Williams, 2000, VI38), we have

$$\mathbb{E}[\mathrm{e}^{-rt}Y_t] - \mathbb{E}[\mathrm{e}^{-rs}Y_s] = -(r+\delta+\mu-\sigma^2)\mathbb{E}\left[\int_s^t \mathrm{e}^{-ru}Y_t\mathrm{d}u\right] + \mathbb{E}\left[\sum_{i=1}^\infty \mathrm{e}^{-r\tau_i}\xi_i\mathbb{I}_{\{s<\tau_i\leq t\}}\right]. \tag{2.17}$$

Through (2.13), we obtain

$$\mathbb{E}\left[\sum_{i=1}^\infty \mathrm{e}^{-r\tau_i}\xi_i\mathbb{I}_{\{s<\tau_i\leq t\}}\right] < \infty. \tag{2.18}$$

Combining (2.16) and (2.18) yields the finite expected discounted costs.

$$\mathbb{E}\left[\sum_{i=1}^\infty \mathrm{e}^{-r\tau_i}C(\xi_i)\mathbb{I}_{\{\tau_i<\infty\}}\right] < \infty \tag{2.19}$$

Therefore, the firm's expected discounted profit function, J, is well defined and finite.

□

We define a set of admissible capital expansion and reduction policy as follows:

Definition 2.1 (Admissible capital expansion and reduction policy). *A capital expansion and reduction policy v are admissible if conditions (2.12)–(2.14) hold. Let $\mathcal{V}$ be a set of admissible capital expansion and reduction policy.*

The firm's problem is to maximize the expected discounted profit J over $\mathcal{V}$.

$$V(y) = \sup_{v \in \mathcal{V}} J(y; v) = J(y; v^*), \tag{2.20}$$

where V is the value function and v^* is the optimal capital expansion and reduction policy. The firm's problem (2.20) is formulated as a stochastic impulse control problem.

3 QVI for the Firm's Problem

Based on the formulation of the firm's problem (2.20), we can guess the following optimal capital expansion and reduction policy. The firm controls the level of y within a region to ensure that, once the level of y reaches a threshold, the firm instantaneously increases the level of y by ξ. Conversely, once the level of y reaches the other threshold, the firm immediately reduces the level of y by ξ'. Recall that the variable y is defined by $y := k/x$. Thus, the above policy implies the following capital expansion and reduction policy. The firm maintains the capital stock level within a given region to ensure that, once the output demand reaches a threshold, the firm purchases the capital and expands the capital by ζ_i. Once the output demand reaches the other threshold, the firm sells the capital and reduces the capital by ζ_i'. To verify this conjecture, we prove that a policy induced by quasi-variational inequalities is an optimal capital expansion and reduction policy for the firm's problem (2.20).

Suppose that $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function. The quasi-variational inequalities of the firm's problem are given as follows.

Definition 3.1 (QVI). *The following relations are referred to as the QVI for the firm's problem:*

$$\mathcal{L}\phi(y) + \pi(y) \leq 0; \tag{3.1}$$

$$\phi(y) \geq \mathcal{M}\phi(y); \tag{3.2}$$

$$[\mathcal{L}\phi(y) + \pi(y)][\mathcal{M}\phi(y) - \phi(y)] = 0, \tag{3.3}$$

where $\mathcal{L}$ is the differential operator and $\mathcal{M}$ is the capital expansion and reduction operator. They are defined as follows:

$$\mathcal{L}\phi(y) := -(\delta + \mu)y\phi'(y) + \frac{1}{2}\sigma^2 y^2 \phi''(y) - (r - \mu)\phi(y), \tag{3.4}$$

$$\mathcal{M}\phi(y) = \sup_{\xi \in \mathbb{R}, y+\xi \in (0,\infty)} \{\phi(y+\xi) - C(\xi)\}. \tag{3.5}$$

It should be noted that the derivatives of the firm's expected discounted profit function are calculated as $\hat{J}_K(k, x) = J'(y)$, $\hat{J}_X(k, x) = J(y) - yJ'(y)$, and $\hat{J}_{XX}(k, x) = (y^2/x)J''(y)$.

The definition of the QVI enables us to divide the interval $(0,\infty)$ into three regions: the continuation region $\mathcal{H}$, capital expansion region $\mathcal{E}$, and capital reduction region $\mathcal{R}$. They are, respectively, given as follows:

$$\mathcal{H} := \{y \in (0,\infty); \phi(y) > \mathcal{M}\phi(y) \text{ and } \mathcal{L}V(y) + \pi(y) = 0\}; \tag{3.6}$$

$$\mathcal{E} := \{y \in (0,\infty); \phi(y) = \mathcal{M}\phi(y) \text{ and } \mathcal{L}V(y) + \pi(y) = 0, \xi > 0\}; \tag{3.7}$$

$$\mathcal{R} := \{y \in (0,\infty); \phi(y) = \mathcal{M}\phi(y) \text{ and } \mathcal{L}V(y) + \pi(y) = 0, \xi < 0\}. \tag{3.8}$$

We define a policy that is derived from the QVI.

Definition 3.2 (QVI policy)**.** *Let ϕ be a solution to QVI (3.1)–(3.3). Then, the following capital expansion and reduction policy $\tilde{v}$ is referred to as a QVI policy:*

$$(\tilde{\tau}_0, \tilde{\xi}_0) = (0,0); \tag{3.9}$$

$$\tilde{\tau}_i = \inf\{t \geq \tilde{\tau}_{i-1}; Y_t \notin \mathcal{H}\}; \tag{3.10}$$

$$\tilde{\xi}_i = \arg\max\left\{\phi\left(Y_{\tilde{\tau}_i^-} + \xi_i\right) - C(\xi_i); \xi_i \in \mathbb{R}, Y_{\tilde{\tau}_i^-} + \xi_i \in (0,\infty)\right\}. \tag{3.11}$$

We verify that the QVI policy is the optimal capital expansion and reduction policy. The following is the well-known verification theorem. We mainly refer to Brekke and Øksendal (1998, Theorem 3.1), Cadenillas and Zapatero (1999, Theorem 3.1), and Wu (2019, Theorem 1).

Theorem 3.1. *(I) Let ϕ be a solution of the QVI. Suppose that ϕ is a C^1-function for $y \in (0,\infty)$ and is a C^2 -function for $y \in (0,\infty) - \mathcal{N}$, where $\mathcal{N}$ is a finite subset of $(0,\infty)$. Suppose that there exists $0 < L < U < \infty$ such that ϕ is linear in $y \in (0,L] \cup [U,\infty)$. Furthermore, we assume that the family $\{\phi(Y_\tau^v)\}_{\tau<\infty}$ is uniformly integrable for all $y \in (0,\infty)$ and $v \in \mathcal{V}$. Then, for all $y \in (0,\infty)$, we obtain*

$$\phi(y) \geq V(y). \tag{3.12}$$

(II) If the QVI-policy corresponding to ϕ is admissible, then it is an optimal impulse control, and for all $y \in (0,\infty)$, we obtain

$$\phi(y) = V(y). \tag{3.13}$$

That is, ϕ is the value function, and $\tilde{v}$ is the corresponding optimal policy.

Proof. (I) The differentiability of ϕ implies its boundedness in the interval (L,U). The first derivative of ϕ, $\phi'(y)$, is bounded by $y \in (0,\infty)$. This is because $\phi'(y)$ is continuous for $y \in (L,U)$ and is constant for $y \in (0,L] \cup [U,\infty)$. Furthermore, condition (2.13) and the property of ϕ: bounded for $y \in (L,U)$ and linear for $y \in (0,L] \cup [U,\infty)$ imply that

$$\lim_{t\to\infty} \mathbb{E}[\mathrm{e}^{-rt}\phi(Y_t)] = 0. \tag{3.14}$$

Choose $v \in \mathcal{V}$. Let $\theta_{i+1} := \tau_i \vee (\tau_{i+1} \wedge s)$ for any $s \geq 0$. Then, through the generalized Dynkin formula and (3.1) that we obtain

$$\mathbb{E}\left[\mathrm{e}^{-r\theta_{i+1}^-}\phi\left(Y_{\theta_{i+1}^-}\right)\right] \leq \mathbb{E}\left[\mathrm{e}^{-r\tau_i}\phi\left(Y_{\tau_i}\right)\right] - \mathbb{E}\left[\int_{\tau_i}^{\theta_{i+1}^-} \mathrm{e}^{-rt}\pi(Y_t)\mathrm{d}t\right]. \tag{3.15}$$

Using $\lim_{s\to\infty}$, we have by the dominated convergence theorem:

$$\mathbb{E}\left[\mathrm{e}^{-r\tau_{i+1}^{-}}\phi\left(Y_{\tau_{i+1}^{-}}\right)\right]\leq\mathbb{E}\left[\mathrm{e}^{-r\tau_i}\phi\left(Y_{\tau_i}\right)\right]-\mathbb{E}\left[\int_{\tau_i}^{\tau_{i+1}^{-}}\mathrm{e}^{-rt}\pi(Y_t)\mathrm{d}t\right].\tag{3.16}$$

Summing from $i=0$ to $i=n$ yields

$$\mathbb{E}\left[\int_0^{\tau_{n+1}^{-}}\pi(Y_t)\mathrm{d}t\right]\leq\phi(y)+\sum_{i=1}^{n}\mathbb{E}\left[\mathrm{e}^{-r\tau_i}\phi\left(Y_{\tau_i}\right)-\mathrm{e}^{-r\tau_i^{-}}\phi\left(Y_{\tau_i^{-}}\right)\right]-\mathbb{E}\left[\mathrm{e}^{-r\tau_{n+1}^{-}}\phi\left(Y_{\tau_{n+1}^{-}}\right)\right].\tag{3.17}$$

After changing the capital level, the variable Y jumps immediately from $Y_{\tau_i^{-}}$ to $Y_{\tau_i}(=Y_{\tau_i^{-}}+\xi_i)$ for all $\tau_i<\infty$; it follows from (3.5) that we obtain

$$\mathcal{M}\phi\left(Y_{\tau_i^{-}}\right)+C(\xi_i)\geq\phi\left(Y_{\tau_i}\right).\tag{3.18}$$

Applying (3.18) to (3.17), we obtain

$$\begin{aligned}\mathbb{E}\left[\int_0^{\tau_{n+1}^{-}}\pi(Y_t)\mathrm{d}t\right]\leq&\phi(y)+\sum_{i=1}^{n}\mathbb{E}\left[\mathrm{e}^{-r\tau_i}\left(\mathcal{M}\phi\left(Y_{\tau_i^{-}}\right)+C(\xi_i)\right)-\mathrm{e}^{-r\tau_i^{-}}\phi\left(Y_{\tau_i^{-}}\right)\right]\\&-\mathbb{E}\left[\mathrm{e}^{-r\tau_{n+1}^{-}}\phi\left(Y_{\tau_{n+1}^{-}}\right)\right].\end{aligned}\tag{3.19}$$

Rewriting (3.19) yields

$$\begin{aligned}\mathbb{E}\left[\int_0^{\tau_{n+1}^{-}}\pi(Y_t)\mathrm{d}t-\sum_{i=1}^{n}\mathrm{e}^{-r\tau_i}C(\xi_i)\right]\leq&\phi(y)+\sum_{i=1}^{n}\mathbb{E}\left[\mathrm{e}^{-r\tau_i}\mathcal{M}\phi\left(Y_{\tau_i^{-}}\right)-\mathrm{e}^{-r\tau_i^{-}}\phi\left(Y_{\tau_i^{-}}\right)\right]\\&-\mathbb{E}\left[\mathrm{e}^{-r\tau_{n+1}^{-}}\phi\left(Y_{\tau_{n+1}^{-}}\right)\right].\end{aligned}\tag{3.20}$$

Through (3.2), we obtain

$$\mathcal{M}\phi\left(Y_{\tau_i^{-}}\right)-\phi\left(Y_{\tau_i^{-}}\right)\leq 0.\tag{3.21}$$

Applying (3.21) to (3.20) leads to the following inequality:

$$\mathbb{E}\left[\int_0^{\tau_{n+1}^{-}}\pi(Y_t)\mathrm{d}t-\sum_{i=1}^{n}\mathrm{e}^{-r\tau_i}C(\xi_i)\right]\leq\phi(y)-\mathbb{E}\left[\mathrm{e}^{-r\tau_{n+1}^{-}}\phi\left(Y_{\tau_{n+1}^{-}}\right)\right].\tag{3.22}$$

Because the family $\{\phi(Y_\tau)\}_{\tau<\infty}$ is uniformly integrable, by letting $n\to\infty$ and using (3.14) and the dominated convergence theorem, we obtain

$$\mathbb{E}\left[\int_0^{\infty}\pi(Y_t)\mathrm{d}t-\sum_{i=1}^{\infty}\mathrm{e}^{-r\tau_i}C(\xi_i)\mathbb{I}_{\tau_i<\infty}\right]\leq\phi(y).\tag{3.23}$$

The left-hand side of (3.23) is the firm's expected discounted profit $J(y;v)$. For the arbitrariness of $v\in\mathcal{V}$, we have

$$\sup_{v\in\mathcal{V}}J(y;v)\leq\phi(y).\tag{3.24}$$

Then, we have $V(y)\leq\phi(y)$.

(II) Assume that the QVI policy $\tilde{v}$ is applied. For the continuation region, (3.1) holds with equality. We repeat that the augmentation in part (I) for $v = \tilde{v}$. Then, all inequalities become equalities. Hence, we have

$$J(y;\tilde{v}) = \phi(y). \tag{3.25}$$

Therefore, $\phi(y) = V(y)$ and $\tilde{v} = v^*$ is optimal; that is, the solution of the QVI is the value function and the QVI policy is optimal.

□

4 Solution of the Firm's Problem

From the analysis of the previous section, we assume that, under a suitable set of sufficient conditions on the given parameters, an optimal capital expansion and reduction policy $v^* \in \mathcal{V}$ is characterized by the following form: once the level of Y reaches $\underline{y}$ (or $\bar{y}$), firm purchases (or sells) the capital to ensure that it instantaneously increases (or decreases) to the other level of Y, $\underset{\sim}{y}$ (or $\tilde{y}$), Hence, the level of Y changes by $\underset{\sim}{y} - \underline{y}$ (or $\tilde{y} - \bar{y}$) at each time τ_i.

Let $v^* = (\tau^*, \xi^*) \in \mathcal{V}$ be an optimal capital expansion and reduction policy such that

$$\tau_i^* := \inf\left\{t > \tau_{i-1}^*; Y_{t^-} \notin \left(\underline{y}, \bar{y}\right)\right\}; \tag{4.1}$$

$$\xi_i^* := Y_{\tau_i} - Y_{\tau_i^-} = \begin{cases} \underset{\sim}{y} - \underline{y}, & Y_{\tau_i^-} = \underline{y}, \\ \tilde{y} - \bar{y}, & Y_{\tau_i^-} = \bar{y}. \end{cases} \tag{4.2}$$

Then, the three regions, namely, the continuation region, capital expansion region, and capital reduction region, are replaced as follows:

$$\mathcal{H} := \{y; \underline{y} < y < \bar{y}\}, \quad \mathcal{E} := \{y; y \le \underline{y}\}, \quad \mathcal{R} := \{y; y \ge \bar{y}\}. \tag{4.3}$$

For $y \in \mathcal{H}$, QVI (3.1)–(3.3) leads to the following ordinary differential equation (ODE):

$$\mathcal{L}\phi(y) + \pi(y) = 0. \tag{4.4}$$

The general solution of the ODE (4.4) is given by

$$\phi(y) = A_1 y^{\gamma_1} + A_2 y^{\gamma_2} + B y^{\alpha}, \qquad y \in \mathcal{H}, \tag{4.5}$$

where A_1 and A_2 are constants to be determined and γ_1 and γ_2 are the solutions to the following characteristic equation:

$$\frac{1}{2}\sigma^2\gamma^2 - \left(\delta + \mu + \frac{1}{2}\sigma^2\right)\gamma - (r - \mu) = 0. \tag{4.6}$$

γ_1 and γ_2 are calculated with

$$\begin{aligned} \gamma_1 &= \frac{\delta+\mu}{\sigma^2} + \frac{1}{2} + \left[\left(\frac{\delta+\mu}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2(r-\mu)}{\sigma^2}\right]^{\frac{1}{2}} > 1; \\ \gamma_2 &= \frac{\delta+\mu}{\sigma^2} + \frac{1}{2} - \left[\left(\frac{\delta+\mu}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2(r-\mu)}{\sigma^2}\right]^{\frac{1}{2}} < 0. \end{aligned} \tag{4.7}$$

B is calculated as

$$B = \frac{1}{(r-\mu) + (\delta+\mu)\alpha - \frac{1}{2}\sigma^2\alpha(\alpha-1)} > 0. \tag{4.8}$$

The first and second terms of (4.5) represent the option value to expand and reduce capital, respectively. This implies that both the constants A_1 and A_2 must be positive.

Furthermore, we assume that the candidate function of the value function seems to satisfy the following equations for $y \in \mathcal{E}$ and $y \in \mathcal{R}$:

$$\phi(y) = \phi(\underset{\sim}{y}) - (c + p(\underset{\sim}{y} - y)), \qquad y \in \mathcal{E}; \tag{4.9}$$

$$\phi(y) = \phi(\tilde{y}) - (c + (1-\lambda)p(\tilde{y} - y)), \qquad y \in \mathcal{R}. \tag{4.10}$$

If the candidate function of the value function is differentiable in $\{\underline{y}, \bar{y}\}$, from equations (4.9) and (4.10), we obtain the following equations:

$$\phi'(\underline{y}) = p; \tag{4.11}$$

$$\phi'(\bar{y}) = (1-\lambda)p. \tag{4.12}$$

By (4.1) and (4.2), the firm's expected discounted profit J is maximized at $\xi = \underset{\sim}{y} - \underline{y}$ or $\xi = \tilde{y} - \bar{y}$. Hence, by the first-order condition for the maximization $\mathrm{d}[\phi(\underline{y} + \xi) - C(\xi)]/\mathrm{d}\xi|_{\xi = \underset{\sim}{y} - \underline{y}} = 0$ or $\mathrm{d}[\phi(\bar{y} + \xi) - C(\xi)]/\mathrm{d}\xi|_{\xi = \tilde{y} - \bar{y}} = 0$, if $\phi(y)$ are differentiable in $\{\underset{\sim}{y}, \tilde{y}\}$, we obtain

$$\phi'(\underset{\sim}{y}) = p; \tag{4.13}$$

$$\phi'(\tilde{y}) = (1-\lambda)p. \tag{4.14}$$

Consequently, we assume that the optimal solution described by (4.1) and (4.2) and the six unknowns, A_1, A_2, $\underline{y}$, $\underset{\sim}{y}$, $\tilde{y}$, $\bar{y}$, are a solution to the simultaneous equations:

$$\phi(\underline{y}) = \phi(\underset{\sim}{y}) - (c + p(\underset{\sim}{y} - \underline{y})), \tag{4.15}$$

$$\phi(\bar{y}) = \phi(\tilde{y}) - (c + (1-\lambda)p(\tilde{y} - \bar{y})), \tag{4.16}$$

and (4.11)–(4.14).

5 Numerical Analysis

We conduct a numerical analysis to obtain useful insights for the firm's manager in this section. First, we numerically obtain the six unknowns: A_1, A_2, $\underline{y}$, $\underset{\sim}{y}$, $\tilde{y}$, and $\bar{y}$. We then examine the effects on the changes in the parameters on the thresholds $\underline{y}$, $\underset{\sim}{y}$, $\tilde{y}$, and $\bar{y}$. We use the following baseline parameter values: $r = 0.05$, $\delta = 0.1$, $\mu = 0.01$, $\sigma = 0.15$, $\alpha = 0.6$, $c = 1$, $p = 10$ and $\lambda = 0.5$. Then, we obtain $A_1 = 2.61159 * 10^{-6}$, $A_2 = 0.0541845$, $\underline{y} = 0.00230798$, $\underset{\sim}{y} = 0.219237$, $\tilde{y} = 1.27266$ and $\bar{y} = 2.91049$.

Figure 1 illustrates the value function of the firm's problem, and Figure 2 shows the continuation, capital expansion, and reduction regions in the $x - k$ plane, respectively.

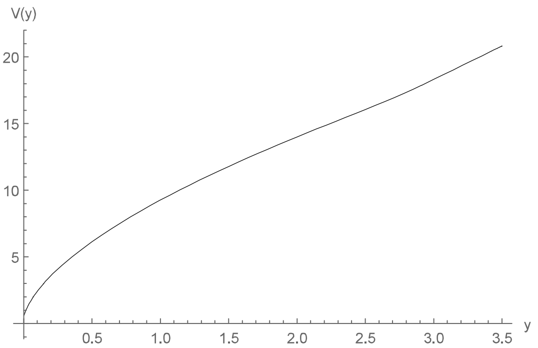

Figure 1: Value function V

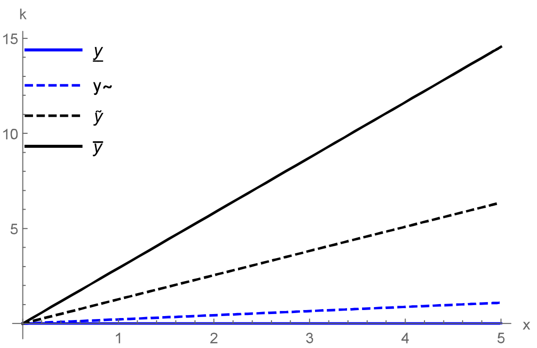

Figure 2: Continuation, expansion, and reduction regions

Figures 3–9 show how each parameter influences the firm's investment decision making. Figure 3 depicts that all thresholds decreases in the discount rate r. This implies that the an increase in the discount rate contracts the capital expansion region $\mathcal{E}$ and expands the capital reduction region $\mathcal{R}$. The magnitude of change in the capital reduction region is larger than that in the capital expansion region. Further, the continuation region $\mathcal{H}$ decreases in the discount rate. The magnitude of capital expansion, $\underset{\sim}{y} - \underline{y}$, and capital reduction, $\bar{y} - \tilde{y}$, decreases in the discount rate. They imply that, when the firm's manager prefers the present to the future, the manager curbs the change of capital.

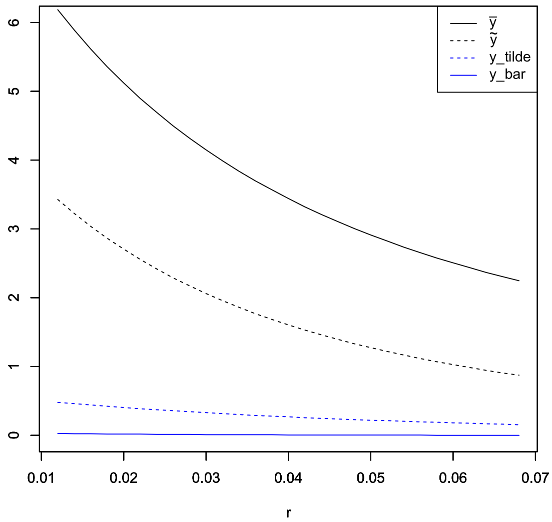

Figure 3: Effect of the changes in the discount rate, r, on the thresholds

Figure 4 illustrates that all thresholds increases in the drift rate of the output demand, μ. This implies that the higher expected growth rate of the output demand expands the capital expansion region $\mathcal{E}$ and contracts the capital reduction region $\mathcal{R}$. Moreover, the magnitude of change in the capital reduction region is larger than that in the capital expansion region. Then, the continuation region $\mathcal{H}$ increases in the drift rate of the output demand. The magnitude of capital expansion, $\underset{\sim}{y} - \underline{y}$, and capital reduction, $\bar{y} - \tilde{y}$, increases in the drift rate.

Figure 5 shows that the three thresholds $\underline{\mathrm{y}}$, $\underset{\sim}{y}$, and $\tilde{y}$ decrease in the output demand risk, σ, while $\bar{y}$ increases in the output demand risk. This implies that the higher output risk contracts the capital expansion region $\mathcal{E}$ and the capital reduction region $\mathcal{R}$, and it expands the continuation region $\mathcal{H}$. The magnitude of capital expansion, $\underset{\sim}{y} - \underline{y}$, decreases in the output demand risk, while that of capital reduction, $\bar{y} - \tilde{y}$, increases in the output risk.

Figure 6 shows that as the thresholds correspond to the capital expansion, $\underline{\mathrm{y}}$ and $\underset{\sim}{y}$ decrease in the output elasticity of the capital, while as the thresholds correspond to the capital reduction, $\tilde{y}$ and $\bar{y}$ increase in the elasticity. This implies that the higher output elasticity of capital contracts the capital expansion region $\mathcal{E}$ and the capital reduction region $\mathcal{R}$, and it expands

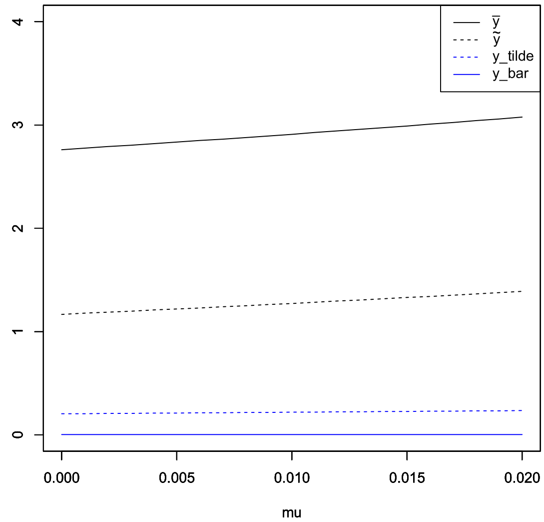

Figure 4: Effect of the changes in the drift rate, μ, on the thresholds

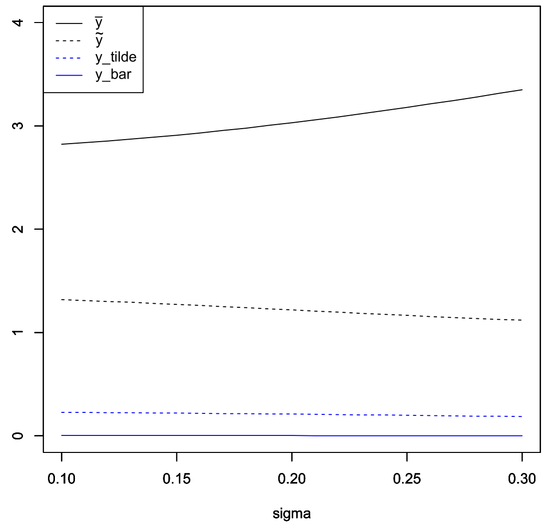

Figure 5: Effect of the changes in the volatility, σ, on the thresholds

the continuation region $\mathcal{H}$. The magnitude of capital expansion, $\underset{\sim}{y} - \underline{y}$, decreases in the output elasticity of capital, while that of capital reduction, $\bar{y} - \tilde{y}$, increases in the elasticity.

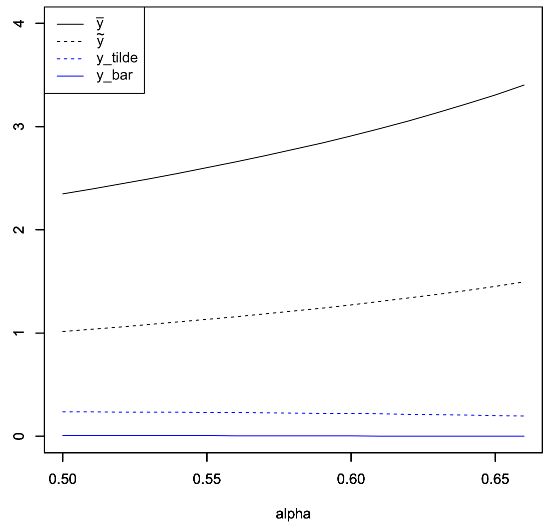

Figure 6: Effect of the changes in the output elasticity of capital, α, on the thresholds

Figure 7 shows that the threshold $\underline{y}$ decreases in the fixed cost c, while the threshold $\bar{y}$ increases in the fixed cost. This implies that the higher fixed cost contracts the capital expansion region $\mathcal{E}$ and the capital reduction region $\mathcal{R}$. The magnitude of capital expansion, $\underset{\sim}{y} - \underline{y}$, and capital reduction, $\bar{y} - \tilde{y}$, increases in the fixed cost. Overall, the continuation region $\mathcal{H}$ is enlarged.

Figure 8 shows that all the thresholds, $\underline{y}$, $\underset{\sim}{y}$, $\tilde{y}$, and $\bar{y}$, decrease in the price of the capital p. This implies that the higher price of capital contracts the capital expansion region $\mathcal{E}$ and expands the capital reduction region $\mathcal{R}$. The magnitude of change in the capital reduction region is larger than that in the capital expansion region. The magnitude of capital expansion, $\underset{\sim}{y} - \underline{y}$, and capital reduction, $\bar{y} - \tilde{y}$, decreases in the price of the capital. Combining these effects, the continuation region $\mathcal{H}$ decreases in the price of the capital.

Figure 9 shows that the thresholds associated with the capital reduction, $\tilde{y}$ and $\bar{y}$, increases in the degree of irreversibility, while the parameters associated with the capital expansion, $\underline{y}$ and $\underset{\sim}{y}$, do not change. This is because the degree of irreversibility is defined by the proportion of the sale price to the purchase price of capital. This implies that the higher degree of irreversibility contracts the capital reduction region $\mathcal{R}$, and then it expands the continuation region $\mathcal{H}$. The magnitude of capital reduction, $\bar{y} - \tilde{y}$, increases in the degree of irreversibility.

6 Conclusion

This study examined the firm's capital expansion and reduction policy with both fixed and proportional costs under the output demand risk. We derived the optimal timing and size of the

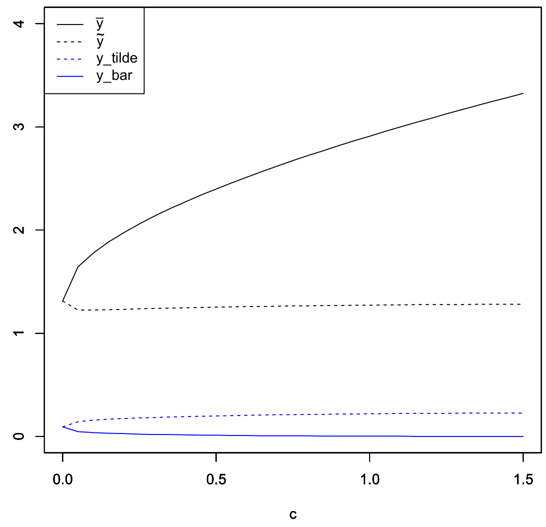

Figure 7: Effect of the changes in the fixed cost, c, on the thresholds

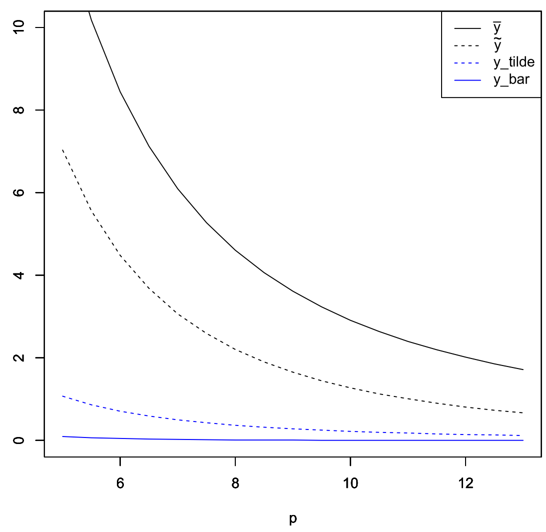

Figure 8: Effect of the changes in the price of capital, p, on the thresholds

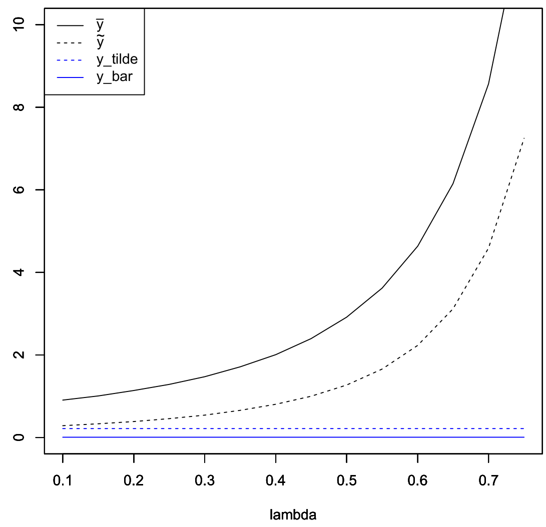

Figure 9: Effect of the changes in the degree of irreversibility of investment, λ, on the thresholds

capital expansion and reduction. Through numerical analysis, we found some useful implications for the firm's manager.

This study has extended some ways to enhance our understanding of a firm's capital investment problems. In this paper, we considered a case in which the firm's manager identifies the distribution of the output demand. In contrast, the real-world business environment is complex and uncertain. Thus, it is difficult for a firm's manager to have a specific demand distribution. To expand our research in the future, we plan to investigate the firm's problem by considering multiple distributions of the output demand.

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