On unilateral contact problems with friction for an elastic body with cracks

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Abstract

This expository article deals with contact problems with friction for a linearized (visco)elasticity in two dimension, which are arising from a wide variety of phenomena in mechanical engineering and concerning with some inverse problems and control problems. Contact conditions for cracks are so-called non-penetration conditions defined as unilateral conditions on the displacements of bodies to exclude non-physical phenomenon such as mutual penetration of crack faces, see [11] for the details. In the present paper, mathematical results obtained in [9] and [5] are introduced and moreover, dynamic unilateral contact problems are discussed.

1 Introduction

Friction problems are very important issues and appearing a lot of phenomena occurring around us. Especially, in material sciences, it is recognized a discipline as "tribology". Tribology encompasses with several science fields of Adhesion, Friction, Lubrication and Wear. One of pioneer of tribology is Leonardo da Vinci who discovered a classical law of friction in 1493. After that Guillaume Amontons and Charles-Augustin Coulomb established a classical laws of friction as follows.

Amontons' First Law: The force of friction is directly proportional to the applied load.

Amontons' Second Law: The force of friction is independent of the apparent area of contact.

Coulomb's Law of Friction: Kinetic friction is independent of the sliding velocity.

They are empirical laws and called Coulomb's Law of Friction briefly. Still today, it is going on modification to apply wide area of tribological phenomena. Let F be the force acting on the interface of two bodies and decomposed into two parts;

$$\boldsymbol{F} = F_n \boldsymbol{n} + \boldsymbol{F_\tau}, \quad F_n = \boldsymbol{F} \cdot \boldsymbol{n},$$

where $\boldsymbol{n}$ is the unit normal on the interface. In the case of unilateral contact which means that the contact is maintained only if the forces press the bodies each other, Coulomb's Law of Friction can be formulated as follows. It is necessary that $F_n \le 0$ and

$$\begin{cases} |\boldsymbol{F_\tau}| < -f\boldsymbol{F_n}: \text{stick} & \Rightarrow & \quad\quad\quad\quad \boldsymbol{V} = \boldsymbol{0}, \\ |\boldsymbol{F_\tau}| = -f\boldsymbol{F_n}: \text{slip} & \Rightarrow & \exists \xi \ge 0 \quad \text{s.t.} \quad \boldsymbol{V} = -\xi \boldsymbol{F_\tau} \quad (\boldsymbol{V} \ne \boldsymbol{0}), \end{cases}$$

where f is the coefficient of friction and $\boldsymbol{V}$ denotes the sliding velocity.

One of most important application of this kind of problems is an earthquake in which fault ruptures occur between different lithospheric plates of the earth's surface under the huge pressure. As a mathematical model, it is described by a system of partial differential equations expressing the motion and deformation of the plates together with the contact and friction conditions at the interface. Contact conditions for cracks are so-called non-penetration conditions defined as unilateral conditions on the displacements of bodies to exclude nonphysical phenomenon such as mutual penetration of crack faces, see [11] for the details. Due to such conditions, there are some difficulties in the mathematical analysis. In this paper, we introduce some theoretical results obtained in [9] and [5], further discuss dynamic unilateral contact problems with friction. A brief outline of the present paper is as follows. In Section 2, the problem for an interfacial crack between two dimensional linearized elastostatic materials is considered. The Coulomb's law of friction and non-penetration condition are assumed to hold on the whole crack surface. Then the existence of the solution is shown by using penalization method. Moreover, asymptotic expansions of the solution near the crack tip are derived. In Section 3, a similar problem treated in [5] is considered. On the crack faces the non-penetration condition and Tresca friction condition which is an approximation of Coulomb friction (e.g. [12])

are imposed. On the other part of the interface, both of adhesion forces and friction force are taken into account. A formula for the derivative of the energy functional with respect to the crack length, which can be represented as a path-independent integral (J-integral), is discussed as well as some numerical results. In Section 4, we mention about dynamic unilateral contact problems. In my knowledge, there are a few mathematical results dealing with the existence of a solution to such kinds of problems (cf. [3, 4, 17]). Then, we overview the related known results and clarify the main difficulty.

2 The interface crack with Coulomb friction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and be divided into two Lipschitz domains

$$\Omega^{(1)} := \Omega \cap \{x_2 > 0\} \quad \text{and} \quad \Omega^{(2)} := \Omega \cap \{x_2 < 0\}$$

by the x_1-axis. We suppose that each $\Omega^{(k)}$ $(k = 1, 2)$ is a dissimilar isotropic homogeneous linearized elasticity. The interface of $\Omega^{(k)}$ $(k = 1, 2)$ is denoted by Γ' and a linear crack Γ lies on the interface Γ'. Two crack tips are located at the origin $\boldsymbol{O} \notin \partial\Omega$ of the coordinate system $\boldsymbol{x} = (x_1, x_2)$ and at a point $\boldsymbol{P} \in \partial\Omega$, see Figure 1 for an illustration of the geometry.

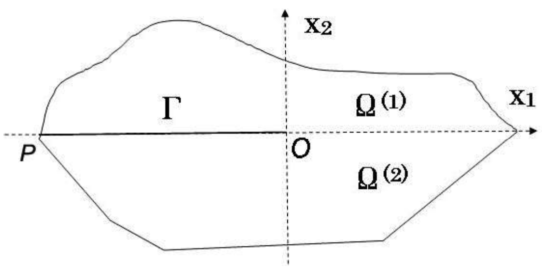

Figure 1: the domain Ω

By $\boldsymbol{u}^{(k)} = (u_i^{(k)})_{i=1,2}$ and $\boldsymbol{\sigma}^{(k)} = (\sigma_{ij}^{(k)})_{i,j=1,2}$ we denote the displacement vector and the stress tensor, respectively. $\lambda^{(k)}$ and $\mu^{(k)}$ are *Lamé* constants satisfying $\mu^{(k)} > 0$ and $\lambda^{(k)} + \mu^{(k)} > 0$. We denote $\tilde{\kappa}^{(k)} = \dfrac{\tilde{\lambda}^{(k)} + 3\mu^{(k)}}{\tilde{\lambda}^{(k)} + \mu^{(k)}}$. The superscripts $k = 1$ and $k = 2$ refer to the materials in $\Omega^{(1)}$ and $\Omega^{(2)}$.

According to [14], the stationary linearized elasticity equation for a homogeneous isotropic material is obtained from the constitutive law, called the generalized Hooke's law, and the equilibrium conditions without any body forces;

$$\mu^{(k)}\triangle \boldsymbol{u}^{(k)} + (\tilde{\lambda}^{(k)} + \mu^{(k)})\nabla(\nabla \cdot \boldsymbol{u}^{(k)}) = \boldsymbol{0} \tag{2.1}$$

$$\tilde{\lambda}^{(k)} = \begin{cases} \lambda^{(k)} & \text{in the state of plane strain,} \\ \dfrac{2\lambda^{(k)}\mu^{(k)}}{\lambda^{(k)} + 2\mu^{(k)}} & \text{in the state of plane stress.} \end{cases}$$

Here the relation between the displacement vector and the stress tensor is given by the form

$$\boldsymbol{\sigma}^{(k)}(\boldsymbol{u}^{(k)}) = \tilde{\lambda}^{(k)}(\nabla \cdot \boldsymbol{u}^{(k)})\boldsymbol{I} + \mu^{(k)}\{\nabla \boldsymbol{u}^{(k)} + (\nabla \boldsymbol{u}^{(k)})^{\mathrm{T}}\}$$

with the second order identity tensor $\boldsymbol{I}$.

Now we consider the following boundary value problem $(*)$: for given surface force $\boldsymbol{g} \in L^2(\partial\Omega)$ such that $\boldsymbol{g} = \boldsymbol{0}$ near $\boldsymbol{P}$, and a small constant friction coefficient $f \in (0,1)$, find $\boldsymbol{u}^{(1)} \in H^1(\Omega^{(1)})$ and $\boldsymbol{u}^{(2)} \in H^1(\Omega^{(2)})$ satisfying

$$(*)\begin{cases} \mu^{(1)}\triangle \boldsymbol{u}^{(1)} + (\tilde{\lambda}^{(1)} + \mu^{(1)})\nabla(\nabla \cdot \boldsymbol{u}^{(1)}) = \boldsymbol{0} & \text{in} \quad \Omega^{(1)}, \\ \mu^{(2)}\triangle \boldsymbol{u}^{(2)} + (\tilde{\lambda}^{(2)} + \mu^{(2)})\nabla(\nabla \cdot \boldsymbol{u}^{(2)}) = \boldsymbol{0} & \text{in} \quad \Omega^{(2)}, \\ \boldsymbol{\sigma}^{(1)}\boldsymbol{n} = \boldsymbol{g} & \text{on} \quad \partial\Omega^{(1)} \cap \partial\Omega, \\ \boldsymbol{\sigma}^{(2)}\boldsymbol{n} = \boldsymbol{g} & \text{on} \quad \partial\Omega^{(2)} \cap \partial\Omega, \\ [\![u_1]\!] = [\![u_2]\!] = [\![\sigma_{12}]\!] = [\![\sigma_{22}]\!] = 0 & \text{on} \quad \Gamma' \setminus \overline{\Gamma}, \\ [\![\sigma_{22}]\!] = 0,\ \sigma_{22}^{(k)} \le 0,\ [\![u_2]\!] \ge 0,\ \sigma_{22}^{(k)}[\![u_2]\!] = 0, & \text{on} \quad \Gamma, \\ [\![\sigma_{12}]\!] = 0,\ |\sigma_{12}^{(k)}| \le -f\sigma_{22}^{(k)},\ \sigma_{12}^{(k)}[\![u_1]\!] + f\sigma_{22}^{(k)}|[\![u_1]\!]| = 0 & \text{on} \quad \Gamma. \end{cases}$$

We denote jump of $\boldsymbol{u}$ at Γ' by $[\![\boldsymbol{u}]\!] := \boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}$ on Γ' and the unit outward normal to $\partial\Omega$ by $\boldsymbol{n} = (n_1, n_2)$. On the conditions at the crack, the boundary traction is continuous across the crack and the non-penetration inequality $[\![u_2]\!] \ge 0$ enforces the complementarity conditions together with Coulomb's

Law of Friction. It is note here that we adopt $[\![u_1]\!]$ instead of the sliding velocity because the problem $(*)$ is a static problem. One can see that conditions on Γ includes the following three states;

Case 1: open crack
$[\![u_2]\!] > 0$, $\sigma_{12}^{(k)} = \sigma_{22}^{(k)} = 0$;

Case 2: Stick state
$[\![u_2]\!] = 0$, $[\![u_1]\!] = 0$, $[\![\sigma_{22}]\!] = [\![\sigma_{12}]\!] = 0$, $\sigma_{22}^{(k)} \leq 0$, $|\sigma_{12}^{(k)}| \leq -f\sigma_{22}^{(k)}$;

Case 3: slip state
$[\![u_2]\!] = 0$, $[\![u_1]\!] \neq 0$, $[\![\sigma_{22}]\!] = [\![\sigma_{12}]\!] = 0$, $\sigma_{22}^{(k)} \leq 0$, $\sigma_{12}^{(k)} \pm f\sigma_{22}^{(k)} = 0$, where the upper sign "+" is taken for $[\![u_1]\!] > 0$ on Γ and the lower sign "−" is taken for $[\![u_1]\!] < 0$ on Γ.

Next, we discuss the existence of the solution of problem $(*)$. Firstly we introduce some notations. Let $\mathcal{E}$ be a bilinear form which represents the strain energy

$$\mathcal{E}_D(\boldsymbol{u}, \boldsymbol{v}) := \int_D \sum_{i,j=1,2} \sigma_{ij}(\boldsymbol{u}) \frac{\partial v_i}{\partial x_j} \, \mathrm{d}\boldsymbol{x}.$$

In what follows we use simplified notations $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ which means that $\boldsymbol{u} = \boldsymbol{u}^{(1)}$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(1)}$ in $\Omega^{(1)}$, $\boldsymbol{u} = \boldsymbol{u}^{(2)}$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(2)}$ in $\Omega^{(2)}$. We denote a space of all rigid displacements by $\mathcal{R} = \{(c_1 + c_0 x_2, c_2 - c_0 x_1) \; \forall \boldsymbol{c} = (c_1, c_2, c_0) \in \mathbb{R}^3\}$. In addition, we assume for $\boldsymbol{g}$ the necessary compatibility condition in the form

$$\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{\rho} \, \mathrm{d}S_{\boldsymbol{x}} = 0, \quad \forall \boldsymbol{\rho} \in \mathcal{R}.$$

We set a solution space which is a convex set on H_1 space,

$$\mathcal{K}_* = \{\boldsymbol{v} \in H^1(\Omega \setminus \overline{\Gamma}) \setminus \mathcal{R} \mid [\![v_2]\!] \geq 0 \quad \text{on } \Gamma\}.$$

Next, we introduce the Hilbert space, so-called Lions-Magenes sapce [15]

$$H_{00}^{1/2}(\Gamma) = \left\{ \boldsymbol{s} \in H_0^{1/2}(\Gamma) \mid \rho^{-1/2} \boldsymbol{s} \in L^2(\Gamma) \right\}$$

where the function $\rho \in C_0^{1,1}(\overline{\Gamma})$ such that $\rho > 0$ and the limit of $\rho(\boldsymbol{x})/\mathrm{dist}(\boldsymbol{x}, \boldsymbol{O}$ or $\boldsymbol{P})$ as $\boldsymbol{x} \to \boldsymbol{O}$ or $\boldsymbol{P}$ exists and attains a nonzero finite value. It follows from the definition that the space $H_{00}^{1/2}(\Gamma)$ is characterized by the following equivalence;

$$\boldsymbol{s} \in H_{00}^{1/2}(\Gamma) \quad \Leftrightarrow \quad \bar{\boldsymbol{s}} = \begin{cases} \boldsymbol{s} & \text{on} \quad \Gamma \\ \boldsymbol{0} & \text{on} \quad \Gamma' \setminus \overline{\Gamma} \end{cases} \in H^{1/2}(\Gamma').$$

By using the Green formulae the problem $(*)$ can be reduced the quasi-variational inequality, see [9] and [13] for the detail of derivation: find $\boldsymbol{u} \in \mathcal{K}_*$ satisfying for an arbitrary $\boldsymbol{v} \in \mathcal{K}_*$

$$\mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - \langle f\sigma_{22}, |[\![v_1]\!]| - |[\![u_1]\!]|\rangle_\Gamma \geq \int_{\partial\Omega} \boldsymbol{g} \cdot (\boldsymbol{v} - \boldsymbol{u}) \mathrm{d}S_{\boldsymbol{x}}, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle_\Gamma$ is the duality pairing between $H_{00}^{-1/2}(\Gamma)$ and $H_{00}^{1/2}(\Gamma)$. It is not obvious to show the existence of solution of (2.2). Let consider the case that $f = 0$ or the load on the crack $F = f\sigma_{22} \geq 0$ is given, which is so-called the Tresca friction problem. Since this problem is described as a convex minimation problem, it is easy solvable by using standard techniques, e.g. [3]. Indeed, in this case (2.2) is rewritten as

$$\mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - \langle F, |[\![v_1]\!]| - |[\![u_1]\!]|\rangle_\Gamma \geq \int_{\partial\Omega} \boldsymbol{g} \cdot (\boldsymbol{v} - \boldsymbol{u}) \mathrm{d}S_{\boldsymbol{x}}. \tag{2.3}$$

We introduce the potential energy functional in the form $P(\boldsymbol{u}) = \Pi(\boldsymbol{u}) + I(\boldsymbol{u})$ where

$$\Pi(\boldsymbol{u}) = \frac{1}{2}\mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}, \boldsymbol{u}) - \int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{u} \, \mathrm{d}S_{\boldsymbol{x}}, \quad I(\boldsymbol{u}) = \langle F, |[\![u_1]\!]|\rangle_\Gamma.$$

Then one can see that I is positive, continuous and convex, and Π is convex and continuous, namely lower semicontinuous, and differentiable, see [13]. Therefore, the problem (2.3) becomes to find the solution of the following variational inequality:

$$\boldsymbol{u} \in \mathcal{K}_*, \quad \Pi'_{\boldsymbol{u}}(\boldsymbol{v} - \boldsymbol{u}) + I(\boldsymbol{v}) - I(\boldsymbol{u}) \geq 0 \quad \forall \boldsymbol{v} \in \mathcal{K}_*,$$

which is equivalent to the minimization problem $P(\boldsymbol{u}) = \inf_{\boldsymbol{v} \in \mathcal{K}_*} P(\boldsymbol{v})$. By virtue of the Korn inequality; there exists a positive constant C_0 such that for all $\boldsymbol{u} \in H^1(\Omega \setminus \overline{\Gamma}) \setminus \mathcal{R}$

$$C_0 \|\boldsymbol{u}\|^2_{H^1(\Omega\setminus\overline{\Gamma})} \leq \mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}, \boldsymbol{u}),$$

the coercivity of the functional Π is ensured. This guarantees the existence of a solution of the minimization problem.

However, in a case of the Coulomb friction (2.2), above standard methods are not valid because convexity is lost, that is, it cannot be described as a minimization problem. To overcome the difficulty, in [16] Tikhonov's fixed point theorem is used and the idea bring many results in frictional contact problems, refer [4]. More precisely, let $\boldsymbol{u}$ be a solution of (2.3). We consider a mapping T: $f\sigma_{22}(\boldsymbol{u}) = T(F)$. Then, one sees that a solution of the quasi-variational inequality (2.2) is obtained as a fixed point of T. Let $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$ be solutions of, $\forall \boldsymbol{v}^k \in \mathcal{K}_*$, $k = 1, 2$,

$$\mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}^k, \boldsymbol{v}^k - \boldsymbol{u}^k) - \langle F^k, |[\![v_1^k]\!]| - |[\![u_1^k]\!]|\rangle_\Gamma \geq \int_{\partial\Omega} \boldsymbol{g}\cdot(\boldsymbol{v}^k - \boldsymbol{u}^k)\,\mathrm{d}S_{\boldsymbol{x}},$$

By summation of two inequalities with $\boldsymbol{v}^1 = \boldsymbol{u}^2$,$\boldsymbol{v}^2 = \boldsymbol{u}^1$

$$\mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}^1 - \boldsymbol{u}^2, \boldsymbol{u}^1 - \boldsymbol{u}^2) \leq \langle F^1 - F^2, |[\![u_1^1]\!]| - |[\![u_1^2]\!]|\rangle_\Gamma$$

Applying the Korn and Hölder inequalities, we have

$$\begin{aligned} \|\sigma_{22}(\boldsymbol{u}^1) - \sigma_{22}(\boldsymbol{u}^2)\|^2_{-\frac{1}{2}} &\leq c\|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_1^2 \\ &\leq c\|F^1 - F^2\|_{-\frac{1}{2}}(\|[\![u_1^1]\!]\|_{\frac{1}{2}} + \|[\![u_1^2]\!]\|_{\frac{1}{2}}) \end{aligned}$$

This implies the Hölder continuity of the mapping T, however, it is not enough for the existence of a fixed point. The common assumptions which guarantee the existence are that the friction coefficient is sufficiently small, and it has a compact support which was used in [13]. In [9] the latter assumption is removed by use of the topological sensitivity technique based on the estimate associated with the Saint-Venant principle. The statement is as follows.

Theorem 1 (Theorem 3.1. in [9]). *If* $f < \dfrac{1}{C_1C_2} \leq 1$ *holds, then there exists a solution* $\boldsymbol{u} \in \mathcal{K}_*$ *of the quasi-variational inequality* (2.2).

Here C_1 and C_2 are positive constants depending on $\lambda^{(k)}$, $\mu^{(k)}$ for $k = 1, 2$, and on the geometry of Ω, and are appeared in the following estimates derived from continuity of the trace operator; there exist C_1 and C_2 such that $1 \leq C_1C_2 < \infty$ and

$$\begin{aligned} &\|[\![\boldsymbol{u}]\!]\|_{H_{00}^{1/2}(\Gamma)} \leq C_1\|\boldsymbol{u}\|_{1,\Omega\setminus\overline{\Gamma}} &&\forall \boldsymbol{u} \in H^1(\Omega\setminus\overline{\Gamma})\setminus\mathcal{R}, \\ &\|\boldsymbol{\sigma}\|_{H_{00}^{-1/2}(\Gamma)} \leq C_2\|\boldsymbol{u}\|_{1,\Omega\setminus\overline{\Gamma}} &&\forall \boldsymbol{u} \in H^1(\Omega\setminus\overline{\Gamma})\setminus\mathcal{R} \quad \text{satisfying} \quad (2.1). \end{aligned}$$

Next, we derive convergent series expansions of the solution of $(*)$ near the crack tip $\boldsymbol{O}$. Due to this, we assume no switches among the three cases open crack, stick state and slip state near the crack tip. We introduce a polar coordinate system (r, θ) with respect to the origin $\boldsymbol{O}$, where $0 < r \ll 1$ and $-\pi < \theta < \pi$.

Proposition 1 ([9], [7]). *There exist complex numbers a_n, b_n satisfying some inequality conditions and a rigid displacement $\boldsymbol{\rho}_0 \in \mathcal{R}$ such that for $k = 1, 2$*

$$\begin{aligned} \boldsymbol{u}^{(k)}(r,\theta) &= \sum_{n=0}^{\infty} r^{n+1}\left\{\mathrm{Re}\left[r^{-s}a_n\right]A_n^{(k)}(\epsilon,\theta) + \mathrm{Im}\left[r^{-s}a_n\right]B_n^{(k)}(\epsilon,\theta)\right\} \\ &\quad + \sum_{n=0}^{\infty} r^{n+1}\left\{\mathrm{Re}\left[b_n\right]C_n^{(k)}(\epsilon,\theta) + \mathrm{Im}\left[b_n\right]D_n^{(k)}(\epsilon,\theta)\right\} + \boldsymbol{\rho}_0. \end{aligned} \tag{2.4}$$

The series is convergent, absolutely in H^1 and uniformly on compact sets in the neighborhood of $\boldsymbol{O}$. Here we can explicitly obtain angular functions $A_n^{(k)}(\epsilon,\theta)$, $B_n^{(k)}(\epsilon,\theta)$, $C_n^{(k)}(\epsilon,\theta)$, $D_n^{(k)}(\epsilon,\theta)$ for $k = 1, 2$ and each cases; open crack, stick state and slip state, respectively. The singularity exponent s is given as follows.

Case 1: open crack $s = \dfrac{1}{2} + i\epsilon$ *with* $\epsilon = \dfrac{1}{2\pi}\log\left(\dfrac{1+\beta}{1-\beta}\right)$ *and a Dundurs parameter* $\beta = \dfrac{\mu^{(2)}(\tilde{\kappa}^{(1)}-1) - \mu^{(1)}(\tilde{\kappa}^{(2)}-1)}{\mu^{(2)}(\tilde{\kappa}^{(1)}+1) - \mu^{(1)}(\tilde{\kappa}^{(2)}+1)}$*;*

Case 2: Stick state $s = 0$*;*

Case 3: slip state $\cot \pi s = \mp f\beta$*,where the upper sign "$-$" is taken for $[\![u_1]\!] > 0$ on Γ and the lower sign "$+$" is taken for $[\![u_1]\!] < 0$ on Γ. Especially, if $\beta \neq 0$, then $0 < s < \frac{1}{2}$. If $\beta = 0$, then $s = \frac{1}{2}$.*

Remark 1. *In Proposition 1, coefficients a_n and b_n satisfy some inequality conditions due to the non-penetration inequality and Coulomb's Law of Friction. For example, in the open crack case, the condition $[\![u_2]\!] > 0$ is rewritten as a condition for a_n in the form*

$$\sum_{n=0}^{\infty}(-1)^n r^{\frac{1}{2}+n}\mathbf{Re}\left[a_n r^{-i\epsilon}\right] > 0. \tag{2.5}$$

The coefficients of leading terms in the expansion (2.4) *(i.e.* a_0*) are called, in fracture mechanics, stress intensity factors. In the case of homogeneous material (i.e.* $\epsilon = 0$*)* (2.5) *leads to* $\mathbf{Re}[a_0] \geq 0$ *which corresponds to the results of [1].*

3 The interface crack with Tresca friction in a spring-type adhesive model

In this section, we introduce results obtained in [5]. Let $\Omega' \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary such that $\partial\Omega' = \overline{\Gamma_N} \cup \overline{\Gamma_D}$, $\Gamma_N \cap \Gamma_D = \emptyset$, $|\Gamma_D| > 0$. The domain Ω' consists of two parts Ω_+ and Ω_- and the interface is denoted as Σ, that is, $\Sigma = \partial\Omega_+ \cap \partial\Omega_-$. We assume that $\partial\Omega_+$ and $\partial\Omega_-$ satisfy the Lipschitz condition and $|\Gamma_D^\pm| > 0$, where $\Gamma_D^\pm = \partial\Omega_\pm \cap \Gamma_D$. In our consideration, the domains Ω_+ and Ω_- correspond to linearized elastic bodies bonded to each other along the part $\Gamma_a \subset \Sigma$. The rest part of Σ is a crack (full delamination) and denoted as Γ_c. Namely, Σ is divided into two parts Γ_c and Γ_a such that $\overline{\Sigma} = \overline{\Gamma_c} \cup \overline{\Gamma_a}$, $\Gamma_c \cap \Gamma_a = \emptyset$. On Γ_c the non-penetration condition and Tresca friction condition are imposed. On Γ_a it is adhesively bonded, which means to take into account both of adhesion forces and friction force. We assume spring-type condition modeling a thin adhesive layer to describe the interaction of bodies. See Figure 2 for an illustration of the general geometry of Ω'.

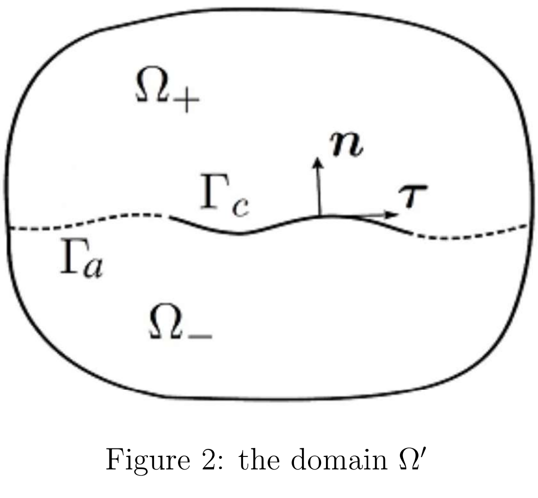

Figure 2: the domain Ω'

We use same notation in Section 2 and the superscripts $+$ and $-$ refer the materials Ω_+ and Ω_-. Also, we denote $\boldsymbol{u}_{\boldsymbol{n}}^{\pm} = \boldsymbol{u}^{\pm} \cdot \boldsymbol{n}$, $\boldsymbol{u}_{\boldsymbol{\tau}}^{\pm} = \boldsymbol{u}^{\pm} \cdot \boldsymbol{\tau}$, $\sigma_{\boldsymbol{n}}^{\pm} = \boldsymbol{\sigma}^{\pm}\boldsymbol{n} \cdot \boldsymbol{n}$, $\sigma_{\boldsymbol{\tau}}^{\pm} = \boldsymbol{\sigma}^{\pm}\boldsymbol{n} \cdot \boldsymbol{\tau}$. Here we consider the following boundary value problem. For given surface force $\boldsymbol{f}^{\pm} \in L^2(\Gamma_N^{\pm})$ $(\Gamma_N^{\pm} = \Gamma_N \cap \partial\Omega_{\pm})$, frictional force $g \in C^1(\overline{\Sigma})$ $(g \geq 0)$ and symmetric positive definite matrix A, our problem is to find $\boldsymbol{u} = (\boldsymbol{u}^+, \boldsymbol{u}^-)$ satisfying

$$
(\dagger)\begin{cases}
\mu^{\pm}\triangle\boldsymbol{u}^{\pm} + (\tilde{\lambda}^{\pm} + \mu^{\pm})\nabla(\nabla\cdot\boldsymbol{u}^{\pm}) = \boldsymbol{0} & \text{in} \quad \Omega_{\pm}, \\
\boldsymbol{\sigma}^{\pm}\boldsymbol{n} = \boldsymbol{f}^{\pm} & \text{on} \quad \Gamma_N^{\pm}, \\
\boldsymbol{u}^{\pm} = \boldsymbol{0} & \text{on} \quad \Gamma_D^{\pm}, \\
[\![\boldsymbol{u}_{\boldsymbol{n}}]\!] \geq 0,\ [\![\sigma_{\boldsymbol{n}}]\!] = 0,\ [\![\sigma_{\boldsymbol{\tau}}]\!] = 0 & \text{on} \quad \Sigma, \\
\sigma_{\boldsymbol{n}} \leq 0,\ \sigma_{\boldsymbol{n}}[\![\boldsymbol{u}_{\boldsymbol{n}}]\!] = 0 & \text{on} \quad \Gamma_c, \\
|\sigma_{\boldsymbol{\tau}}| \leq g,\ \sigma_{\boldsymbol{\tau}}[\![\boldsymbol{u}_{\boldsymbol{\tau}}]\!] = g|[\![\boldsymbol{u}_{\boldsymbol{\tau}}]\!]| & \text{on} \quad \Gamma_c, \\
\sigma_{\boldsymbol{n}} - A[\![\boldsymbol{u}]\!]\cdot\boldsymbol{n} \leq 0,\ (\sigma_{\boldsymbol{n}} - A[\![\boldsymbol{u}]\!]\cdot\boldsymbol{n})[\![\boldsymbol{u}_{\boldsymbol{n}}]\!] = 0 & \text{on} \quad \Gamma_a, \\
|\sigma_{\boldsymbol{\tau}} - A[\![\boldsymbol{u}]\!]\cdot\boldsymbol{\tau}| \leq g,\ (\sigma_{\boldsymbol{\tau}} - A[\![\boldsymbol{u}]\!]\cdot\boldsymbol{\tau})[\![\boldsymbol{u}_{\boldsymbol{\tau}}]\!] = g|[\![\boldsymbol{u}_{\boldsymbol{\tau}}]\!]| & \text{on} \quad \Gamma_a.
\end{cases}
$$

Similar to the problem $(*)$, we assume a non-penetration condition on Γ_c and also a frictional condition with a given frictional force g, so-called Tresca friction condition, instead of the Coulomb's friction. Moreover, on Γ_a an adhesion force described as A is taken into account.

Next, we consider the problem $(\dagger)$ in the weak sense by using the potential functional. The total energy is defined as

$$
\Pi(\boldsymbol{u}^+, \boldsymbol{u}^-) = P_+(\boldsymbol{u}^+) + P_-(\boldsymbol{u}^-) + S(\boldsymbol{u}^+, \boldsymbol{u}^-) + F(\boldsymbol{u}^+, \boldsymbol{u}^-),
$$

where $P_{\pm}(\boldsymbol{u}^{\pm}) = \dfrac{1}{2}\mathcal{E}_{\Omega_{\pm}}(\boldsymbol{u}^{\pm}, \boldsymbol{u}^{\pm}) - \displaystyle\int_{\Gamma_N^{\pm}} \boldsymbol{f}\cdot\boldsymbol{u}^{\pm}\ \mathrm{d}S_{\boldsymbol{x}}$ is the potential and surface energy, $S(\boldsymbol{u}^+, \boldsymbol{u}^-) = \dfrac{1}{2}\displaystyle\int_{\Gamma_a} A[\![\boldsymbol{u}]\!]\cdot[\![\boldsymbol{u}]\!]\ \mathrm{d}S_{\boldsymbol{x}}$ is the adhesive-layer energy and $F(\boldsymbol{u}^+, \boldsymbol{u}^-) := \displaystyle\int_{\Sigma} g|[\![\boldsymbol{u}_{\boldsymbol{\tau}}]\!]|\ \mathrm{d}S_{\boldsymbol{x}}$ is the frictional energy. Now we define the spaces $V_{\pm} = \{\boldsymbol{u}^{\pm} \in H^1(\Omega_{\pm}) :\ \boldsymbol{u}^{\pm} = \boldsymbol{0} \quad \text{on } \Gamma_D^{\pm}\}$ and the convex closed set of admissible displacements

$$
\mathcal{K}_{\dagger} = \{\boldsymbol{u} = (\boldsymbol{u}^+, \boldsymbol{u}^-) \in V_+ \times V_- :\ [\![\boldsymbol{u}_{\boldsymbol{n}}]\!] \geq 0 \quad \text{on } \Sigma\}.
$$

Then the problem (†) can be reduced to find $(\boldsymbol{u}^+, \boldsymbol{u}^-) \in \mathcal{K}_\dagger$ such that

$$\Pi(\boldsymbol{u}^+, \boldsymbol{u}^-) = \inf_{(\boldsymbol{v}^+, \boldsymbol{v}^-) \in \mathcal{K}_\dagger} \Pi(\boldsymbol{v}^+, \boldsymbol{v}^-). \tag{3.1}$$

It follows from the calculus of variation that there exists a unique solution of the minimization problem (3.1) and it yields a variational inequality

$$\begin{aligned} &\mathcal{E}_{\Omega_+}(\boldsymbol{u}^+, \boldsymbol{v}^+ - \boldsymbol{u}^+) + \mathcal{E}_{\Omega_-}(\boldsymbol{u}^-, \boldsymbol{v}^- - \boldsymbol{u}^-) \\ &\qquad + \int_{\Gamma_a} A[\![\boldsymbol{u}]\!] \cdot [\![\boldsymbol{v} - \boldsymbol{u}]\!] \, \mathrm{d}S_{\boldsymbol{x}} + \int_{\Sigma} g(|[\![\boldsymbol{v}_{\boldsymbol{T}}]\!]| - |[\![\boldsymbol{u}_{\boldsymbol{T}}]\!]|) \, \mathrm{d}S_{\boldsymbol{x}} \\ &\geq \int_{\Gamma_N^+} \boldsymbol{f}^+ \cdot (\boldsymbol{v}^+ - \boldsymbol{u}^+) \, \mathrm{d}S_{\boldsymbol{x}} + \int_{\Gamma_N^-} \boldsymbol{f}^- \cdot (\boldsymbol{v}^- - \boldsymbol{u}^-) \, \mathrm{d}S_{\boldsymbol{x}} \end{aligned}$$

for an arbitrary $(\boldsymbol{v}^+, \boldsymbol{v}^-) \in \mathcal{K}_\dagger$, which is necessary and sufficient optimality condition for solvability of the problem (3.1).

Moreover, the derivative of the energy functional with respect to the crack length is calculated in [5] and it can be represented as a path-independent integral, so-called J-integral. In [5], by using a non-overlapping domain decomposition method for the bonded structure we have done some numerical experiments.

4 Dynamic contact on crack faces with friction

Mathematical analysis for contact problems with friction has been well studied in the static case, e.g. Section 2 and 3. One can say that dynamic models are more realistic than static models, however, the analysis is indeed so difficult. In reference to well-organized books [3, 4], we overview the related known results and clarify the main difficulty.

In the dynamic case the inertial term is added into the static governing equation (2.1),

$$\rho \boldsymbol{u}_{tt} = \mu \triangle \boldsymbol{u} + (\tilde{\lambda} + \mu) \nabla (\nabla \cdot \boldsymbol{u}). \tag{4.1}$$

Also, taking into account the viscosity effect (material has short memory), we consider a viscoelastic model (so-called the Kelvin-Voigt model), that is, replacing the Cauchy stress by the sum of elastic stress and dissipative stress (e.g. [2, 6, 8, 10]). As a consequence, it yields the system of linearized viscoelasticity equations

$$\rho \boldsymbol{u}_{tt} = \mu \triangle \boldsymbol{u} + (\tilde{\lambda} + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \eta \triangle \boldsymbol{u} + (\zeta + \eta)\nabla(\nabla \cdot \boldsymbol{u}). \tag{4.2}$$

Here η and ζ are viscosity coefficients.

If we consider the crack problem, on the crack the following conditions are imposed.

$$[\![\sigma_{\boldsymbol{n}}]\!] = 0, \quad \sigma_{\boldsymbol{n}} \leq 0, \quad [\![u_{\boldsymbol{n}}]\!] \geq 0, \quad [\![u_{\boldsymbol{n}}]\!]\sigma_{\boldsymbol{n}} = 0, \tag{4.3}$$

$$[\![\boldsymbol{\sigma_\tau}]\!] = \boldsymbol{0}, \quad |\boldsymbol{\sigma_\tau}| \leq g, \quad [\![(\boldsymbol{u}_t)_{\boldsymbol{\tau}}]\!] \cdot \boldsymbol{\sigma_\tau} = g|[\![(\boldsymbol{u}_t)_{\boldsymbol{\tau}}]\!]|. \tag{4.4}$$

(4.3) is the non-penetration condition as mentioned in the static case. (4.4) implies the friction condition; Tresca friction condition if the frictional force g is given; Coulomb's law of friction if g is given by $-\tilde{g}\sigma_{\boldsymbol{n}}$. One can say that (4.4) is a reasonable condition depending on the sliding velocity in contrast with the static case depending on a tangential part of displacements. In the variational form of the problems, the terms

$$\rho \int_\Omega \boldsymbol{u}_{tt} \cdot (\boldsymbol{v} - \boldsymbol{u}_t)\, \mathrm{d}\boldsymbol{x}, \quad \mathcal{E}_\Omega(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}_t), \quad \int_\Gamma g(|[\![\boldsymbol{v}_\tau]\!]| - |[\![(\boldsymbol{u}_t)_\tau]\!]|)\, \mathrm{d}S_{\boldsymbol{x}}$$

appear and the term coming from viscosity is added in the case of viscoelasticity. One of main difficulties is due to the hyperbolic property of a dynamic elastic equation. As a way to avoid it, we consider a viscoelastic problem because a smoothing effect for the solution can be expected. Additionally, it is difficult because the contact condition (4.3) is written in the displacement, however the friction condition (4.4) is expressed in the velocity.

The following table shows the results of dynamic models of contact problems with or without friction in books [3] and [4], but not crack problems.

	elastic	viscoelastic
contact	× (Sec. 4.1.3 in [4]) wave, half sp.	○ Sec. 4.2.2 in [4]
friction	Tresca: Sec. 5.5 in [3] Coulomb: open	Tresca: Sec. 4.4 in [4] Coulomb: open
contact + friction	×	Tresca: Sec. 4.4.1 in [4] Coulomb: Contact condition in velocities (Chap. 5 in [4]) Coulomb: non-local ([17])

For dynamic contact problems without friction, mathematical analysis for the elastic problem is so hard. Therefore, we only have the results for the special case such as the wave equation in a half-space. However, in the case of viscoelasticity, existence results for the contact problems have been given owing to the viscosity term. For frictional problems, in the case of given friction (Tresca friction), existence results of a variational solution have been shown in both of elastic [3] and viscoelastic [4] cases. Then, extension of the results to crack problems can be expected. As regard of Coulomb friction case, we still have as open problems. For dynamic contact problems with friction, elastic problem is so difficult because even the contact problem is left unresolved. For viscoelasticity, the existence results are obtained in a given friction case [4]. For the Coulomb friction case, if we assume the contact condition in velocities not displacements, then the existence of the solution can be shown. Indeed, by taking the velocity as the test function we can get the good estimate of the solution and gain the regularity (cf. [2] and [6]). In this case the existence proof is similar to that in the static case. If the contact condition is formulated in the displacements, then the regularity is proved for the displacements only; and the possible gain of regularity is not sufficient in order to get the required regularity of the velocities. It seems that is main difficulty of this problem. As one of other ways to overcome the difficulty, instead of Coulomb's friction, there is a non-local condition, also called averaged friction, which is a kind of approximation of Coulomb's friction condition by mollifying the traction in the friction term, see [17] for the details.

In future research we will consider to extend the results to crack problems and also the possibility of application to fault rupture in earthquakes as well as inverse problems.

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