Chandrasekhar polynomials – A brief review

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Abstract. A review on the Chandrasekhar polynomials is given. The polynomials often appear in transport theory. The relation to the method of rotated reference frames for the three-dimensional radiative transport equation is clarified.

1. Introduction

The Chandrasekhar polynomials play an important role in one-dimensional transport theory (see [5, 9] and references therein). Recently, the appearance of the polynomials has been recognized even for the three-dimensional radiative transport equation [10, 11].

We begin with the one-dimensional transport equation. Let μ_t, μ_s be constants such that $\mu_t > \mu_s \geq 0$. Let μ be the third component of vector $\theta \in \mathbb{S}^2$, i.e., μ is the cosine of the polar angle of θ. Let Ω be an interval on the real axis. We write the transport equation as

$$\left(\mu\frac{\partial}{\partial z}+\mu_t\right)I(z,\theta)=\mu_s\int_{\mathbb{S}^2}p(\theta,\theta')I(z,\theta')\,d\theta',\quad (z,\theta)\in\Omega\times\mathbb{S}^2.$$

The solution $I(z,\theta)$ will be uniquely determined if suitable boundary conditions are imposed. We assume that the scattering phase function $p(\theta,\theta')$ is given by

$$p(\theta,\theta')=\frac{1}{4\pi}\sum_{l=0}^{L}\beta_lP_l(\theta\cdot\theta')=\sum_{l=0}^{L}\sum_{m=-l}^{l}\frac{\beta_l}{2l+1}Y_{lm}(\theta)Y_{lm}^*(\theta'),$$

where P_l are Legendre polynomials, Y_{lm} are spherical harmonics, and the symbol $*$ means complex conjugate. The coefficient $\beta_0 = 1$ and for $1 \leq l \leq L$, $|\beta_l| < 2l + 1$. Using associated Legendre polynomials $P_l^m(\mu)$, spherical harmonics are given by

$$Y_{lm}(\theta)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\mu)e^{im\varphi},$$

where $\varphi \in [0, 2\pi)$ is the azimuthal angle of θ. We note that this $p(\theta,\theta')$ implies scatterers are spherically symmetric. In optics, coefficients β_l are often given by $\beta_l = (2l + 1)\mathrm{g}^l$ with the anisotropy factor $\mathrm{g} \in (-1, 1)$ [3].

By changing the spatial variable as $x = \mu_t z$, we can rewrite the transport equation as

$$\left(\mu\frac{\partial}{\partial x}+1\right)\psi(x,\theta)=\varpi\int_{\mathbb{S}^2}p(\theta,\theta')\psi(x,\theta')\,d\theta',\quad (x,\theta)\in\Omega\times\mathbb{S}^2,$$

where $\varpi = \mu_s/\mu_t \in [0, 1)$ is called the albedo for single scattering and $\psi(x,\theta) = I(x/\mu_t,\theta)$. Chandrasekhar's polynomials appear when the solution $\psi(x,\theta)$ to the homogeneous equation is sought assuming the form

$$\psi(x,\theta)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}f_{lm}(\nu)Y_{lm}(\theta)e^{-x/\nu},$$

where $\nu\in\mathbb{R}$ is a parameter and $f_{lm}(\nu)$ are coefficients which will be later related to Chandrasekhar's polynomials. See [1] for the equivalence between the method of discrete ordinates and the spherical-harmonic expansion.

Let us introduce h_l as

$$h_l=\begin{cases}2l+1-\varpi\beta_l, & 0\le l\le L,\\ 2l+1, & l\ge L+1.\end{cases}$$

We note the relation

$$\int_{-1}^{1}\mu P_l^m(\mu)P_{l'}^m(\mu)\,d\mu=\frac{2}{4(l+1)^2-1}\frac{(l+1+m)!}{(l-m)!}\delta_{l+1,l'}+\frac{2}{4l^2-1}\frac{(l+m)!}{(l-1-m)!}\delta_{l-1,l'}.$$

By substituting the assumed form of $\psi(x,\theta)$ into the homogeneous transport equation, we obtain

$$\mu\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'}f_{l'm'}(\nu)Y_{l'm'}(\theta)-\nu\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'}\frac{h_{l'}}{2l'+1}f_{l'm'}(\nu)Y_{l'm'}(\theta)=0.$$

Then by multiplying $Y_{lm}^*(\theta)$ and integrating over θ, we obtain

$$\sum_{l'=|m|}^{\infty}\sqrt{\frac{l^2-m^2}{4l^2-1}}\delta_{l-1,l'}f_{l'm}(\nu)-\frac{\nu h_l}{2l+1}f_{lm}(\nu)+\sum_{l'=|m|}^{\infty}\sqrt{\frac{(l+1)^2-m^2}{4(l+1)^2-1}}\delta_{l+1,l'}f_{l'm}(\nu)=0.$$

Let us define $f_{lm}(\nu)=0$ for $l<|m|$. If we multiply $\sqrt{2l+1}$ in the above equation, we obtain

$$\sqrt{\frac{l^2-m^2}{2l-1}}f_{l-1,m}(\nu)-\frac{\nu h_l}{\sqrt{2l+1}}f_{lm}(\nu)+\sqrt{\frac{(l+1)^2-m^2}{2(l+1)+1}}f_{l+1,m}(\nu)=0.$$

2. Chandrasekhar polynomials

Let $x\in\mathbb{R}$. Chandrasekhar introduced polynomials $G_l^m(x)$ which satisfy the following three-term recurrence relation [2, 4].

$$(l+m)G_{l-1}^m(x)-h_lxG_l^m(x)+(l-m+1)G_{l+1}^m(x)=0,\quad l\ge m\ge 0,$$

with $G_m^m(x)=(2m-1)!!$. See [8] for the case $m=0$.

Then the normalized Chandrasekhar polynomials $g_l^m(x)$ ($l\ge|m|$) were introduced [6, 7]. By setting [15]

$$g_l^m(x)=\sqrt{\frac{(l-m)!}{(l+m)!}}G_l^m(x),$$

we see that g_l^m satisfy the following three-term recurrence relation.

$$\sqrt{l^2-m^2}g_{l-1}^m(x)-h_lxg_l^m(x)+\sqrt{(l+1)^2-m^2}g_{l+1}^m(x)=0,\quad l\ge|m|.$$

Indeed, the three-term recurrence relation for f_{lm} is recovered if we put $g_l^m = f_{lm}/\sqrt{2l+1}$. We set the initial term as

$$g_m^m(x) = \frac{(2m-1)!!}{\sqrt{(2m)!}} = \frac{\sqrt{(2m)!}}{2^m m!}, \quad m \geq 0.$$

Moreover, $g_l^{-m}(x)$ and $g_l^m(-x)$ are related to $g_l^m(x)$ as

$$g_l^{-m}(x) = (-1)^m g_l^m(x), \quad g_l^m(-x) = (-1)^{l+m} g_l^m(x).$$

3. Eigenproblem

To avoid tedious calculations, in this section we assume m is nonnegative: $m = 0, 1, \ldots$. It is straightforward to extend results below to the case of negative m. It is also possible to write $p(\theta, \theta')$ only with $m \geq 0$ making use of the formula $P_l^{-m}(\mu) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(\mu)$. Let us introduce

$$\sigma_l = \frac{\mu_t h_l}{2l+1},$$

and

$$y_l^m(x) = \sqrt{(2l+1)\sigma_l}\, g_l^m(x).$$

Using the new notation, the three-term recurrence relation for g_l^m becomes

$$b_l(m) y_{l-1}^m(x) - \frac{x}{\mu_t} y_l^m(x) + b_{l+1}(m) y_{l+1}^m(x) = 0,$$

where

$$b_l(m) = \sqrt{\frac{l^2 - m^2}{(4l^2-1)\sigma_l \sigma_{l-1}}}.$$

By imposing the truncation condition

$$g_{M+1}^m(\xi) = 0, \qquad M = l_{\max} + m \quad \text{or} \quad l_{\max},$$

where M determines the highest degree of P_l^m used to express $\psi(x, \theta)$, we arrive at the eigenproblem

$$B(m) Y_\xi(m) = \frac{\xi}{\mu_t} Y_\xi(m),$$

where $Y_\xi(m) = (y_m^m(\xi), y_{m+1}^m(\xi), \ldots, y_M^m(\xi))^T$. The tridiagonal matrix $B(m)$ is given by [12, 14]

$$\{B(m)\}_{ll'} = b_l(m)\delta_{l',l-1} + b_{l'}(m)\delta_{l',l+1}.$$

In the method of rotated reference frames [12, 14], eigenmodes are labeled by eigenvalues of $B(m)$. The number of rows and columns of $B(m)$ is $l_{\max} + 1$ when $M = l_{\max} + m$ and is $l_{\max} - m + 1$ when $M = l_{\max}$. Since $B(m)$ is a symmetric tridiagonal matrix with nonzero off-diagonal elements, its eigenvalues are distinct. Also if ξ/μ_t is an eigenvalue for $y_l^m(\xi)$, then $-\xi/\mu_t$ is another eigenvalue and $y_l^m(-\xi) = (-1)^l y_l^m(\xi)$ [12]. Essentially the same tridiagonal matrix W was introduced in [15]. Elements of W are given by $\{W\}_{ll'} = w_l(m)\delta_{l',l-1} + w_{l'}(m)\delta_{l',l+1}$, where $w_l(m) = \sqrt{(l^2 - m^2)/(h_l h_{l-1})}$. Let ξ_j/μ_t ($j = 1, \ldots, l_{\max}+1$) denote eigenvalues of $B(m)$. We note that $\{\xi_j\}$ are eigenvalues of W.

For simplicity, hereafter, we suppose $M = l_{\max} + m$ and $l_{\max} \geq 1$ is an odd integer. There are $(l_{\max}+1)/2$ positive eigenvalues and $(l_{\max}+1)/2$ negative eigenvalues for each m. Then we can write eigenvalues as

$$\xi_1 > \xi_2 > \cdots > \xi_{\frac{l_{\max}+1}{2}} > 0 > \xi_{\frac{l_{\max}+1}{2}+1} > \cdots > \xi_{l_{\max}+1},$$

and $\xi_{l_{\max}+2-j} = -\xi_j$ $(j = 1, \ldots, (l_{\max}+1)/2)$.

The following lemmas hold.

Lemma 3.1 (Orthogonality [13, 15])**.** *We have*

$$\frac{1}{Z_j} \sum_{l=m}^{l_{\max}+m} y_l^m(\xi_i) y_l^m(\xi_j) = \delta_{ij}, \quad i, j = 1, 2, \ldots, l_{\max}+1,$$

where $Z_j = \sum_{l=m}^{l_{\max}+m} [y_l^m(\xi_j)]^2$.

Proof. Eigenvectors corresponding to two distinct eigenvalues of a symmetric real matrix are orthogonal. □

Lemma 3.2 (Completeness [13, 15])**.** *We have*

$$\sum_{j=1}^{l_{\max}+1} \frac{1}{Z_j} y_l^m(\xi_j) y_{l'}^m(\xi_j) = \delta_{ll'}, \quad l, l' = m, m+1, \ldots, m+l_{\max},$$

where $Z_j = \sum_{l=m}^{l_{\max}+m} [y_l^m(\xi_j)]^2$.

Proof. Let us introduce vectors $X_j = Y_{\xi_j}(m)/\sqrt{Z_j}$ and matrix $X = (X_1, \ldots, X_{l_{\max}+1})$. Then X is an orthogonal matrix: $X^{-1} = X^T$. Next we introduce matrices $Z = \mathrm{diag}(\sqrt{Z_1}, \ldots, \sqrt{Z_{l_{\max}+1}})$ and $Y = (Y_{\xi_1}(m), \ldots, Y_{\xi_{l_{\max}+1}}(m))$. The matrix Y is expressed as $Y = XZ$.

Let us consider

$$\sum_{j=1}^{l_{\max}+1} D_j y_{l'}^m(\xi_j) = \delta_{ll'}, \quad l, l' = m, m+1, \ldots, m+l_{\max}.$$

To find D_j $(j = 1, \ldots, l_{\max}+1)$, we introduce vectors $D = (D_1, \ldots, D_{l_{\max}+1})^T$ and $F = (\delta_{m,l}, \delta_{m+1,l}, \ldots, \delta_{m+l_{\max},l})^T$, and write the relation as $YD = F$. Since $D = Z^{-1}X^T F$, we obtain

$$D_j = \frac{1}{Z_j} y_l^m(\xi_j).$$

This completes the proof. □

Remark 3.3. If we define $\left|y_{\xi_j}(m)\right\rangle = Y_{\xi_j}(m)/\sqrt{Z_j}$ and $\left\langle l \middle| y_{\xi_j}(m)\right\rangle = y_l^m(\xi_j)/\sqrt{Z_j}$, then the orthogonality and completeness in Lemma 3.1 and Lemma 3.2 are equivalently expressed as

$$\left\langle y_{\xi_i}(m) \middle| y_{\xi_j}(m)\right\rangle = \delta_{ij}, \qquad \sum_{j=1}^{l_{\max}+1} \left|y_{\xi_j}(m)\right\rangle \left\langle y_{\xi_j}(m)\right| = 1.$$

4. Concluding remarks

In this paper, we focused on the case $\mu_a > 0$, i.e., $\varpi < 1$. It is possible to consider the conservative (nonabsorbing) case $\mu_a = 0$ but it must be done separately. When $\varpi = 1$, $\sigma_0 = 0$ and the element $b_1(0)$ becomes infinity. In this case, we need to remove the top left part of $B(0)$.

In application, the numerical evaluation of the Chandrasekhar polynomials is important. Various numerical techniques have been developed [5, 6, 7].

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