# Nonexistence results for Levi-flat real hypersurfaces

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## 1 Introduction

A real hypersurface $M$ (of class at least $\mathcal{C}^2$) in a complex manifold is called Levi-flat if its Levi-form vanishes identically or, eqivalently, if it admits a foliation by complex hypersurfaces. Another equivalent formulation is that $M$ is locally pseudoconvex from both sides.

Levi-flat real hypersurfaces are locally equivalent to each other, thus only global properties are of interest from the viewpoint of classification results.

In several complex variables, the first nontrivial examples appeared when looking for examples of locally pseudoconvex domains (in a complex manifold) that are not Stein. In fact Grauert described a class of Levi-flat real hypersurfaces as tubular neighborhoods of the zero section of a generically chosen line bundle over a non-rational Riemann surface [G]. In these examples, the Levi-flat hypersurfaces arise as the boundary of a pseudoconvex domain admitting only constant holomorphic functions. On the other hand, there are also examples of compact Levi-flat real hypersurfaces bounding Stein domains. For example, the product of an annulus and the punctured plane is bilomorphically equivalent to a domain in $\mathbb{P}^1 \times \{\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})\}$ with Levi-flat boundary [O1]. Further examples of complex surfaces that can be cut into two Stein domains along smooth Levi-flat real hypersurface can be found in [N]. From [A] one even obtains examples of Levi-flat hypersurfaces in complex surfaces having hyperconvex complement.

These examples above show that Levi-flat hypersurfaces can be of quite different nature and therefore explain a certain interest in the classification of compact Levi-flat real hypersurfaces. Let us also mention that some of these constructions can be extended to higher dimensions.

On the other hand, the study of Levi-flat real hypersurfaces is related to basic questions in dynamical systems and foliation theory: Levi-flats arise as stable sets of holomorphic foliations, and a real-analytic Levi-flat real hypersurface extends to a holomorphic foliation leaving $M$ invariant. Relating to this, a famous open problem is whether or not $\mathbb{CP}^2$ contains a smooth

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Levi-flat real hypersurface. This problem arose as part of a conjecture that, for any codimension one holomorphic foliation on $\mathbb{CP}^2$ (with singularities), any leaf accumulates to a singular point of the foliation [C-L-S]. This problem is still open. It is only known that if $\mathbb{CP}^2$ admits a smooth Levi-flat real hypersurface, then it has to satisfy a restrictive curvature condition [A-B].

In the following, we shall be interested in Levi-flat real hypersurfaces from the viewpoint of its normal bundle:

Given a Levi-flat real hypersurface $M$ in a complex manifold $X$ of dimension $n$, we call $N_M^{1,0} = (T_X^{1,0})_{|M}/T^{1,0}M$ the holomorphic normal bundle of $M$. The restriction of $N_M^{1,0}$ to each $(n-1)$-dimensional complex submanifold of $M$ has a structure of a holomorphic line bundle induced from that of $T_X^{1,0}$.

**Acknowledgements.** I wish to express my thanks to Adachi Masanori and Takeo Ohsawa for organizing the RIMS Symposium on "Topology of pseudoconvex domains and analysis of reproducing kernels" in November 2017, where we discussed several questions related to this manuscript.

## 2 Nonexistence results

For $n \geq 3$, it is known that there does not exist any smooth real Levi-flat hypersurface $M$ in $\mathbb{CP}^n$. This was first proved by LinsNeto in [LN] for real-analytic $M$ and by Siu in [S] for $\mathcal{C}^{12}$-smooth $M$. For further improvements concerning the regularity, we refer the reader to [I-M] and [C-S].

The proofs of the above-mentioned results essentially exploited the positivity of $T^{1,0}\mathbb{CP}^n$. Brunella's main observation [Br] was that the positivity of the normal bundle itself is enough to ensure that the complement of $M$ is pseudoconvex. If $X = \mathbb{CP}^n$, or if $X$ admits a hermitian metric of positive curvature, then the normal bundle $N_M^{1,0}$ is automatically positive (it is a quotient of $T^{1,0}X$, and therefore more positive than $T^{1,0}X$).

This led Brunella to prove that if $X$ is a compact Kähler manifold with $\dim X \geq 3$, and if $M$ is a smooth Levi-flat real hypersurface such that there exists a holomorphic foliation on a neighborhood of $M$ leaving $M$ invariant, then the normal bundle of this foliation does not admit any fiber metric with positive curvature.

*Sketch of Brunella's proof in* [Br]. Assume that $X$ is a connected compact Kähler manifold of dimension $n \geq 3$, and let $M$ be a smooth Levi-flat real hypersurface such that there exists a holomorphic foliation on a neighborhood of $M$ leaving $M$ invariant. Under the assumption that the normal bundle of this foliation admits a fiber metric with positive curvature, Brunella shows that $X \setminus M$ is strongly pseudoconvex. Then the argument is as follows: Since the normal bundle of the foliation is topologically trivial, its curvature form $\theta$ is $d$-exact on a tubular neighborhood $U$ of $M$. Thus $\theta = d\beta$ on $U$, where the primitive $\beta = \beta^{1,0} + \beta^{0,1}$ can be chosen of real type ($\overline{\beta}^{1,0} = \beta^{0,1}$) and one has $\overline{\partial}\beta^{0,1} = 0$. Since $\dim X \geq 3$, the vanishing theorem of Gauert and Riemenschneider combined with Serre's duality implies that the $\overline{\partial}$-cohomology with compact support $H^{0,2}(X \setminus M)$ is zero. This means that one can extend $\beta^{0,1}$ $\overline{\partial}$-closed to $X$. Hodge symmetry on the Kähler manifold $X$ means $H^{0,1}(X) \simeq \overline{H^{1,0}(X)}$. Hence $\beta^{0,1} = \eta + \overline{\partial}\alpha$, with $\partial\eta = 0$. But then $\partial\beta^{0,1} = \partial\overline{\partial}\alpha$. Therefore, setting $\phi = i(\overline{\alpha} - \alpha)$, one obtains $\theta = i\partial\overline{\partial}\phi$. The existence of a potential for the positive curvature form is, however, a contradiction to the maximum principle on the leaves of the foliation. □

Ohsawa generalized this in [O2] to a nonexistence result for smooth Levi-flat real hypersurfaces admitting a fiber metric whose curvature form is semi-positive of rank $\geq 2$ along the leaves of $M$ (in any compact Kähler manifold).

Recently we have obtained a generalized version of Brunella's result in the sense that we are able to drop the compact Kähler assumption on the ambient $X$ (Theorem 2.1). This was conjectured in [O3, Conjecture 5.1]. The full proof will appear elsewhere.

**Theorem 2.1**
*Let $X$ be a complex manifold of dimension $n \geq 3$. Then there does not exist a smooth compact Levi-flat real hypersurface $M$ in $X$ such that the normal bundle to the Levi foliation admits a Hermitian metric with positive curvature along the leaves.*

*Sketch of the proof.* Our proof follows the general idea of Brunella explained above. We assume by contradiction that there exists a smooth compact Levi-flat real hypersurface $M$ in $X$ such that the normal bundle to the Levi foliation admits a Hermitian metric with positive curvature along the leaves. However, since our $M$ is not embedded in a compact Kähler manifold, we have to make several important modifications. Since $M$ has a tubular neighborhood which is pseudoconcave (of dimension $\geq 3$), this tubular neighborhood can be compactified to a compact manifold $\tilde{X}$ by a theorem of Andreotti/Siu and Rossi. Then $\tilde{X} \setminus M$ is a strongly pseudoconvex manifold, and we can even arrange that it carries a complete Kähler metric.

By means of $L^2$-estimates on $\tilde{X} \setminus M$, we then extend the normal bundle to $M$ to a holomorphic line bundle over $\tilde{X}$. We also show that $CR$ sections of high tensor powers of the normal bundle extend to holomorphic sections over $\tilde{X}$, again by means of solving some Cauchy-problem for the $\overline{\partial}$-equation using $L^2$-estimates. This permits us to find sufficiently many sections that provide a holomorphic embedding of a tubular neighborhood of $M$ into a compact Kähler manifold. This proves the nonexistence of such $M$ as before.

# 3 Examples of Levi-flats with positive normal bundle

The following example from [Br, Example 4.2] and [O3, Theorem 5.1] shows that Theorem 2.1 cannot hold for $n = 2$, even for $X$ compact Kähler:

Let $\Sigma$ be a compact Riemann surface of genus $g \geq 2$. Let $\mathbb{D}$ be the open unit disc, and let $\Gamma$ be a discrete subgroup of $\mathrm{Aut}\mathbb{D} \subset \mathrm{Aut}\mathbb{CP}^1$ such that $\Sigma \simeq \mathbb{D}/\Gamma$. Then $\Gamma$ also acts on $\mathbb{D} \times \mathbb{CP}^1$ by

$$(z, w) \mapsto (\gamma(z), \gamma(w)), \ \ \gamma \in \Gamma.$$

The quotient $X = (\mathbb{D} \times \mathbb{CP}^1)/\Gamma$ is a compact complex surface, ruled over $\Sigma$ (and hence projective). Let $\pi : \mathbb{D} \times \mathbb{CP}^1 \longrightarrow X$ denote the projection.

From the horizontal foliation on $\mathbb{D} \times \mathbb{CP}^1$, we get a holomorphic foliation $\mathcal{F}$ on $X$. $\pi(\mathbb{D} \times \{w\}), \ w \in \mathbb{CP}^1$ are the leaves of $\mathcal{F}$. $M = \mathbb{D} \times S^1/\Gamma$ is a real analytic Levi-flat hypersurface invariant by $\mathcal{F}$.

The Bergman metric induces a metric with positive curvature on the normal bundle of $M$. We recall the construction from [O3]: The Bergman metric

$$\frac{1}{(1 - |w|^2)^2} dw \otimes d\overline{w}$$

on $\mathbb{D} \cup (\mathbb{CP}^1 \setminus \overline{\mathbb{D}})$ is a fiber metric of $N_M^{1,0}$ on $X \setminus \pi(\mathbb{D} \times S^1)$, because it is invariant under $\Gamma$. We define the smooth function $\rho$ by

$$\rho(z, w) = \begin{cases} \left(1 - \left|\dfrac{z - w}{1 - z\overline{w}}\right|^2\right)^2 \text{ if } z, w \in \mathbb{D} \\ \left(1 - \left|\dfrac{1 - z\overline{w}}{z - w}\right|^2\right)^2 \text{ if } z \in \mathbb{D}, w \in \mathbb{CP}^1 \setminus \mathbb{D} \end{cases}$$

Multiplying $(1 - |w|^2)^{-2} dw \otimes d\overline{w}$ by $\rho$, one obtains a smooth fiber metric of $N_M^{1,0}$ that has positive curvature along the leaves (a standard computation shows that the curvature is twice the Bergman metric along the leaves).

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