Asymptotic solution of Bowen equation for perturbed potentials defined on shift spaces

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Abstract

We study the asymptotic solution of the equation of the pressure function $s \mapsto P(s\phi_\epsilon)$ for a perturbed potential ϕ_ϵ defined on a shift space with countable state. We show that if the perturbed potential ϕ_ϵ has an asymptotic expansion for a small parameter ϵ and some conditions are satisfied, then the solution $s = s(\epsilon)$ of $P(s\phi_\epsilon) = 0$ has also an asymptotic behaviour with same order. In addition, we also give the case where the order of the expansion of the solution $s = s(\epsilon)$ is less than the order of the expansion of the perturbed potential ϕ_ϵ. Our results can be applied to problems concerning asymptotic behaviors of Hausdorff dimensions obtained from Bowen formula.

1 Preliminaries

In this section we will recall the notion of thermodynamic formalism and some facts of Ruelle transfer operators which were manly introduced by Sarig [4, 5, 6].

Let $G = (V, E, i(\cdot), t(\cdot))$ be a directed multigraph endowed with countable vertex set V, countable edge set E, and two maps $i(\cdot)$ and $t(\cdot)$ from E to V. For each $e \in E$, $i(e)$ is called the initial vertex of e and $t(e)$ called the terminal vertex of e. Denoted by E^∞ the one-sided shift space $\{\omega = \omega_0\omega_1 \cdots \in \prod_{k=0}^\infty E : t(\omega_k) = i(\omega_{k+1}) \text{ for any } k \geq 0\}$. The shift transformation $\sigma : E^\infty \to E^\infty$ is defined by $(\sigma\omega)_k = \omega_{k+1}$ for any $k \geq 0$. For $\theta \in (0, 1)$, a metric d_θ on E^∞ is given by $d_\theta(\omega, \upsilon) = \theta^{\inf\{k \geq 0 : \omega_k \neq \upsilon_k\}}$. The metric space (E^∞, d_θ) is compact if E is finite. On the other hand, this metric space is complete and separable and however may be not compact when E is infinite.

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let $C(E^\infty, \mathbb{K})$ be the set of all $\mathbb{K}$-valued continuous functions on E^∞, and $F_\theta(E^\infty, \mathbb{K})$ the set of all $\mathbb{K}$-valued d_θ-Lipschitz continuous functions on E^∞. We define $C_b(E^\infty, \mathbb{K})$ as the set of all functions $f \in C(E^\infty, \mathbb{K})$ with $\|f\|_\infty < \infty$ and $F_{\theta,b}(E^\infty, \mathbb{K})$ as the set of all functions $f \in F_\theta(E^\infty, \mathbb{K})$ with $\|f\|_\theta < \infty$, where $\|f\|_\theta = \|f\|_\infty + [f]_\theta$, $\|f\|_\infty = \sup_{\omega \in E^\infty} |f(\omega)|$ and $[f]_\theta = \sup\{|f(\omega) - f(\upsilon)|/d_\theta(\omega, \upsilon) : \omega, \upsilon \in E^\infty, \ \omega \neq \upsilon, \ \omega_0 = \upsilon_0\}$. It is known that if E is finite then the equalities $C(E^\infty, \mathbb{K}) = C_b(E^\infty, \mathbb{K})$ and $F_\theta(E^\infty, \mathbb{K}) = F_{\theta,b}(E^\infty, \mathbb{K})$ hold. For simplicity, the notation $\mathbb{K}$ is omitted from these definitions when $\mathbb{K} = \mathbb{C}$.

The incidence matrix $A = (A(ee'))$ of E^∞ is an $E \times E$ zero-one matrix defined by $A(ee') = 1$ if $t(e) = i(e')$ and $A(ee') = 0$ if $t(e) \neq i(e')$. The matrix A is said to be *finitely irreducible* if there exists a finite subset F of $\bigcup_{k=0}^{\infty} E^k$ such that for any $e, e' \in E$, ewe' is a path on the graph G for some $w \in F$. This matrix A is called *finitely primitive* if there exist an integer $n \geq 1$ and a finite subset F of E^k for some $k \geq 0$ such that for any $e, e' \in E$, ewe' is a path on the graph G for some $w \in F$. Note that A is finitely irreducible if and only if the dynamics (E^∞, σ) is topologically transitive and A has the *big images and pre-images property* [5], i.e. there exists a finite set F of E such that for any $e \in E$, $A(e'e)A(ee'') = 1$ for some $e', e'' \in F$. Similarity, A is finitely primitive if and only if (E^∞, σ) is topologically mixing and A has the big images and pre-images property.

Assume that the incidence matrix of E^∞ is finitely irreducible and $\psi : E^\infty \to \mathbb{R}$ is in $F_\theta(E^\infty, \mathbb{R})$. We recall the *topological pressure* $P(\psi)$ of ψ defined by

$$P(\psi) = \lim_{k \to \infty} \frac{1}{k} \log \sum_{w \in E^k \,:\, [w] \neq \emptyset} \exp\Big(\sup_{\omega \in [w]} \sum_{j=0}^{k-1} \psi(\sigma^j \omega)\Big),$$

where $[w] = \{\omega \in E^\infty : \omega_0 \cdots \omega_{k-1} = w\}$ denotes the cylinder of a word $w \in E^k$. This limit exists in $(-\infty, +\infty]$ (see [2]). A σ-invariant Borel probability measure μ on E^∞ is said to be a *Gibbs measure* of a function $\psi : E^\infty \to \mathbb{R}$ if there exist constants $c \geq 1$ and $P \in \mathbb{R}$ such that for any $\omega \in E^\infty$ and $k \geq 1$

$$c^{-1} \leq \frac{\mu([\omega_0 \omega_1 \cdots \omega_{k-1}])}{\exp(-kP + \sum_{j=0}^{k-1} \psi(\sigma^j \omega))} \leq c.$$

For the existence of this measure, see Theorem 1.1 below (see also [5]).

For a real-valued function ψ defined on E^∞, the Ruelle operator $\mathcal{L}_\psi$ associated to ψ is defined by

$$\mathcal{L}_\psi f(\omega) = \sum_{e \in E \,:\, t(e) = i(\omega_0)} e^{\psi(e \cdot \omega)} f(e \cdot \omega)$$

if this series converges in $\mathbb{C}$ for a complex-valued function f on E^∞ and for $\omega \in E^\infty$. Here $e \cdot \omega$ is the concatenation of e and ω, i.e. $e \cdot \omega = e\omega_0\omega_1 \cdots \in E^\infty$. It is known that if E^∞ is finitely irreducible and ψ is in $F_\theta(E^\infty, \mathbb{R})$ with finite pressure, then $\mathcal{L}_\psi$ is a bounded linear operator both on $F_{\theta,b}(E^\infty)$ and on $C_b(E^\infty)$.

The following is a version of Ruelle-Perron-Frobenius Theorem for $\mathcal{L}_\psi$:

Theorem 1.1 ([1, 4]) *Let $G = (V, E, i(\cdot), t(\cdot))$ be a directed multigraph such that A is finitely irreducible. Assume that $\psi \in F_\theta(E^\infty, \mathbb{R})$ with $P(\psi) < \infty$. Then there exists a unique triplet $(\lambda, h, \nu) \in \mathbb{R} \times F_{\theta,b}(E^\infty) \times C_b(E^\infty)^*$ such that the following are satisfied:*

(1) The number λ is positive and a simple maximal eigenvalue of the operator $\mathcal{L}_\psi : F_{\theta,b}(E^\infty) \to F_{\theta,b}(E^\infty)$.

(2) The operator $\mathcal{L}_\psi : F_{\theta,b}(E^\infty) \to F_{\theta,b}(E^\infty)$ has the decomposition

$$\mathcal{L}_\psi = \lambda \mathcal{P} + \mathcal{R}$$

with $\mathcal{PR} = \mathcal{RP} = O$. Here the operator $\mathcal{P}$ is a projection onto the one-dimensional eigenspace of the eigenvalue λ. Moreover, this has the form $\mathcal{P}f = \int_{E^\infty} fh\, d\nu$ for $f \in C_b(E^\infty)$, where $h \in F_{\theta,b}(E^\infty, \mathbb{R})$ is the corresponding eigenfunction of λ and ν is the corresponding eigenvector of λ of the dual $\mathcal{L}_\psi^*$ with $\nu(h) = 1$. In particular, h satisfies $0 < \inf_\omega h(\omega) \le \sup_\omega h(\omega) < \infty$ and ν is a Borel probability measure on E^∞.

(3) The spectrum of $\mathcal{R} : F_{\theta,b}(E^\infty) \to F_{\theta,b}(E^\infty)$ is contained in $\{z \in \mathbb{C} : |z - \lambda| \ge \rho\}$ for some small $\rho > 0$.

Note that the eigenvalue λ is equal to $\exp(P(\psi))$ and $h\nu$ becomes the Gibbs measure of the potential ψ. For simplicity, we sometimes call h the Perron eigenfunction of $\mathcal{L}_\varphi$ and ν the Perron eigenvalue of $\mathcal{L}_\varphi^*$.

2 Known result : the case when E is finite

We give an asymptotic solution of $P(s\varphi(\epsilon, \cdot)) = 0$ for $s \in \mathbb{R}$ under the finite state space given in [7]. Recall that if $\sharp E < +\infty$, $\varphi \in F_\theta(E^\infty, \mathbb{R})$ and $\varphi < 0$ are satisfied, then the equation $P(s\varphi) = 0$ has a unique solution $s \ge 0$. Then we have the following.

Theorem 2.1 ([7, Theorem 2.6]) *Assume that E is finite and the incidence matrix A of E^∞ is (finitely) irreducible. Assume also that functions $g(\epsilon, \cdot) : E^\infty \to \mathbb{R}$ has the form $g(\epsilon, \cdot) = g + g_1\epsilon + \cdots + g_n\epsilon^n + \tilde{g}_n(\epsilon, \cdot)\epsilon^n$ for some Hölder continuous functions $g = g_0, g_1, g_2, \ldots, g_n, \tilde{g}_n(\epsilon, \cdot)$ from the metric space (E^∞, d_θ) to $\mathbb{R}$ with $0 < \|g\|_\infty < 1$ and $\lim_{\epsilon \to 0} \|\tilde{g}_n(\epsilon, \cdot)\|_\infty = 0$. Take a unique solution $s = s_0 \ge 0$ of the equation $P(s \log |g|) = 0$. Then for any small $\epsilon > 0$, the solution $s = s(\epsilon)$ of the equation $P(s \log |g(\epsilon, \cdot)|) = 0$ exists uniquely. Moreover, there exist numbers $s_1, s_2, \ldots, s_n \in \mathbb{R}$ such that $s(\epsilon)$ has an n-order asymptotic expansion*

$$s(\epsilon) = s_0 + s_1\epsilon + \cdots + s_n\epsilon^n + o(\epsilon^n).$$

as $\epsilon \to 0$.

Corollary 2.2 *Under the same conditions of Theorem 2.1, we also assume that the remainder $\tilde{g}_n(\epsilon, \cdot)$ of $g(\epsilon, \cdot)$ is equal to zero. Then the solution $s(\epsilon)$ has an m-order asymptotic expansion for any $m \ge 1$.*

Remark 2.3 Each coefficient s_k in Theorem 2.1 is precisely decided (see the proof of Theorem 2.6 in [7]). Indeed, this number is given for $k = 1, 2, \dots, n$ inductively by

$$s_k = \frac{-1}{\mu(\log|g|)} \left(\sum_{i=1}^{k-1} \nu_{k-i}(\mathcal{L}_{s_0 \log|g|}((\log|g|)h))s_i + \sum_{j=1}^{k} \nu_{k-j}(\mathcal{M}_j h) \right),$$

where μ is the Gibbs measure of $s_0 \log|g|$, $\mathcal{M}_1, \dots, \mathcal{M}_n$ are coefficients of the expansion $\mathcal{L}_{s(\epsilon)\log|g(\epsilon,\cdot)|} = \mathcal{L}_{s_0 \log|g|} + \sum_{j=1}^{n} \mathcal{M}_j \epsilon^j + (s(\epsilon) - s_0)\mathcal{L}_{s_0 \log|g|}((\log|g|) \ \cdot \ ) + o(\epsilon^n)$ of the Ruelle operator $\mathcal{L}_{s(\epsilon)\log|g(\epsilon,\cdot)|}$, $\nu_0, \cdots, \nu_n$ are coefficients of the expansion $\nu(\epsilon, f) = \nu_0(f) + \sum_{i=1}^{n-1} \nu_i(f)\epsilon^i + o(\epsilon^{n-1})$ for $f \in F_{\theta,b}(E^\infty)$ of the Perron eigenvector $\nu(\epsilon, \cdot)$ of the dual of $\mathcal{L}_{s(\epsilon)\log|g(\epsilon,\cdot)|}$, and h is the Perron eigenfunction of $\mathcal{L}_{s_0 \log|g|}$.

3 Main results : the case when E is infinite

First we will state one of our main results. Assume that there exist functions $g, g_1, \dots, g_n \in F_{\theta,b}(E^\infty, \mathbb{R})$, $g(\epsilon, \cdot), \tilde{g}_n(\epsilon, \cdot) \in F_{\theta(\epsilon)}(E^\infty, \mathbb{R})$ and numbers $c_1, c_2, c_3 > 0$, $q \in (0, 1]$ and $c_4(\epsilon) > 0$ with $\lim_{\epsilon \to 0} c_4(\epsilon) = 0$ such that

$$g(\epsilon, \cdot) = g + g_1\epsilon + \dots + g_n\epsilon^n + \tilde{g}_n(\epsilon, \cdot)\epsilon^n \tag{1}$$

$$\|g\|_\infty < 1 \tag{2}$$

$$\text{either } \inf_{\omega\in[e]} g(\omega) > 0 \text{ or } \sup_{\omega\in[e]} g(\omega) < 0 \quad \text{for each } e \in E \tag{3}$$

$$||g(\omega)| - |g(\upsilon)|| \le c_1|g(\omega)|d_\theta(\omega, \upsilon) \quad \text{for } \omega, \upsilon \in E^\infty \text{ with } \omega_0 = \upsilon_0 \tag{4}$$

and the inequalities

$$|g_k(\omega)| \le c_2|g(\omega)|^q \quad \text{for } k = 1, 2, \dots, n \text{ and } \omega \in E^\infty \tag{5}$$

$$|g_k(\omega) - g_k(\upsilon)| \le c_3|g(\omega)|^q d_\theta(\omega, \upsilon) \quad \text{for } \omega, \upsilon \in E^\infty \text{ with } \omega_0 = \upsilon_0 \tag{6}$$

$$|\tilde{g}_n(\epsilon, \omega)| \le c_4(\epsilon)|g(\omega)|^q \quad \text{for } \omega \in E^\infty \tag{7}$$

hold for any $\omega \in E^\infty$, $k = 1, 2, \dots, n$ and any small $\epsilon > 0$. Let

$$\underline{s} := \inf\{s \ge 0 \ : \ P(s\log|g|) < +\infty\}$$

and put $S(n) := \max(\underline{s} + (1-q)n, \underline{s}/q)$. We also assume that

$$\text{there exists } s_0 > S(n) \text{ such that } P(s_0 \log|g|) = 0. \tag{8}$$

Then we have the following:

Theorem 3.1 ([8]) *Fix a nonnegative integer n. Let $G = (V, E, i(\cdot), t(\cdot))$ be a graph multigraph satisfying that V, E are countable and the incidence matrix of E^∞ is finitely* *irreducible. Assume that the conditions (1)-(8) are satisfied. Then exists a unique solution* $s = s(\epsilon)$ *of the equation* $P(s \log |g(\epsilon, \cdot)|) = 0$ *for any small* $\epsilon > 0$*, and there exist numbers* $s_1, \ldots, s_n \in \mathbb{R}$ *such that* $s(\epsilon)$ *has an n-order asymptotic expansion*

$$s(\epsilon) = s_0 + s_1\epsilon + \cdots + s_n\epsilon^n + \tilde{s}_n(\epsilon)\epsilon^n$$

with $\tilde{s}_n(\epsilon) \to 0$ *as* $\epsilon \to 0$*.*

Note that each coefficient s_k is decided as well as in Remark 2.3.

Next we give a sufficient condition for a case that $s(\epsilon)$ does not have an $(n+1)$-order asymptotic expansion under the conditions the conditions (1)-(8). We introduce the following conditions:

$$E \text{ is infinitely countable} \tag{9}$$

$$\tilde{g}_n(\epsilon, \cdot) \equiv 0 \tag{10}$$

$$q < 1 \tag{11}$$

$$|g(\omega)| \leq |g(\epsilon, \omega)| \text{ for any } \omega \in E^\infty \text{ and for any small } \epsilon > 0 \tag{12}$$

$$\text{there exist } q_0 \in [q, \frac{nq+1}{n+1}) \text{ and } c_5 > 0 \text{ such that } g_1(\omega) \geq c_5 \frac{|g(\omega)|^{q_0}}{\mathrm{sign}(g(\omega))} \tag{13}$$

$$s_0 < \underline{q} + (1 - q_0)(n+1). \tag{14}$$

Proposition 3.2 ([8]) *Under the same conditions of Theorem 3.1, we also assume the conditions (9)-(14) are satisfied. Then the remainder of the expansion* $s(\epsilon) = s_0 + s_1\epsilon + \cdots + s_n\epsilon^n + \tilde{s}_n(\epsilon)\epsilon^n$ *satisfies* $\lim_{\epsilon \to \infty} |\tilde{s}_n(\epsilon)|/\epsilon = +\infty$*. (Compare with Corollary 2.2).*

4 Exmaples

4.1 Nussbaum-Priyadarshi-Lunel's infinite iterative function systems

We refer to [3]. We assume the following (i)-(ix):

(i) $G = (\{v\}, E = \{1, 2, \cdots\})$ is an infinitely directed graph with singleton vertex.

(ii) (J, d) is a compact metric space. Moreover, J is *perfect set*, namely for any $x \in J$ there exists a sequence $x_k \in J$ with $x_k \neq x$ $(k \geq 1)$ such that $\lim_{k \to \infty} d(x_k, x) = 0$.

(iii) For any $e \in E$, $T_e : J \to J$ is a Lipschitz map satisfying $\sup_{e \in E} \mathrm{Lip}(T_e) =: r < 1$.

(iv) Each T_e is an *infinitesimal similitude* on J, i.e. for each $x \in J$, for any sequences (x_k) and (y_k) on J with $x_k \neq y_k$ for each $k \geq 1$ and $x_k \to x$ and $y_k \to x$, the limit

$$\lim_{k \to \infty} \frac{d(T_e(x_k), T_e(y_k))}{d(x_k, y_k)} =: DT_e(x)$$

exists in $\mathbb{R}$ and is independent of the particular sequences (x_k) and (y_k).

(v) $DT_e(x) > 0$ For any $e \in E$ and $x \in J$.

(vi) There exist constants $c > 0$ and $\beta > 0$ such that for any $x, y \in J$, $|DT_e(x) - DT_e(y)| \leq c|DT_e(x)|d(x,y)^\beta$.

(vii) There exist $t > 0$ and $x \in J$ such that $\sum_{e \in E}(DT_e(x))^t < +\infty$.

(viii) For any $\eta > 0$ there exists $c(\eta) \geq 1$ with $\lim_{\eta \to +0} c(\eta) = 1$ such that for each $e \in E$ and $x, y \in J$ with $0 < d(x,y) < \eta$,

$$c(\eta)^{-1} DT_e(x) \leq \frac{d(T_e(x), T_e(y))}{d(x,y)} \leq c(\eta) DT_e(x).$$

(ix) For each $k \geq 1$, the limit set K_k of the finite iterated function system $(T_1, T_2, \ldots, T_k)$ satisfies that the restricted map $T_e(K_k) \cap T_{e'}(K_k) = \emptyset$ for each $1 \leq e < e' \leq k$ with $e \neq e'$, and $T_e|_{K_k} : K_k \to J$ is one to one for $1 \leq e \leq k$.

Such a system is firstly introduced by Nussbaum, Priyadarshi and Lunel in [3]. For convenience, we call such a system $(J, (T_e))$ an *NPL system.*

The coding map $\pi : E^\infty \to J$ is defined as $\{\pi\omega\} = \{\bigcap_{k=0}^\infty T_{\omega_0} \circ \cdots \circ T_{\omega_k}(J)\}$. Let K be the limit set of the system $(G, J, (O_e))$ which is given by $K = \pi(E^\infty)$. Put $\varphi(\omega) = \log DT_{\omega_0}(\pi\sigma\omega)$. Then a version of Bowen's formula is described as follows:

Theorem 4.1 *Assume that* $(J, (T_e))$ *is an NPL system and* K *is its limit set. Then we have* $\dim_H K = \inf\{s > 0 : P(s\varphi) < 0\}$. *Moreover, if* $(J, (T_e))$ *is strongly regular, i.e.* $0 < P(s\varphi) < +\infty$ *for some* $s > 0$, *then* $\dim_H K = s$ *if and only if* $P(s\varphi) = 0$.

Fix $n \geq 0$. To formulate asymptotic perturbation of NPL systems, we consider the following conditions:

(I) A pair $(J, (T_e))$ is a strongly regular NPL system satisfying that J is a compact subset of a Banach space $(X, \|\cdot\|)$. Moreover, there exists an open connected subset O of X containing J such that T_e is extended to a map of $C^n(O, X)$ and DT_e is extended to a map of $C^{n+\beta}(O, \mathbb{R})$.

(II) A pair $(J, (T_e(\epsilon, \cdot)))$ is an NPL system with a small parameter $\epsilon > 0$ satisfying that there exists a number $s \in (\underline{s}/\dim_H K, 1]$ if $n = 0$ or $s \in (1 - (\dim_H K - \underline{s})/n, 1]$ if $n \geq 1$ such that the following conditions (a)-(d) are satisfied:

(a) For each $e \in E$, $T_e(\epsilon, \cdot)$ has the n-asymptotic expansion:

$$T_e(\epsilon, \cdot) = T_e + \textstyle\sum_{k=1}^n T_{e,k}\epsilon^k + \tilde{T}_{e,n}(\epsilon, \cdot)\epsilon^n \quad \text{on } J$$

for some mappings $T_{e,k} \in C^{n-k+1}(O, X)$ $(k = 1, 2, \ldots, n)$ and $\tilde{T}_{e,n}(\epsilon, \cdot) \in C^1(O, X)$ with $\sup_{e \in E} \sup_{x \in J} \|\tilde{T}_{e,n}(\epsilon, x)\| \to 0$.

(b) For each $e \in E$, $DT_e(\epsilon, \cdot)$ has an n-order asymptotic expansion:

$$DT_e(\epsilon, \cdot) = DT_e + \sum_{k=1}^n S_{e,k}\epsilon^k + \tilde{S}_{e,n}(\epsilon, \cdot)\epsilon^n \quad \text{on } J$$

for some mappings $S_{e,k} \in C^{n-k+\beta}(O, \mathbb{R})$ $(k = 1, 2, \ldots, n)$ and $\tilde{S}_{e,n}(\epsilon, \cdot) \in C^{\beta(\epsilon)}(O, \mathbb{R})$ with $\sup_{e\in E} \sup_{x\in J} |\tilde{S}_{e,n}(\epsilon, x)| \to 0$ with $\beta(\epsilon) > 0$ for $\epsilon > 0$.

(c) There exist $q \in (\max(1 - (\dim_H K - \underline{s})/n, \underline{s}/\dim_H K), 1]$ and a constant $c > 0$ such that for any $e \in E$, $l = 0, 1, \ldots, n$, $x \in J$, $y \in O$ and $k = 1, 2, \ldots, n - l + 1$,

$$\begin{aligned}\|T_{e,l}^{(n-l+1)}(x) - T_{e,l}^{(n-l+1)}(y)\| &\le c(DT_e(x))^q \|x - y\|^\beta, \\ \|T_{e,l}^{(k)}(x)\| &\le c(DT_e(x))^q.\end{aligned}$$

(d) There exists a map $c(\epsilon) > 0$ with $\lim_{\epsilon \to 0} c(\epsilon) = 0$ such that for any $e \in E$, $x \in J$ and $\epsilon > 0$

$$\|\tilde{S}_{e,n}(\epsilon, x)\| \le c(\epsilon)(DT_e(x))^q.$$

By applying Theorem 3.1 to the function $g(\epsilon, \omega) := \log DT_{\omega_0}(\epsilon, \pi(\epsilon, \sigma\omega))$, we obtain the following, where $\pi(\epsilon, \cdot)$ denotes the coding map of $(J, (T_e(\epsilon, \cdot)))$.

Theorem 4.2 ([8]) *Under the above conditions (I) and (II) for NPL systems $(J, (T_e(\epsilon, \cdot)))$ with small parameter $\epsilon > 0$, the Hausdorff dimension of the limit set $K(\epsilon)$ of $(J, (T_e(\epsilon, \cdot)))$ has the form $\dim_H K(\epsilon) = \dim_H K + s_1\epsilon + \cdots + s_n\epsilon^n + o(\epsilon^n)$ for some numbers $s_1, \ldots, s_n \in \mathbb{R}$.*

4.2 Linear countable IFS

Let $a > 1$. Let E be the set of all positive integers. We take an infinite graph $G = (\{v\}, E)$, $J_v = [0, 1]$ and $O_v = (-\eta, 1 + \eta)$ for a small $\eta > 0$. For $e \in E$ and $\epsilon \ge 0$, we define a function $T_e(\epsilon, \cdot) : O_v \to O_v$ by

$$T_e(\epsilon, x) = \Big(\frac{1}{5^e} + \frac{1}{a^e}\epsilon\Big)x + b(e),$$

where we choose $b(e) = 1 - 1/2^{e-1}$. Put $g(\epsilon, \omega) = (1/5^{\omega_0}) + 1/a^{\omega_0}\epsilon$. Denoted by $K(\epsilon)$ the limit set of the IFS $(T_e(\epsilon, \cdot))_{e\in E}$ which is defined as $K(\epsilon) = \bigcup_{\omega\in E^\infty} \bigcap_{n=0}^\infty T_{\omega_0}(\epsilon, \cdot) \circ \cdots \circ T_{\omega_n}(\epsilon, \cdot)(J_v)$. It is not hard to check that the function $g(\epsilon, \cdot)$ satisfies all conditions (1)-(8) by putting $g(\omega) = 1/5^{\omega_0}$, $g_1(\omega) = 1/a^{\omega_0}$, $g_2 = \cdots = g_n = 0$, and $\tilde{g}_n(\epsilon, \cdot) = 0$. Moreover the topological pressure of $s \log |g|$ has the equation $P(s \log |g|) = \log \sum_{e\in E} (1/5^e)^s$ for $s > 0$. Therefore, a Bowen formula [2] implies that $P(s(0) \log |g|) = 0$ if and only if $\sum_{e\in E}(1/5^e)^{s(0)} = 1$ if and only if $\dim_H K(0) = s(0) = \log 2/ \log 5$. Moreover, $\underline{s} = \inf\{s : P(s \log |g|) < +\infty\}$ is equal to 0.

Theorem 4.3 **([8])** *Assume the above conditions for $T_e(\epsilon,\cdot)$. Then we have the following:*

(1) If $a \geq 5$ then the Hausdorff dimension $s(\epsilon) = \dim_H K(\epsilon)$ has an n-order asymptotic expansion $s(\epsilon) = s(0) + s_1\epsilon + \cdots + s_n\epsilon^n + o(\epsilon^n)$ for any $n \geq 0$. Each coefficient s_k $(k = 1, 2, \ldots, n)$ is decided as

$$s_k = \sum_{\substack{0\le v\le u, 0\le q\le u-v:\\ (v,q)\neq(0,1)}} \sum_{j=0}^{\min(v,q)} s_{q,u-v} \frac{a_{v,j,s(0)}}{(q-j)!}(-\log 5)^{q-j} \sum_{e=1}^{\infty} e^{q-j}\left(\frac{5^v}{2a^v}\right)^e, \tag{15}$$

where $s_{q,u-v}$ and $a_{v,j,s(0)}$ are defined by

$$(s(\epsilon)-s(0))^k = \begin{cases} s_{1,0} + s_{1,1}\epsilon + \cdots + s_{1,n-1}\epsilon^{n-1} + o(\epsilon^{n-1}) & (k=1) \\ s_{k,0} + s_{k,1}\epsilon + \cdots + s_{k,n-1}\epsilon^{n-1} + s_{k,n}\epsilon^n + o(\epsilon^n) & (k\geq 2) \end{cases} \quad \textit{with}$$

$$s_{k,i} = \begin{cases} 1 & (k=i=0) \\ \displaystyle\sum_{\substack{j_1,\cdots,j_{i-1}\ge 0:\\ j_1+\cdots+j_{i-1}=k \\ j_1+2j_2+\cdots+(i-1)j_{i-1}=i}} \frac{s_1^{j_1}\cdots s_{i-1}^{j_{i-1}}}{j_1!\cdots j_{i-1}!} & (k\geq 1 \textit{ and } k\le i\le n) \\ 0 & (\textit{otherwise}), \end{cases}$$

and

$$\binom{t}{v} = \sum_{j=0}^{v} a_{v,j,s(0)}(t-s(0))^j \quad \textit{with}$$

$$a_{v,j,s(0)} = \begin{cases} \binom{s(0)}{v} & (j=0) \\ \displaystyle\sum_{\substack{0\le i_1,\cdots,i_{v-j}\le v-1 \\ i_1<\cdots<i_{v-j}}} \frac{1}{v!}\prod_{p=1}^{v-j}(s(0)-i_p) & (l\geq 1 \textit{ and } 0\le j<v) \\ 1/v! & (v\geq 1 \textit{ and } j=v) \\ 0 & (v<j), \end{cases}$$

where $\binom{t}{v}$ is the binomial coefficient. In particular,

$$s_1 = \frac{\log 2}{(\log 5)^2}\frac{5}{4a-10}$$
$$s_2 = \frac{25\log 2}{(\log 5)^3}\left(\frac{1}{2(2a-5)^2} - \frac{a\log 2}{(2a-5)(4a^2-5)^2} + \frac{\log(2/5)}{8a^2-100}\right).$$

(2) If $1 < a < 5$ then take the largest integer $k \geq 0$ satisfying $a \leq 5/2^{1/(k+1)}$. In this case,

$s(\epsilon)$ has the form

$$s(\epsilon) = \begin{cases} s(0) + s_1\epsilon + \cdots + s_k\epsilon^k + \hat{s}(\epsilon)\epsilon^{k+1}\log\epsilon & (a = 5/2^{1/(k+1)} \text{ for some } k \geq 0) \\ s(0) + s_1\epsilon + \cdots + s_k\epsilon^k + \hat{s}(\epsilon)\epsilon^{\frac{\log 2}{\log(5/a)}} & (\text{otherwise}) \end{cases}$$

with $|\hat{s}(\epsilon)| \asymp 1$ as $\epsilon \to 0$, i.e. $c^{-1} \leq |\hat{s}(\epsilon)| \leq c$ for any $\epsilon > 0$ for some $c \geq 1$, where each s_i is defined by (15).

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