

ON THE CONTINUITY OF LYAPUNOV EXPONENTS FOR SYSTEMS WITH ADDITIVE NOISE

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ABSTRACT. In this short essay I show a small result on continuity of Lyapunov exponents for systems with additive noise.

1. INTRODUCTION

Noise induced phenomena are an important research theme in mathematics, applied mathematics and physics: Noise Induced Order [5], Stochastic Resonance and Anomalous Diffusion [1, 6] are some of the observed phenomena, both in models and in experiments.

In [2] we presented a computer assisted proof of Noise Induced Order for the Matsumoto-Tsuda model of the Belosouov-Zhabotinsky reaction, and we proved some continuity results for the Lyapunov exponent of the system.

In this paper I would like to present a self-contained proof of the Lipschitz continuity of the Lyapunov exponent when the annealed transfer operator is mixing; this is a generalization of the arguments in [2].

This proof is quite general and the only hypothesis are the regularity of the noise kernel and the fact that for some noise size the system is mixing; another condition for the proof to work is that the weak norm has to be L^1 , but this is pretty natural. I will present it for one-dimensional systems and Bounded Variation noise but the proof can be easily extended to higher dimension and more regular kernels.

2. STATEMENT OF THE RESULTS

Let $T : [0, 1] \rightarrow [0, 1]$, I want to study the “Lyapunov exponent” of the system where I replace T^n by $(T + \xi_n) \circ (T + \xi_{n-1}) \circ \dots \circ (T + \xi_1)$, where (ξ_n, \dots, ξ_1) are chosen in the interval $[-\epsilon, \epsilon]$ using a probability distribution with density function ρ_ϵ .

The noise may take orbits outside the interval $[0, 1]$; this may reflect on the behaviour of the annealed operator of the random dynamical system therefore I suppose that we have **periodic boundary conditions**, i.e., as in [4], if $f \in L^1([0, 1])$, letting \hat{f} be its periodic extension, i.e.,

$$\hat{f}(x+k) = f(x) \quad k \in \mathbb{Z}$$

I denote by abuse of notation, for $x \in [0, 1]$

$$\rho_\epsilon * f(x) := \rho_\epsilon * \hat{f}(x).$$

Let m be the Lebesgue measure on $[0, 1]$; for each ϵ the behaviour of the system with noise can be described by a Markov chain, such that for each Borel subset E of $[0, 1]$ we have that the probability of x entering E is

$$(1) \quad P_\epsilon(x, E) = \int_E \rho_\epsilon(y - T(x)) dm(y).$$

From here on, suppose that $\log(|T'|)$ is in $L^1(m)$.

Definition 1. If T is a nonsingular Borel measurable dynamical system on X then it induces a linear operator on the space of signed Borel measures $L : SM(X) \rightarrow SM(X)$, defined on $\mu \in SM(X)$ by

$$L\mu(A) = \mu(T^{-1}A),$$

the **transfer operator**.

Remark 1. It is worth observing that $L(\delta_x) = \delta_{T(x)}$; we can think of the transfer operator as the operator that embodies how mass is transferred around by the dynamics on X .

Definition 2. Let ρ be a BV density function and

$$\rho_\epsilon(x) = \frac{1}{\epsilon} \rho\left(\frac{x}{\epsilon}\right)$$

then, we call

$$L_\epsilon = \rho_\epsilon * L,$$

the **annealed transfer operator** associated to the random dynamical system with noise size ϵ . The annealed transfer operator is the operator that encodes how densities evolve under the action of the Markov chain (1).

Remark 2. The annealed transfer operator gives us information on the behaviour of the system averaged over all possible noise realizations. We can observe this by duality, let $\phi \in L^\infty([0, 1])$

$$\langle \phi, L_\epsilon \delta_x \rangle = \int \phi(T(x) - \xi) d\rho_\epsilon(\xi).$$

We are taking the average (weighted by ρ_ϵ) of ϕ over all the possible images of the point x .

If ρ is in BV the (annealed) random dynamical system with noise of size ϵ has (at least) one stationary measure with density in BV (Lemma 1). For the moment being, suppose that for each ϵ the operator L_ϵ has a unique fixed point f_ϵ . The object of study of this essay is the well definedness and continuity of the following function.

Definition 3. The function

$$\lambda(\epsilon) = \int_{-1}^1 \log(|T'(x)|) f_\epsilon dm.$$

is the “Lyapunov exponent in function of the noise”.

Definition 4. In the following we will denote by V_0 the space of average 0 signed measures. By abuse of notation we also denote by V_0 the intersection of average 0 signed measures with more regular Banach spaces as L_1 and BV (this will be clear from context).

I will show the following result.

Theorem 1. Let L_{ϵ_0} be the annealed transfer operator associated to the dynamical system T with additive noise with kernel ρ_{ϵ_0} , for a fixed noise size ϵ_0 . Suppose

$$\|L_{\epsilon_0}^n|_{V_0}\|_{BV} \leq C_0 \eta_0^n,$$

with $\eta_0 < 1$, $C_0 > 0$.

Then, for all $\epsilon > \epsilon_0$ there exist C_ϵ and η_ϵ such that

$$\|L_\epsilon^n|_{V_0}\|_{BV} \leq C_\epsilon \eta_\epsilon^n.$$

This implies that the stationary measure f_ϵ is unique for $\epsilon > \epsilon_0$; moreover, the stationary measure f_ϵ varies continuously (in BV norm) with respect to the noise

size, which in turn implies that the function $\lambda(\epsilon)$ is well-defined and locally Lipschitz continuous for all $\epsilon > \epsilon_0$.

3. WELL DEFINEDNESS AND CONTINUITY OF $\lambda(\epsilon)$

This is a simple inequality, where the regularization properties of convolution have strong consequences on the regularity of the fixed points of the operator L_ϵ . The same inequality works for any “regularity” seminorm, i.e., if the noise is of regularity C^k (or even analytic), the fixed point of the annealed transfer operator has the same regularity.

In the following section I show how one of the consequences of the uniform correlation decay in BV norm is that the operator L_ϵ has a unique stationary measure. This shows that the function $\lambda(\epsilon)$ is well defined.

3.1. Regularity of the stationary measure. We state this results with respect to the variation seminorm $\text{Var}(\cdot)$; this lemma easily generalizes to higher regularity.

Lemma 1. *Suppose $\|\rho_\epsilon\|_{BV} < \infty$. Then L_ϵ preserves BV and if f_ϵ is a stationary density in L^1 , then f_ϵ is in BV and*

$$\|f_\epsilon\|_{BV} \leq \|\rho_\epsilon\|_{BV}$$

Proof. Both statements follows from the properties of convolution, for any $g \in L^1$ we have that:

$$\|L_\epsilon g\|_{BV} \leq \|\rho_\epsilon\|_{BV} \|g\|_{L^1},$$

i.e. $\|L_\epsilon\|_{L^1 \rightarrow BV} \leq \|\rho_\epsilon\|_{BV}$ which in turn implies that

$$\|L_\epsilon\|_{BV} \leq \|\rho_\epsilon\|_{BV}.$$

If f_ϵ is a fixed point then

$$\|f_\epsilon\|_{BV} = \|L_\epsilon f_\epsilon\|_{BV} \leq \|\rho_\epsilon\|_{BV} \|f_\epsilon\|_{L^1} \leq \|\rho_\epsilon\|_{BV}.$$

□

Remark 3. Indeed, a stronger result is true, i.e., if \mathcal{M} is the space of signed measures endowed with the Wasserstein norm [2], we have that L_ϵ is a continuous operator from \mathcal{M} to BV , i.e., any stationary measure is in BV and has a density.

3.2. Decay of correlations in L^1 and BV for L_ϵ .

Lemma 2. *Suppose L_ϵ has exponential decay of correlations with respect to bounded variation observables. Then, there exists N such that*

$$\|L_\epsilon^N|_{V_0}\|_{L^1} \leq \frac{1}{2}.$$

Proof. Decay of correlations for bounded variations observables guarantees that

$$\|L_\epsilon^n|_{V_0}\|_{BV} \leq C\theta^n.$$

Observing that $\|L_\epsilon^n f\|_{BV} \leq \|\rho_\epsilon\|_{BV} \|f\|_{L^1}$ we have that

$$\|L_\epsilon^{n+1}|_{V_0}\|_{L^1} \leq \left(\frac{C \cdot \text{Var}(\rho_\epsilon)}{\theta} \right) \theta^{n+1},$$

by choosing $N > (-1 - \log_2(C) - \log_2(\text{Var}(\rho_\epsilon)))/\log_2(\theta)$ we have the thesis. □

The following Lemma is a generalization of a result in [2]: if for some noise amplitude the operator contracts L^1 , then for all bigger amplitudes the annealed operators also contracts L^1 .

Lemma 3. *Let $\rho_\epsilon = \epsilon^{-1}\rho(x/\epsilon)$ and N_ϵ the associated noise operator. If $\epsilon < \hat{\epsilon}$ and $\|(N_\epsilon L)^i\|_{V_0 \rightarrow L^1} \leq C_i < 1$, then*

$$(2) \quad \|(N_{\hat{\epsilon}} L)^i\|_{V_0 \rightarrow L^1} \leq C_i (\epsilon/\hat{\epsilon})^i + [1 - (\epsilon/\hat{\epsilon})^i] < 1.$$

Proof. We observe that

$$\frac{1}{\epsilon + \epsilon} \rho\left(\frac{x}{\epsilon + \epsilon}\right) = \frac{\epsilon}{\epsilon + \epsilon} \rho\left(\frac{x}{\epsilon}\right) + \frac{\epsilon}{\epsilon + \epsilon} \frac{1}{\epsilon} \left(\rho\left(\frac{x}{\epsilon + \epsilon}\right) - \rho\left(\frac{x}{\epsilon}\right)\right);$$

moreover, the convolution operator

$$Mf = \frac{1}{\epsilon} \left(\rho\left(\frac{x}{\epsilon + \epsilon}\right) - \rho\left(\frac{x}{\epsilon}\right)\right) * f$$

is a Markov operator.

Therefore

$$N_{\hat{\epsilon}} Lf = \frac{\epsilon}{\hat{\epsilon}} N_\epsilon Lf + \left(1 - \frac{\epsilon}{\hat{\epsilon}}\right) MLf,$$

and

$$(N_{\hat{\epsilon}} L)^i f = \left(\frac{\epsilon}{\hat{\epsilon}}\right)^i (N_\epsilon L)^i f + \left(1 - \left(\frac{\epsilon}{\hat{\epsilon}}\right)^i\right) Qf$$

for a suitable Markov operator Q . Taking the L^1 norm, the result follows. \square

This result is strong, but it has even stronger consequences, proved in the next lemma, i.e., if we see L^1 contraction for a noise amplitude ϵ we have BV contraction for all bigger noise amplitudes.

Lemma 4. *Suppose $\|L_\epsilon^n|_{V_0}\|_{L^1} \leq C\theta^n$; then*

$$\|L_\epsilon^n|_{V_0}\|_{BV} \leq \left(\frac{C\|\rho_\epsilon\|_{BV}}{\theta}\right)\theta^n.$$

Proof. Let $f \in V_0$, the proof is a chain of inequalities,

$$\begin{aligned} \|L_\epsilon^{n+1} f\|_{BV} &\leq \|\rho_\epsilon\|_{BV} \|L\|_{L^1} \|L_\epsilon^n f\|_{L^1} \\ &\leq \|\rho_\epsilon\|_{BV} C\theta^n \|f\|_{L^1} \\ &\leq \|\rho_\epsilon\|_{BV} C\theta^n \|f\|_{BV}. \end{aligned}$$

\square

3.3. Unicity of the stationary measure and well-definedness of $\lambda(\epsilon)$. From the results on the decay of correlation, we obtain the unicity of the stationary measure and therefore the fact that $\lambda(\epsilon)$ is well-defined.

Corollary 1. *Let $T : [0, 1] \rightarrow [0, 1]$ be a dynamical system such that there exists ϵ_0 such that for all $\epsilon \in [0, \epsilon_0)$ the operators L_ϵ have a uniform bound on the decay of correlations in BV , then the function $\lambda(\epsilon)$ is well defined.*

Proof. For all ϵ any stationary measure has density f_ϵ in $BV \subset L^1$. Due to the lemmas above, for all ϵ there exists an N such that $\|L_\epsilon^N|_{V_0}\|_{L^1} < 1/2$. By contradiction, suppose there are two stationary densities, f_ϵ and g_ϵ , then

$$\|f_\epsilon - g_\epsilon\|_{L^1} = \|L_\epsilon^N(f_\epsilon - g_\epsilon)\|_{L^1} \leq \frac{1}{2} \|f_\epsilon - g_\epsilon\|_{L^1},$$

which implies that f_ϵ coincides with g_ϵ . \square

3.4. Continuity in BV of the stationary measure with respect to the noise size. In this subsection we prove the continuity of the stationary measure with respect to the noise size at a fixed noise size $\epsilon > 0$.

Corollary 2. *Let $\rho_\epsilon = \epsilon^{-1}\rho(x/\epsilon)$ and N_ϵ the associated noise operator. Then, for $\hat{\epsilon} > \epsilon > 0$ we have that*

$$\|L_{\hat{\epsilon}} - L_\epsilon\|_{L^1} \leq 2 \left(1 - \frac{\epsilon}{\hat{\epsilon}}\right)$$

and that

$$\|L_{\hat{\epsilon}} - L_\epsilon\|_{L^1 \rightarrow BV} \leq 3 \left(1 - \frac{\epsilon}{\hat{\epsilon}}\right) \|\rho_\epsilon\|_{BV}$$

Proof. From the proof of Lemma 3 we have that

$$L_{\hat{\epsilon}}f - \frac{\epsilon}{\hat{\epsilon}}L_\epsilon f = \left(1 - \frac{\epsilon}{\hat{\epsilon}}\right) M L f,$$

where M is a Markov operator. Then

$$\|L_{\hat{\epsilon}} - L_\epsilon\|_{L^1} \leq \|L_\epsilon - \frac{\epsilon}{\hat{\epsilon}}L_\epsilon\|_{L^1} + \|L_{\hat{\epsilon}} - \frac{\epsilon}{\hat{\epsilon}}L_\epsilon\|_{L^1} \leq 2 \left(1 - \frac{\epsilon}{\hat{\epsilon}}\right).$$

Similarly

$$\begin{aligned} \|L_{\hat{\epsilon}} - L_\epsilon\|_{L^1 \rightarrow BV} &\leq \|L_\epsilon - \frac{\epsilon}{\hat{\epsilon}}L_\epsilon\|_{L^1 \rightarrow BV} + \|L_{\hat{\epsilon}} - \frac{\epsilon}{\hat{\epsilon}}L_\epsilon\|_{L^1 \rightarrow BV} \\ &\leq 3 \left(1 - \frac{\epsilon}{\hat{\epsilon}}\right) \|\rho_\epsilon\|_{BV}, \end{aligned}$$

since

$$\frac{1}{\epsilon} \|\rho\left(\frac{x}{\hat{\epsilon}}\right) - \rho\left(\frac{x}{\epsilon}\right)\|_{BV} \leq 2 \|\rho_\epsilon\|_{BV}.$$

□

Now, we want to estimate the distance between the fixed points of L_ϵ and $L_{\hat{\epsilon}}$. This proofs follows the line of Theorem 1 in [3].

Theorem 2. *Let L_ϵ and $L_{\hat{\epsilon}}$ be the annealed operators for the noise of size ϵ and $\hat{\epsilon}$ respectively. Suppose moreover that $\|L_\epsilon^N|_{V_0}\|_{BV} \leq C\theta_\epsilon^N$ for $\theta_\epsilon < 1$. Suppose f_ϵ and $f_{\hat{\epsilon}}$ are the unique fixed points of L_ϵ and $L_{\hat{\epsilon}}$ respectively. Then*

$$\|f_{\hat{\epsilon}} - f_\epsilon\|_{BV} \leq \frac{C}{1 - \theta_\epsilon} 3 \left(1 - \frac{\epsilon}{\hat{\epsilon}}\right) \|\rho_\epsilon\|_{BV}.$$

Proof. Let N be such that $C\theta_\epsilon^N < 1/2$, then

$$\begin{aligned} \|f_\epsilon - f_{\hat{\epsilon}}\|_{BV} &= \|L_\epsilon^N f_\epsilon - L_{\hat{\epsilon}}^N f_{\hat{\epsilon}}\|_{BV} \\ &\leq \|L_\epsilon^N f_\epsilon - L_\epsilon^N f_{\hat{\epsilon}}\|_{BV} + \|L_\epsilon^N f_{\hat{\epsilon}} - L_{\hat{\epsilon}}^N f_{\hat{\epsilon}}\|_{BV}, \end{aligned}$$

which implies that

$$\begin{aligned} \|f_\epsilon - f_{\hat{\epsilon}}\|_{BV} &\leq 2 \|L_\epsilon^N f_{\hat{\epsilon}} - L_{\hat{\epsilon}}^N f_{\hat{\epsilon}}\|_{BV} \\ &\leq \sum_{k=0}^{N-1} \|L_\epsilon^k|_{V_0}\|_{BV} \|L_\epsilon - L_{\hat{\epsilon}}\|_{L^1 \rightarrow BV} \|L_\epsilon^{N-k-1} f_{\hat{\epsilon}}\|_{L^1} \\ &\leq \sum_{k=0}^{N-1} C\theta_\epsilon^k \cdot 3 \left(1 - \frac{\epsilon}{\hat{\epsilon}}\right) \|\rho_\epsilon\|_{BV}, \end{aligned}$$

which implies the thesis since $\|f_{\hat{\epsilon}}\|_{L^1} = 1$.

□

4. PROOF OF THEOREM 1

Proof. Let $\epsilon, \hat{\epsilon} > \epsilon_0$ then, by Theorem 2 we have

$$\begin{aligned} |\lambda(\epsilon) - \lambda(\hat{\epsilon})| &= \int \log(|T'|)(f_\epsilon - f_{\hat{\epsilon}}) dm \\ &\leq \|\log(|T'|)\|_{L^1} \|f_\epsilon - f_{\hat{\epsilon}}\|_{BV} \\ &\leq \|\log(|T'|)\|_{L^1} \frac{C}{1 - \theta_\epsilon} 3 \left(1 - \frac{\epsilon}{\hat{\epsilon}}\right) \|\rho_\epsilon\|_{BV}. \end{aligned}$$

□

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