

# On the existence of absolutely continuous $\sigma$ -finite invariant measures for random dynamical systems

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## 1 Introduction

Finding an absolutely continuous finite or  $\sigma$ -finite infinite invariant measure ( $\sigma$ -finite acim, for short) for a given system (described by a transformation or a Markov process) is one of the classical problems in ergodic theory. Thus, there are lots of previous researches for this problem (see for example [A97, DS66, Fog69, HK64, In12, In20, Sch95, Th80] and references therein). However, necessary and sufficient conditions for the existence of a  $\sigma$ -finite acim have still not been well-known. In this paper, for a given Markov operator over a probability space, we give some equivalent conditions for the existence of a  $\sigma$ -finite acim with certain support property. One of the equivalent conditions is weak almost periodicity of the jump operator with respect to some sweep-out set (which implies the Jacobs-de Leeuw-Glicksberg splitting theorem [E06]). Here the method of jump operators is generalization of the method of jump transformations established in [Sch95, Th80]. Because we consider general Markov operators, we can apply our setting not only to deterministic systems but also to random dynamical systems represented by null-preserving transition probabilities. Our result is applicable to certain one-dimensional random dynamical system arising from intermittent Markov maps with uniformly contractive part.

To be more precise, we consider a probability space  $(X, \mathcal{F}, m)$  and a *Markov operator*  $P$  defined on  $L^1 = L^1(X, \mathcal{F}, m)$  into itself, i.e.,  $P$  satisfies  $Pf \geq 0$  and  $\|Pf\|_1 = \|f\|_1$  whenever  $f \in L^1_+ = \{g \in L^1 : g \geq 0\}$ . The adjoint operator of  $P$  is denoted by  $P^*$  which is defined on  $L^\infty$ . Then a finite (resp.  $\sigma$ -finite) measure  $\mu$  on  $(X, \mathcal{F})$  is said to be a *finite* (resp.  *$\sigma$ -finite*) *acim* if  $\mu$  is absolutely continuous w.r.t.  $m$  ( $\mu \ll m$ ) and the Radon-Nikodym derivative  $d\mu/dm$  is a (not identically zero) fixed point of  $P$ . Notice that the domain of  $P$  can be extended to the set of all non-negative measurable functions and the definition of a  $\sigma$ -finite acim makes sense even if  $\mu$  is a  $\sigma$ -finite infinite measure. When we have a null-preserving transition probability  $\mathbb{P}(x, A)$  for  $x \in X$  and  $A \in \mathcal{F}$  (i.e.,  $\mathbb{P}(x, N) = 0$  for a.e.  $x$  if  $m(N) = 0$ ), which describes our system, the corresponding

Markov operator  $P$  is given by,

$$\int_A Pf dm = \int_X f(x) \mathbb{P}(x, A) dm(x)$$

for each  $f \in L^1$  and  $A \in \mathcal{F}$ . If we consider a deterministic system, given by a non-singular transformation  $T : X \rightarrow X$  (i.e.,  $m \circ T^{-1} \ll m$ ), the corresponding transition probability is  $\mathbb{P}(x, A) = 1_{T^{-1}A}(x)$ . Then the Markov operator associated to a given non-singular transformation is called the Perron-Frobenius operator  $P$  given by

$$\int_X Pf \cdot g dm = \int_X f \cdot g \circ T dm$$

for  $f \in L^1$  and  $g \in L^\infty$ . We will show the existence of a  $\sigma$ -finite acim for a Markov operator in the next section. That is, we can apply our results to both non-singular transformations and null-preserving transition probabilities.

## 2 Main Result

In this section, we present our main results. Our results Theorem 2.2 and Theorem 2.6 give equivalent conditions for the existence of a finite or  $\sigma$ -finite acim with the maximal support condition for a given Markov operator. Here, “the maximal support condition” means the support of the invariant measure contains a proper sweep-out set (see Definition 2.1). That is, almost all trajectories under the process will eventually concentrate on the support of the invariant measure. Throughout this section  $\text{supp}\mu$  denotes the support of  $\mu$ , i.e.,  $\text{supp}\mu = \{\frac{d\mu}{dm} > 0\}$ .

In order to state Theorem 2.2, we need the following definition of a sweep-out set.

**Definition 2.1** (A sweep-out set). *For a Markov operator over  $L^1(X, \mathcal{F}, m)$ , a set  $E \in \mathcal{F}$  is called a ( $P$ -) sweep-out set (w.r.t.  $m$ ) if  $\lim_{n \rightarrow \infty} (P^* I_{E^c})^n 1_X(x) = 0$   $m$ -a.e.  $x \in X$  where  $I_{E^c}$  denotes the restriction operator on  $E^c$ .*

Recall that a Markov operator  $P$  is called *weakly almost periodic* if for any  $f \in L^1$  the sequence of functions  $\{P^n f\}$  is weakly precompact. In the following Theorem 2.2, weak almost periodicity of a Markov operator plays a key role as an equivalent condition for the existence of a finite acim with the maximal support condition.

**Theorem 2.2** ([T]). *Let  $P$  be a Markov operator over a probability space  $(X, \mathcal{F}, m)$ . Then the followings are equivalent.*

1. *There exists a finite acim  $\mu$  for  $P$  s.t.  $\text{supp}\mu$  is a sweep-out set;*
2.  *$\{P^n 1_X\}_n$  is weakly precompact;*
3.  *$P$  is weakly almost periodic.*

**Remark 2.3.** (1) *The condition 1 in Theorem 2.2 can be paraphrased:*

1'. There exists a finite acim  $\mu$  for  $P$  s.t.  $\lim_{n \rightarrow \infty} P^{*n}1_{\text{supp}\mu}(x) = 1$   $m$ -a.e.  $x \in X$ .

(2) The condition in 1 “ $\text{supp}\mu$  is a sweep-out set” is a necessary condition of ergodicity of  $(P, m)$ , where  $(P, m)$  is called ergodic if  $E \in \mathcal{F}$  with  $P^*1_E = 1_E$  implies  $E = \emptyset$  or  $X \pmod{m}$ .

We prepare the methods of inducing and jump to state Theorem 2.6, equivalent conditions for the existence of a  $\sigma$ -finite acim. The following definition of the induced operator or the jump operator is the generalization of the induced transformation or the jump transformation (see [A97, Fog69, In20, Sch95, Th80, T] for details).

**Definition 2.4** (The induced operator/The jump operator). For a Markov operator  $P$  with a sweep-out set  $E$ , the induced operator  $P_E$  is defined by

$$P_E = I_E P \sum_{n \geq 0} (I_{E^c} P)^n,$$

and the jump operator  $\hat{P}_E$  is defined by

$$\hat{P}_E = P I_E \sum_{n \geq 0} (P I_{E^c})^n.$$

**Remark 2.5.** (1) The induced operator  $P_E$  and the jump operator  $\hat{P}_E$  are also Markov operators over  $L^1(X, \mathcal{F}, m)$  as long as  $E$  is sweep-out.

(2) When  $P$  is the Perron-Frobenius operator for some non-singular transformation, the restricted induced operator  $P_E I_E$  (defined on  $L^1(E, \mathcal{F} \cap E, m|_E)$ ) and the jump operator  $\hat{P}_E$  are the Perron-Frobenius operators corresponding to the induced transformation and the jump transformation, respectively.

The following theorem give equivalent conditions for the existence of a  $\sigma$ -finite (it might be infinite) acim with the maximal support condition. Equivalent conditions are characterized by the methods of induced operator and jump operator respectively.

**Theorem 2.6** ([T]). Let  $P$  be a Markov operator over a probability space  $(X, \mathcal{F}, m)$ . Then the followings are equivalent.

1. There exists a  $\sigma$ -finite acim  $\mu$  for  $P$  s.t.  $\text{supp}\mu$  contains a  $P$ -sweep-out set  $A$  w.r.t.  $m$  with  $\mu(A) < \infty$ ;
2. There exists a sweep-out set  $E$  s.t. the induced operator  $P_E$  admits a finite acim  $\mu_E$  with  $\text{supp}\mu_E = E \pmod{m}$ ;
3. There exists a sweep-out set  $E$  s.t. the jump operator  $\hat{P}_E$  is weakly almost periodic.

**Remark 2.7.** We can apply Theorem 2.2 to the condition 3 in Theorem 2.6. That is, we only have to check weak precompactness of  $\{\hat{P}_E^n 1_X\}_n$ .

### 3 Example of Random Dynamical System

In this section, we apply Theorem 2.6 to certain one-dimensional random dynamical system. Our random dynamical system is random iteration of non-uniformly expanding maps which have uniformly contractive part on average. Throughout this section, our phase space  $(X, \mathcal{B}(X), \lambda)$  is the unit interval with the Lebesgue measure.

Let  $I$  be an at most countable non-empty subset of  $\mathbb{N}$  and for each  $i \in I$ ,  $J_i$  be also an at most countable non-empty subset of  $\mathbb{N}$ . We consider  $\{T_i : i \in I\}$  a family of piecewise linear Markov maps on the unit interval  $X = [0, 1]$  with the Lebesgue measure  $\lambda$  satisfying:

(a-1)  $T_i|_{X_n} : X_n \rightarrow X_{n-1}$  for  $n \geq 2$  and  $i \in I$ , given by

$$T_i|_{X_n}(x) = \frac{n+1}{n-1}x - \frac{1}{n(n-1)};$$

(a-2)  $T_i|_{X_1} : X_1 \rightarrow \bigcup_{k \in J_i} X_k$ , a surjective and monotonically increasing map which is piecewise linear in the sense

$$T_i|_{X_1} = \frac{\sum_{k \in J_i} \lambda(X_k)}{\lambda(X_1)}$$

whenever the derivative can be defined, for each  $i \in I$

where  $X_n = \left(\frac{1}{n+1}, \frac{1}{n}\right]$  is the cylinder of rank one for  $n \geq 1$ . Then the point 0 is the common fixed point for all  $T_i$  where the derivative of all  $T_i$  is one. We call a family of transformations  $\{T_i : i \in I\}$  with the above conditions (a-1) and (a-2) *piecewise linear intermittent Markov maps* (with index  $\{J_i\}_{i \in I}$ ).

By using piecewise linear intermittent Markov maps  $\{T_i : i \in I\}$ , we define our random dynamical system. For a given probability vector  $\{p_i\}_{i \in I}$  (i.e.,  $p_i \geq 0$  and  $\sum_{i \in I} p_i = 1$ ), the random iteration of piecewise linear intermittent Markov maps of  $\{T_i, p_i : i \in I\}$  is given by the following transition probability:

$$\mathbb{P}(x, E) = \sum_{i \in I} p_i 1_E(T_i x) \quad (x \in X, E \in \mathcal{B}(X)). \quad (3.1)$$

That is, each transformation  $T_i$  will be chosen with probability  $p_i$  and the selected transformation will be applied to the system. For the transition probability given by (3.1), we can define the Markov operator  $P$  on  $L^1 = L^1(X, \mathcal{B}, \lambda)$  since each transformation is non-singular w.r.t.  $\lambda$ , which is determined by

$$\int_A P f d\lambda = \int_X f(x) \mathbb{P}(x, A) dm(x) \quad (A \in \mathcal{B}(X), f \in L^1).$$

Equivalently, the Markov operator associated to this random dynamical system is given by  $P = \sum_{i \in I} p_i P_i$  where each  $P_i$  is the Perron-Frobenius operator corresponding to

$T_i$ . We remark that the random iteration of piecewise linear intermittent Markov maps  $\{T_i, p_i : i \in I\}$  may not satisfy expanding property on average in the following sense:

$$\sup_{X \setminus U_0} \sum_{i \in I} \frac{p_i}{|T_i'|} (x) < 1 \quad (3.2)$$

where  $U_0$  is a small neighborhood of the indifferent fixed point 0. Indeed, if  $1 \notin J_i$  for any  $i \in I$  then the average of derivatives of  $\{T_i, p_i : i \in I\}$  is obviously strictly less than one on  $X_1$ .

Then by Theorem 2.6 we can show the existence of a  $\sigma$ -finite acim for random iterations of piecewise linear intermittent Markov maps:

**Proposition 3.1.** *Any random iteration of piecewise linear intermittent Markov maps  $\{T_i, p_i : i \in I\}$  admits a  $\sigma$ -finite acim.*

**Remark 3.2.** *If  $\{T_i, p_i : i \in I\}$  satisfies expanding property on average in (3.2) sense, then the existence of a  $\sigma$ -finite acim was already shown in [In20] via the method of inducing. However, for random iterations of non-uniformly expanding maps which do not satisfy expanding property on average, statistical properties including the existence of a  $\sigma$ -finite acim are not well-studied. Therefore, our result could be interpreted as the first step toward the direction of non-uniformly expanding random maps with uniformly contractive part.*

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