Characterisations of quasi-metric semilattices and applications to intrinsic entropies

Nicolò Zava
University of Udine

1 Introduction

The main objects of this paper are generalised quasi-metric spaces. A *generalised quasi-metric space* is given by a set X and a *generalised quasi-metric* d which is a map $d\colon X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfying the following properties:

(QM1) $d(x,y) = d(y,x) = 0$ if and only if $x = y$, for every $x, y \in X$ (indistancy implies identity);

(QM2) $d(x,y) \leq d(x,z) + d(z,y)$, for every $x,y,z \in X$ (triangular inequality).

The pair (X,d) is a *quasi-metric space* and d is a *quasi-metric* if d does not assume the value ∞. Quasi-metric spaces were introduced and studied for the first time in [32], even though some traces can be already found in [15]. In this paper we focus on the applications of this notion to both domain theory and dynamics.

Scott's breakthrough work used order theory and non-Hausdorff topologies to describe partial objects in computation ([27, 28], see also the survey [1] and the monograph [14]). Matthews then defined partial metric spaces to provide a "metric setting" to Scott's ideas ([19]). Partial metrics are, intuitively, metrics for which the distance from one point to itself need not be zero. They were defined to parametrise computations searching for a denotational semantics to lambda-calculus. Later on, O'Neill generalised partial metrics to include more examples ([22]).

Matthews noticed that, for a set X, partial metrics are equivalent to the quasi-metrics d (a quasi-metric is a generalised quasi-metric that does not assume the value ∞) that are weighted. Weighted quasi-metric spaces were studied in [18]. In general, characterisations of partial metric spaces are hard to obtain ([17, 18]). To tackle the problem, Schellekens noticed that important examples have an underlying further structure: they are indeed semilattices if they are equipped with the specialisation order ([26]). Hence, motivated by the observation, he studied weighted invariant quasi-metric semilattices. He then proved that those objects satisfying the descending path condition, and that the weights are semi-co-valuations (particular real-valued maps introduced by Schellekens, [26], and, before it, by Nakamura, [21]). Moreover, Schellekens provided conditions under which an invariant quasi-metric semilattice satisfying a certain property, called descending path condition, is weighted ([25]), and he produced quasi-metrics out of semi-co-valuations ([26]). The latter process is close to Nakamura's construction of a metric from a semi-co-valuation ([21]), and generalises the classical correspondence between metrics and valuations in lattices ([3]).

The second application that came out recently concerns dynamics, and, more precisely, entropies. In 1865 Clausius defined the notion of entropy in physics, but it was only in 1948 that Shannon introduced it in mathematics, and, more precisely, in information theory ([29]). Inspired by that concept, several other entropies have been introduced and studied in mathematics so far. Intuitively, entropies associate to every self-morphism of a space a positive, possibly infinite, real value estimating its impact to the properties of the object. Let us mention Kolmogorov ([16]) and Sinai's ([30]) measure-theoretic entropy in ergodic theory, and Adler, Konheim and McAndrew's topological entropy ([2]) in compact spaces. In algebraic dynamics, we can cite again the work of Adler, Konheim and McAndrew ([2]), the entropy defined by Weiss in [31], and the one introduced by Peters ([23, 24]). Recently, coarse entropy was defined in metric spaces ([20]) and independently in coarse spaces ([34]).

Among the various entropies that have been defined in the last decades, let us mention a distinguished class, called intrinsic entropies. Intuitively, those are entropies for whose computation one needs to focus on some points satisfying particular properties and called inert. First examples are the intrinsic algebraic entropy for endomorphisms of abelian groups ([10]), the algebraic entropy and the topological entropy of continuous endomorphisms of locally linearly compact vector spaces ([5, 6]). Moreover, intrinsic characterisations were provided for the topological entropy of totally disconnected locally compact groups ([13]) and the algebraic entropy of locally compact strongly compactly covered groups ([12]).

In front of a plethora of entropies, the need for a unifying approach emerged. A first attempt was provided by Dikranjan and Giordano Bruno ([8]). They introduced the semigroup entropy of norm-contractive homomorphisms of normed semigroups and used it to recover several different definitions. However, that approach was not suitable to generalise intrinsic entropies. Later on, Castellano, Dikranjan, Freni, Giordano Bruno and Toller ([4]) proposed a different method to capture those intrinsic entropies. They introduced semilattice entropy of contractive endomorphisms of invariant generalised quasi-metric semilattices and recovered several intrinsic entropies.

In this paper we summarise the definition and characterisations of weighted quasi-metric spaces and semilattices. Moreover, we sketch the applications of their extensions to intrinsic entropy. More precisely, In Section 2 we introduce weighted quasi-metrics and provide their characterisation, due to Matthews, using partial metrics. In §2.1 we briefly mention the extension of partial metrics due to O'Neill and the corresponding notion of weakly weighted quasi-metrics. In Section 3 we focus on generalised quasi-metric semilattices. To do it, we recall the definition of specialisation order, and then introduce two fundamental properties: invariance and descending path condition. Section 4 is devoted to present classical characterisations of weighted invariant quasi-metric semilattices. In §4.1 we study the relation with the descending path condition and in §4.2 the connection with semi-co-valuations. Finally, in Section 5, we further motivate the interest in invariant generalised quasi-metric semilattices by describing their use in systematising intrinsic entropies. In §5.2 we define the intrinsic semilattice entropy and in §5.1 we show how it can be used to recover the intrinsic algebraic entropy.

Notation

We denote by $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$, $\mathbb{R}_{\leq 0} = \{x \in \mathbb{R} \mid x \leq 0\}$, and $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$.

If $f, g\colon X \to \mathbb{R}$ are maps from a set X and $c \in \mathbb{R}$, we write:

- $f + g$ for the map $h\colon X \to \mathbb{R}$ defined by $h(x) = f(x) + g(x)$, for every $x \in X$;
- c for the constant map having image $\{c\} \subseteq \mathbb{R}$;
- $f \geq 0$ if, for every $x \in X$, $f(x) \geq 0$.

2 Weighted quasi-metrics and partial metrics

Definition 2.1 ([19])**.** Let (X, d) be a quasi-metric space. Then the quasi-metric space X and the quasi-metric d are said to be *weighted* provided that there exists a *weight* w, which is a map $w\colon X \to \mathbb{R}_{\geq 0}$ satisfying, for every $x, y \in X$,

$$d(x, y) + w(x) = d(y, x) + w(y). \tag{1}$$

If d is a weighted quasi-metric there are infinitely many weights showing it. However, we can always choose a canonic representative, as Proposition 2.2 shows.

A map $f\colon X \to \mathbb{R}_{\geq 0}$ from a set X is called *fading* if $\inf\{f(x) \mid x \in X\} = 0$.

Let X and Y be two set. We write Y^X for the family of all maps $f\colon X \to Y$. We define an equivalence relation $\sim$ on the set $\mathbb{R}^X$ as follows: for every $f, g \in \mathbb{R}^X$, $f \sim g$ if $f - g$ is constant. With some abuse of notation, in the sequel we consider $\mathbb{R}^X_{\geq 0}$ as a subset of $\mathbb{R}^X$ implying the trivial identification.

Proposition 2.2. *Let (X, d) be a weighted quasi-metric space and w be a weight.*

(a) If $w'\colon X \to \mathbb{R}_{\geq 0}$, then w' is a weight of d if and only there exists $c \in \mathbb{R}$ such that $w' = w + c$. Equivalently, $[w]_\sim \cap \mathbb{R}^X_{\geq 0}$ consists of all the weights of d.

(b) There is a unique weight w^0 of d which is fading.

Proof. As for item (a), it is trivial to verify (1) for the map $w + c$. Conversely, if w' is a weight, $w - w'$ turns out to be constant if we compare (1) for w and w'.

Using item (a), the conclusion (b) descends. In fact, if w is a weight, we can define $w^0 = w - c$, where $c = \inf\{w(x) \mid x \in X\}$. Then $w^0 > 0$, it is a weight because of item (a) and it is fading. Again, because of item (a), w^0 is trivially the unique weight with the desired properties. □

Because of Proposition 2.2(b), for every weighted quasi-metric a canonic weight can be chosen, which is the fading weight.

Matthews introduced weighted quasi-metric spaces to characterise partial metrics.

Definition 2.3 ([19])**.** A *partial metric space* is given by a pair (X, p) where X is a set, and $p\colon X \times X \to \mathbb{R}_{\geq 0}$ is a *partial metric*, i.e., a map satisfying the following properties:

(PM1) $p(x, y) = p(y, x)$, for every $x, y \in X$ (symmetry);
(PM2) $p(x, x) = p(y, y) = p(x, y)$ if and only if $x = y$, for every $x, y \in X$ (identity condition);
(PM3) $p(x, x) \leq p(x, y)$, for every $x, y \in X$ (small self-distance);
(PM4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$, for every $x, y, z \in X$ (triangular inequality).

The following counterpart of Proposition 2.2 regarding partial metrics is easy to show.

Proposition 2.4. *Let (X, p) be a partial metric space. Then $[p]_\sim \cap \mathbb{R}_{\geq 0}^{X \times X}$ consists of partial metrics. Moreover, there exists a fading partial metric p^0 in $[p]_\sim \cap \mathbb{R}_{\geq 0}^{X \times X}$.*

Proof. The first part of the claim is trivial. As for the second one, the proof is similar to that of Proposition 2.2(b). □

Before providing examples of weighted quasi-metric spaces and partial metric spaces, let us state the correspondence theorem provided by Matthews in [19].

Theorem 2.5. *Let X be a set. Then there exists a one-to-one correspondence between equivalence classes of partial metrics of X and weighted quasi-metrics.*

Proof. Let p be a partial metric. We define the map $d_p \colon X \times X \to \mathbb{R}_{\geq 0}$ as follows: for every $x, y \in X$, $d_p(x, y) = p(x, y) - p(x, x)$. It is not hard to check that d is actually a quasi-metric. Moreover, it is weighted by the map w_p defined by $w_p(x) = p(x, x)$, for every $x \in X$. Furthermore, note that, if p' is another partial metric such that $p \sim p'$, then $d_p = d_{p'}$.

Conversely, let d be a quasi-metric weighted by w. Set $p_{d,w}(x, y) = d(x, y) + w(x)$, for every $x, y \in X$. Then $p_{d,w}$ is a partial metric. Moreover, if w' is a weight of d, Proposition 2.2(a) implies that $w \sim w'$, and thus $p_{d,w} \sim p_{d,w'}$.

Finally, note that, if p is a partial metric, $p_{d_p, w_p} \sim p$, and, if d is a quasi-metric weighted by w, $d_{p_{d,w}} = d$ and $w \sim w_{p_{d,w}}$. □

Theorem 2.5 states that weighted quasi-metrics and partial metrics are essentially equivalent. There is a trade-off between the symmetry and the fact that self-distance is always null.

Let us also mention that Theorem 2.5 and Propositions 2.2 and 2.4 imply that there is a one-to-one correspondence between fading partial metrics and quasi-metrics with fading weights.

Let us now give some examples of those spaces.

Example 2.6 ([4, 9, 26])**.** (a) On the pair $\mathbb{S} = \{0, 1\}$ the quasi-metric d defined by $d(0, 1) = 0$ and $d(1, 0) = 1$ is a weighted quasi-metric ($w(0) = 1$ and $w(1) = 0$).
(b) On $\mathbb{R}_{\leq 0}$ define the quasi-metric d as follows: for every $x, y \in \mathbb{R}$,

$$d(x, y) = \max\{x - y, 0\}. \tag{2}$$

Then $\mathbb{R}_{\leq 0}$ is weighted by the weight $w = -id$, for every $x \in X$. If we extend the quasi-metric d on the entire real line using again the definition 2, we obtain a quasi-metric that is not weighted. This statement will follow from what we prove in §4.

(c) Let Σ be a not necessarily finite alphabet and Σ^* be the family of all words, both finite and infinite, on the alphabet Σ. More precisely,

$$\Sigma^* = \{(\sigma_n)_{n\in[0,m)} \mid m \in \mathbb{N}\cup\{\infty\}, \text{ and } \forall n \in [0,m), \sigma_n \in \Sigma\}.$$

On Σ^* we define the partial metric p as follows: for every $x, x' \in \Sigma^*$, we set $p(x,x') = 2^{-l(x,x')}$, where $l(x,x')$ is the length of the longest common prefix of x and x'. We call p the *Baire partial metric*.
According to Theorem 2.5, p induces a weighted quasi-metric d_p which can be described as follows: for every $x, x' \in \Sigma^*$,

$$d_p(x,x') = p(x,x') - p(x,x) = 2^{-l(x,x')} - 2^{-|x|},$$

where $|x|$ is the length of the word x.

(d) Let X be a set. We define a generalised quasi-metric d on $\mathcal{P}(X)$ as follows: for every $A, B \subseteq X$,

$$d(A,B) = |(A\cup B)\setminus A| = |B\setminus A|.$$

Then d is a quasi-metric if and only if X is finite. Moreover, we will prove in the sequel (§4) that, if X is finite, then d is weighted.

(e) Let G be an abelian group and $L(G)$ be the lattice of all subgroups of G. We define a generalised quasi-metric d on $L(G)$ as follows: for every $H, K \leq G$,

$$d(H,K) = \log|H+K : H| = \log|K : H\cap K|.$$

Then d is a quasi-metric if and only if G is finite. In that case, d is weighted, as we will prove in §4.
To the generalised quasi-metric d we can associate a generalised metric d^s, called *symmetrisation of* d, defined by $d^s(H,K) = \max\{d(H,K), d(K,H)\}$, for every pair of subgroups H, K of G. The generalised metric space $(L(G), d^s)$ is studied in [11].

There are easy examples of finite quasi-metric spaces that are not weighted. Before introducing them, let us define a class of generalised quasi-metric spaces. Given a directed graph $\Gamma = (V,E)$ and two vertices $x, y \in V$, a *directed path P connecting x to y* is a finite subset of edges $\{(z_{i-1}, z_i)\}_{i=1,\dots,n}$ such that $z_0 = x$ and $z_n = y$. We define the *path generalised quasi-metric* d_Γ as follows: for every $x, y \in V$,

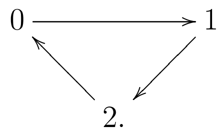
$$d_\Gamma(x,y) = \begin{cases} \inf\{|P| \mid P \text{ is a directed path connecting } x \text{ to } y\} & \text{if } x \neq y, \\ 0 & \text{otherwise.}\end{cases}$$

In general, d_Γ is a generalised quasi-metric as there may be no directed path connecting a vertex to another one. More precisely, d_Γ is a quasi-metric if and only if Γ is *strongly connected*, i.e., for every pair of vertices x and y there are a path connecting x to y and a path connecting y to x.

Example 2.7. Let $X = \{0,1,2\}$. On X, let us consider two quasi-metrics d_1 and d_2. For every $x, y \in X$, we define

$$d_1(x,y) = \begin{cases} 0 & \text{if } x \leq y, \\ 1 & \text{otherwise.}\end{cases}$$

The quasi-metric d_2 is the path quasi-metric associated to the following strongly connected directed graph:



It is an easy exercise to see that neither d_1 nor d_2 are weighted. As for d_1 we will provide in §4 a proof of a more general statement.

2.1 Extensions of partial metrics and weighted quasi-metrics

O'Neill in [22] provided a more general definition of partial metrics, that we call weak partial metric as in [7]. A *weak partial metric space* is given by a set X and a *weak partial metric*, which is a map $p\colon X \times X \to \mathbb{R}$ satisfying (PM1)–(PM4). The only difference between a weak partial metric and a partial metric is the codomain: we allow distances to be negative.

In [7] we study the correspondent notion of weakly weighted quasi-metric spaces. A quasi-metric space (X, d) is *weakly weighted* if there exists a *weak weight*, which is a map $w\colon X \to \mathbb{R}$ for which (1) is satisfied. Again, the difference between weak weights and weights is the codomain. Weakly weighted quasi-metrics form a class strictly including weighted quasi-metrics, as the following example shows.

Example 2.8. On the real line, take the quasi-metric d defined as in Example 2.6(b). We claimed that d is not weighted. However, it is weakly weighted by the map $w = -id$.

Most of the results stated for weighted quasi-metrics can be adapted for weakly weighted quasi-metrics. For example, let us mention the following two results.

Proposition 2.9. *Let (X, d) be a weakly weighted quasi-metric space and w be a weak weight. If $w'\colon X \to \mathbb{R}$, then w' is a weak weight of d if and only there exists $c \in \mathbb{R}$ such that $w' = w + c$. Equivalently, $[w]_\sim \cap \mathbb{R}^X_{\geq 0}$ consists of all the weak weights of d.*

We cannot define a canonic weak weight as we did for weights in Proposition 2.2 since there is no fading weak weight in general.

Theorem 2.10. *Let X be a set. Then there exists a one-to-one correspondence between equivalence classes of weak partial metrics of X and weakly weighted quasi-metrics.*

Proof. The correspondence defined in the proof of Theorem 2.5 can be readily adapted in this context. □

3 Invariant generalised quasi-metric semilattices

In this section we deal with generalised quasi-metric spaces. We will soon notice that every generalised quasi-metric space can be broken down into quasi-metric subspaces.

Let us recall that a subset Y of a generalised quasi metric space (X,d) inherits a generalised quasi-metric $d|_{Y\times Y}$ as the restriction of d. For the sake of brevity, we denote it by $d|_Y$.

Let (X,d) be a generalised quasi-metric space and $x\in X$. Following [33], we define the *connected component* $\mathcal{Q}(x)$ *of* x as follows:

$$\mathcal{Q}(x)=\{y\in X\mid d(x,y)<\infty,\ \text{and}\ d(y,x)<\infty\}.$$

Then $d|_{\mathcal{Q}(x)}$ is a quasi-metric. Moreover, for every subset Y of X, $d|_Y$ is a quasi-metric if and only if either $Y=\emptyset$ or for any point $y\in Y$ we have that $Y\subseteq\mathcal{Q}(y)$.

Let $(X,\leq)$ be a partial order. If $x,y\in X$, their *meet* $x\wedge y$ (*join* $x\vee y$) is the maximum of all the elements that are below both x and y (the minimum of all the elements that are above both x and y, respectively) provided that it exists. Then X is a *meet* (*join*) *semilattice* if the meet (join, respectively) of every pair of points exists. In the sequel, we refer to meet semilattices simply as semilattices as we do not use join semilattices in this paper.

To every generalised quasi-metric space (X,d) we can associate its *specialisation order* $\leq_d$, which is a partial order defined by $x\leq_d y$ if $d(x,y)=0$, for every $x,y\in X$.

Fact 3.1 (Monotonicity)**.** *Let (X,d) be a generalised quasi-metric space, and $x,x',y,y'\in X$. If $x\leq_d x'$ and $y'\leq_d y$, then $d(x,y)\leq d(x',y')$.*

Proof. Let $x,x',y,y'\in X$ as in the statement. Applying the triangular inequality, we obtain that

$$d(x,y)\leq d(x,x')+d(x',y')+d(y',y)=d(x',y')$$

because of the definition of the specialisation order. □

Proposition 3.2. *Let (X,d) be a weighted quasi-metric space and w be a weight. Then w is strictly decreasing.*

Proof. Let $x,y\in X$ be two distinct points satisfying $x\leq_d y$. Since $\leq_d$ is a partial order, $d(y,x)>0$. Then

$$w(x)-w(y)=d(y,x)-d(x,y)=d(y,x)>0,$$

which implies the desired property. □

Definition 3.3. A generalised quasi-metric space (X,d) is a *generalised quasi-metric semilattice* if $(X,\leq_d)$ is a semilattice. It is a *quasi-metric semilattice* if it is a generalised quasi-metric semilattice and d is a quasi-metric.

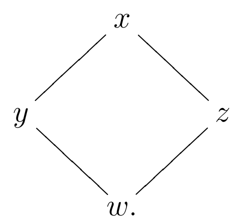
In the sequel, if there is no risk of ambiguity, for a generalised quasi-metric semilattice we denote its specialisation order simply by $\leq$.

It is not true in general that connected components of generalised quasi-metric semilattices are subsemilattices. Consider the following example.

Example 3.4. Let $X = \{x, y, z, w\}$ and define a generalised quasi-metric d on X as follows:

$$d(x, y) = d(x, z) = d(y, z) = d(z, y) = 1, \quad d(x, w) = d(y, w) = d(z, w) = \infty,$$

and all the other distances are set to zero. Then (X, d) is a generalised quasi-metric semilattice as the Hasse diagram of $(X, \leq_d)$ is



(3)

However, the connected component of x is not a subsemilattice as $y \wedge z = w \notin \mathcal{Q}(x)$.

We define and characterise a property immediately ensuring that every connected component of a generalised quasi-metric semilattice is a subsemilattice.

Proposition 3.5 ([9])**.** *For a generalised quasi-metric semilattice (X, d) the following properties are equivalent:*
(a) $d(x, y) = d(x, x \wedge y)$, for every $x, y \in X$;
(b) $d(x, \cdot)\colon X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is subadditive *(i.e., for every $y, z \in X$, $d(x, y \wedge z) \leq d(x, y) + d(x, z)$), for every $x \in X$;*
(c) $d(x \wedge z, y \wedge z) \leq d(x, y)$, for every $x, y, z \in X$.

Proof. In this proof we apply several times the definition of the specialisation order.

(a)→(b) Let $x, y, z \in X$. Using the invariance and the monotonicity of d we obtain the following chain of inequalities

$$\begin{aligned} d(x, y \wedge z) &= d(x, x \wedge y \wedge z) \leq d(x, x \wedge y) + d(x \wedge y, x \wedge y \wedge z) = \\ &= d(x, y) + d(x \wedge y, z) \leq d(x, y) + d(x, z). \end{aligned}$$

(b)→(c) For every $x, y, z \in X$, applying the subadditivity and the monotonicity we obtain that

$$d(x \wedge z, y \wedge z) \leq d(x \wedge z, y) + d(x \wedge z, z) = d(x \wedge z, y) \leq d(x, y).$$

(c)→(a) Let $x, y \in X$. The triangular inequality implies that

$$d(x, y) \leq d(x, x \wedge y) + d(x \wedge y, y) = d(x, x \wedge y).$$

Conversely,

$$d(x, x \wedge y) = d(x \wedge x, x \wedge y) \leq d(x, y).$$ □

A generalised quasi-metric semilattice satisfying the properties enlisted in Proposition 3.5 is said to be *invariant* ([26]). Let us recall that a map $f\colon (X, d_X) \to (Y, d_Y)$ between generalised quasi-metric spaces is *contractive* if $d_Y(f(x), f(y)) \leq d_X(x, y)$, for every $x, y \in X$. Proposition 3.5(c) implies that the shifts $s_x\colon y \mapsto x \wedge y$ are contractive.

Fact 3.6. *Let (X, d) be an invariant generalised quasi-metric semilattice. Then every connected component of X is a quasi-metric subsemilattice.*

Proof. Let $x \in X$ and $y \in \mathcal{Q}(x)$. We have already noticed that $d|_{\mathcal{Q}(x)}$ is a quasi-metric. We claim that $x \wedge y \in \mathcal{Q}(x)$. As $x \wedge y \leq x$, $d(x \wedge y, x) = 0$. Moreover, $d(x, x \wedge y) = d(x, y) < \infty$ since X is invariant and $y \in \mathcal{Q}(x)$. □

Every semilattice can be endowed with a quasi-metric making it a quasi-metric semilattice.

Proposition 3.7. *Let $(X, \leq)$ be a semilattice. Then there exists a quasi-metric of X whose specialisation order coincides with $\leq$.*

Proof. Let $c \in \mathbb{R}_{>0}$. We define a quasi-metric $d^c_\leq$ as follows:

$$d^c_\leq(x, y) = \begin{cases} 0 & \text{if } x \leq y, \\ c & \text{otherwise.} \end{cases} \tag{4}$$

Then $d^c_\leq$ satisfies the desired properties. □

For a given semilattice $(X, \leq)$ we can apply the construction (4) also putting $c = \infty$. In that case, $d^\infty_\leq$ is a generalised quasi-metric whose specialisation order coincides with $\leq$.

Note that, the quasi-metric space (X, d_1) constructed in Example 2.7 coincides with the semilattice X endowed with the usual order $(0 < 1 < 2)$ and the quasi-metric $d^1_\leq$.

Let us state another important property for generalised quasi-metric semilattices.

Definition 3.8. For a generalised quasi-metric semilattice (X, d) we define the *descending path condition* as the property
(DPC) $d(x, z) = d(x, y) + d(y, z)$, for every $x, y, z \in X$ satisfying $z \leq y \leq x$.

The next easy result discuss the monotonicity property of invariance and (DPC).

Proposition 3.9. *Let X be a generalised quasi-metric semilattice and $Y \subseteq X$ be a subsemilattice endowed with the generalised quasi-metric inherited by X. Then the following implications hold:*
(a) if X is invariant, then so is Y;
(b) if X satisfies (DPC), then so does Y.

The descending path condition can be checked on each connected component separately.

Theorem 3.10 ([9])**.** *Let X be an invariant generalised quasi-metric semilattice. Then X satisfies (DPC) if and only if each connected component satisfies (DPC).*

Proof. If X satisfies (DPC), then trivially each subspace satisfies the same property. Assume now that every connected component satisfies (DPC), and let $x, y, z \in X$ satisfy $z \leq y \leq x$. The triangular inequality implies that $d(x, z) \leq d(x, y) + d(y, z)$. Hence, if

$d(x,z) = \infty$, there is nothing to prove. If $d(x,z) < \infty$, then monotonicity implies that $d(x,y) < \infty$ and $d(y,z) < \infty$. Thus the three points x, y, z belong to the same connected component and we can apply the hypothesis. □

Invariance and descending path condition are independent properties, as the following examples show.

Example 3.11. (a) Consider the quasi-metric space (X, d_1) described in Example 2.7 which is a quasi-metric semilattice, as noticed before. Then X is invariant, but it does not satisfy (DPC).

(b) Let $X = \{x, y, z, w\}$ be the semilattice whose Hasse diagram is (3). Define a quasi-metric d on X as follows:

$$d(x,y) = d(x,z) = d(y,z) = d(z,y) = 1, \quad d(y,w) = d(z,w) = 2, \quad d(x,w) = 3$$

and all the other distances are set to 0. Then (X, d) is not invariant ($d(y,z) = 1 \neq 2 = d(y,w)$) even though it satisfies (DPC).

Let us now discuss the generalised quasi-metric spaces provided in Example 2.6.

Example 3.12. All the generalised quasi-metric spaces defined in Example 2.6 are invariant generalised quasi-metric semilattices satisfying (DPC). Let us discuss each one of them separately.

(a) The specialisation order of the quasi-metric d on $\mathbb{S}$ is the usual order $0 < 1$, which induces a semilattice. Moreover, d trivially satisfies invariance and (DPC).

(b) On $\mathbb{R}$, the quasi-metric d induces the usual order, and thus $(\mathbb{R}, d)$ is a quasi-metric semilattice. The meet of two elements is their minimum. Moreover, it is invariant and satisfies (DPC). According to Proposition 3.9, also $\mathbb{R}_{\leq 0}$ satisfies the same properties.

(c) In the notation of Example 2.6(c), if $x, x' \in \Sigma^*$ satisfy $d_p(x, x') = 0$, then $l(x, x') = |x|$ and thus x is a substring of x'. Thus, for every pair $x, y \in \Sigma^*$, $x \wedge y$ is the longest prefix common to both x and y. Hence, (Σ^*, d_p) is a quasi-metric semilattice. Moreover, it is easy to verify that it is invariant. As for the descending path condition, if $x, y, z \in \Sigma^*$ satisfy $z \leq y \leq x$, then

$$d_p(x,z) = 2^{-|z|} - 2^{-|x|} = 2^{-|z|} - 2^{-|y|} + 2^{-|y|} - 2^{-|x|} = d_p(x,y) + d_p(y,z).$$

(d) Let X be a set and d be the generalised quasi-metric on $\mathcal{P}(X)$ defined in Example 2.6(d). Then $A \leq B$ if and only if $d(A, B) = 0$ if and only if $B \subseteq A$, and so $\leq = \supseteq$. The meet of two elements is their union and d is trivially invariant. It also satisfies the descending path condition. In fact, if $C \leq B \leq A$ (equivalently, $A \subseteq B \subseteq C$), the equality $C \setminus A = (C \setminus B) \sqcup (B \setminus A)$ implies the desired claim.

(e) Let G be an abelian group and d be the generalised quasi-metric on $L(G)$ as in Example 2.6(e). For every pair of subgroups H, K of G, $d(H, K) = 0$ if and only if $|H + K : H| = 1$ or, equivalently, $K \subseteq H$. Thus $\leq = \supseteq$, and the meet of two subgroups is their sum, which is a subgroup since G is abelian. Again d is trivially invariant. It also satisfies (DPC), which is implied by the equality $|H : L| = |H : K| \cdot |K : L|$ that holds for every triple of subgroups $H, K, L \in L(G)$ satisfying $L \subseteq K \subseteq H$.

4 Characterisations of weighted invariant quasi-metric semilattices

In this section, following the work of Schellekens ([26, 25]), we provide two different characterisations of weighted invariant quasi-metric semilattices.

4.1 Inner characterisation

Let us note the following implication.

Proposition 4.1. *Let X be an invariant quasi-metric semilattice. If X is weighted, then it satisfies (DPC).*

Proof. Let d be the quasi-metric on X and w be a weight. Take three points $x, y, z \in X$ satisfying $z \leq y \leq x$. Then

$$\begin{aligned} d(x,z) - d(x,y) - d(y,z) &= d(x,z) + w(x) - d(x,y) - w(x) - d(y,z) - w(y) + w(y) = \\ &= d(z,x) + w(z) - d(y,x) - w(y) - d(z,y) - w(z) + w(y) = \\ &= 0, \end{aligned}$$

which implies that (DPC) holds. □

As the quasi-metric semilattice (X, d_1) defined in Example 2.7 does not satisfy (DPC), it is not weighted.

Since there are invariant quasi-metric semilattices satisfying (DPC) that are not weighted (see Examples 2.6(b) and 3.12(b)), we cannot revert the implication stated in Proposition 4.1 in general. However, following [7], we can provide a clearer connection to weakly weighted quasi-metrics from which the desired result easily descends.

Let X be an invariant quasi-metric semilattice satisfying (DPC), and $x \in X$. We define the map $w_x \colon X \to \mathbb{R}$ as follows: for every $y \in X$,

$$w_x(y) = d(x,y) - d(y,x).$$

Let us first state a lemma concerning the maps w_x.

Lemma 4.2. *Let (X, d) be an invariant quasi-metric semilattice satisfying (DPC). Then, for every $x, y, z \in X$,*

$$w_x(y) = d(x, x \wedge y \wedge z) - d(y, x \wedge y \wedge z).$$

Proof. Let $x, y, z \in X$. Then invariance and (DPC) imply that

$$\begin{aligned} w_x(y) &= d(x,y) - d(y,x) = d(x, x \wedge y) - d(y, x \wedge y) = \\ &= d(x, x \wedge y) + d(x \wedge y, x \wedge y \wedge z) - d(y, x \wedge y) - d(x \wedge y, x \wedge y \wedge z) = \\ &= d(x, x \wedge y \wedge z) - d(y, x \wedge y \wedge z). \end{aligned}$$ □

Theorem 4.3. *Let (X, d) be an invariant quasi-metric semilattice satisfying (DPC). Then, for every $x \in X$, w_x is a weak weight of d (i.e., it satisfies* (1)*).*

Proof. Fix $x \in X$. We claim that, for every $y, z \in X$, $d(y, z) + w_x(y) = d(z, y) + w_x(z)$ or, equivalently, that $w_x(y) - w_x(z) = d(z, y) - d(y, z)$. Applying Lemma 4.2 and (DPC), we obtain the following chain

$$\begin{aligned} w_x(y) - w_x(z) &= d(x, x \wedge y \wedge z) - d(y, x \wedge y \wedge z) - d(x, x \wedge y \wedge z) + d(z, x \wedge y \wedge z) = \\ &= d(z, x \wedge y \wedge z) - d(y, x \wedge y \wedge z) = w_z(y) = d(z, y) - d(y, z), \end{aligned}$$

which concludes the proof. □

The previous result, however, does not provide conditions under which the quasi-metric semilattice is weighted. In fact, w_x can assume also negative values.

Theorem 4.4. *Let (X, d) be an invariant quasi-metric semilattice satisfying (DPC). Then the following properties are equivalent:*
(a) X is weighted;
(b) for every $x \in X$, there exists $c_x \in \mathbb{R}$ such that $d(x, y) - d(y, x) = w_x(y) \geq c_x$, for every $y \in X$;
(c) there exist $x \in X$ and $c \in \mathbb{R}$ such that $d(x, y) - d(y, x) = w_x(y) \geq c$, for every $y \in X$.

Proof. The implication (b)→(c) is trivial. As for the implication (c)→(a), we consider the map $w = w_x - c$. Because of the hypothesis, $w \geq 0$. Then Theorem 4.3 implies the claim as w satisfies (1) since w_x does.

Let us prove the remaining implication (a)→(b). Assume, by contradiction, that d is weighted by the weight w and there exists $x \in X$ such that, for every $c \in \mathbb{R}$, $w_x(y_c) < c$ for some $y_c \in X$. Then (1) implies that, for every $c \in \mathbb{R}$,

$$w(x) = d(y_c, x) - d(x, y_c) + w(y_c) = -w_x(y_c) + w(y_c) > -c$$

as $w \geq 0$, which is a contradiction. □

Corollary 4.5. *Let X be an invariant quasi-metric semilattice satisfying (DPC). If X has a top element $\top$, then X is weighted.*

Proof. It follows by applying Theorem 4.4 according to the fact that

$$w_\top(x) = d(\top, x) - d(x, \top) = d(\top, x) \geq 0. \qquad \square$$

We now apply Theorem 4.3 and Corollary 4.5 to the examples we introduced.

Example 4.6. In this example we adopt the notation used in Examples 2.6 and 3.12.

(a) As $\mathbb{S}$ has top element 1, the map d is weighted by the map $w_1(\cdot) = d(1, \cdot)$, which coincides with the one provided in Example 2.6(a).

(b) On $(\mathbb{R}, d)$, we consider the weak weight w_0 which associates to every point $x \in \mathbb{R}$,

$$w_0(x) = d(0, x) - d(x, 0) = \begin{cases} 0 - x & \text{if } x \geq 0, \\ |x| - 0 & \text{otherwise.} \end{cases}$$

Then w_0 coincides with $-id$.
The same map w_0 restricted to $\mathbb{R}_{\leq 0}$ provides a weight for that subspace. Moreover, in that context, 0 is the top element of the semilattice.

(c) On the quasi-metric space (Σ^*, d_p), we consider the weak weight w_ε, where ε is the empty word. More explicitly, for every $x \in \Sigma^*$,

$$w_\varepsilon(x) = d_p(\varepsilon, x) - d_p(x, \varepsilon) = (2^0 - 2^0) - (2^0 - 2^{-|x|}) = 2^{-|x|} - 1.$$

Even though the map w_ε is not a weight, as it assumes negative values, d is weighted as $w_\varepsilon + 1 > 0$. Moreover, $w_\varepsilon + 1$ coincides with the map w_p defined according to Theorem 2.5. Note that Σ^* is weighted even though it does not have top element.

(d) For every finite set X, the quasi-metric space $\mathcal{P}(X)$ has a top element given by $\emptyset$. Then the map $w_\emptyset(A) = |A|$, defined for every $A \subseteq X$, is a weight.

(e) For every finite abelian group G, the quasi-metric space $L(G)$ has a the trivial subgroup $\{0\}$ as top element. Thus it is weighted by the map $w_{\{0\}}(H) = |H|$.

4.2 Semi-co-valuations

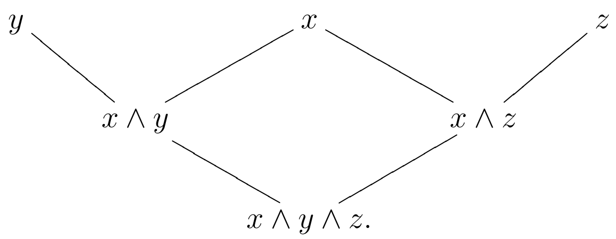
Let us introduce the notion of meet co-valuation that generalises Brikhoff's valuation on a lattice ([3]).

Definition 4.7 ([26])**.** Let X be a semilattice. A *meet co-valuation* is a map $f\colon X \to \mathbb{R}_{\geq 0}$ satisfying the following property:

$$f(x) + f(x \wedge y \wedge z) \leq f(x \wedge y) + f(x \wedge z). \tag{5}$$

In [26] there are several other notions close to meet co-valuations. Moreover, we refer to [9] and [7] for their characterisation. For consistency with the notation adopted in [26] and [7], we call meet co-valuations also *semi-co-valuations*.

We represent the points involved in (5):



We note that (5) links the values on the upper-left and the lower-right sides of the square formed in the diagram above.

Fact 4.8. *Let f be a semi-co-valuation of the semilattice X. Then f is decreasing.*

Proof. Let $x, y \in X$ satisfying $x \leq y$. Then (5) implies that

$$f(y) = f(y) - f(y \wedge x) \leq f(y \wedge x) - f(y \wedge x \wedge x) = f(x) - f(x),$$

and so f is decreasing. □

It is not true in general that a semi-co-valuation is strictly decreasing. For example, note that every constant map $c\colon X \to \mathbb{R}_{\geq 0}$ collapsing a semilattice into a positive value is a semi-co-valuation.

Note that, if f is a semi-co-valuation, then $[f]_\sim \cap \mathbb{R}^X_{\geq 0}$ is a family of semi-co-valuations. Moreover, there exists a canonic representative f^0 of the family $[f]_\sim \cap \mathbb{R}^X_{\geq 0}$, which is given by the unique fading semi-co-valuation equivalent to f. This easy observation is consistent with Proposition 2.2 in virtue of the following result.

Theorem 4.9. *Let $(X, \leq)$ be a semilattice. Then there is a one-to-one correspondence between invariant weighted quasi-metrics satisfying $\leq_d = \leq$ and equivalence classes of strictly decreasing semi-co-valuations.*

Proof. Let (X, d) be an invariant quasi-metric semilattice weighted by w. We claim that w is a semi-co-valuation. Let $x, y, z \in X$. Invariance implies that $d(x \wedge z, x \wedge y \wedge z) \leq d(x, x \wedge y)$. Since $d(x \wedge y \wedge z, x \wedge z) = 0 = d(x \wedge y, x)$, (1) implies that

$$w(x \wedge y \wedge z) - w(x \wedge z) = d(x \wedge z, x \wedge y \wedge z) \leq d(x, x \wedge y) = w(x \wedge y) - w(x),$$

which is equivalent to (5).

Given a strictly decreasing semi-co-valuation f of X, we define the map $d_f \colon X \times X \to \mathbb{R}_{\geq 0}$ as follows: for every $x, y \in X$,

$$d_f(x, y) = f(x \wedge y) - f(x).$$

Then d_f is actually a quasi-metric, thanks to (5) and to the fact that f is strictly decreasing. In fact, for every $x, y, z \in X$,

$$\begin{aligned} d_f(x, y) &= f(x \wedge y) - f(x) = f(x \wedge z) - f(x) + f(x \wedge y) - f(x \wedge z) = \\ &= d_f(x, z) + f(x \wedge y) - f(x \wedge z), \end{aligned}$$

and the chain

$$f(z) + f(x \wedge y) \leq f(z) + f(x \wedge y \wedge z) \leq f(x \wedge z) + f(y \wedge z),$$

due to Fact 4.8 and (5), implies the triangular inequality. Moreover, $d_f(x, y) = 0$ if and only if $f(x) = f(x \wedge y)$ and, since f is strictly decreasing and $x \wedge y \leq x$, $x \wedge y = x$, which implies $x \leq y$. Thus, $\leq_{d_f} = \leq$. Furthermore, d_f is invariant by construction and it is weighted by the map f itself.

Finally, it is a one-to-one correspondence as a weight of d_f is a semi-co-valuation w satisfying $w \sim f$ and, if g is a semi-co-valuation satisfying $f \sim g$, then $d_f = d_g$. □

Equivalently, Theorem 4.9 can be rewritten as follows: *Let $(X, \leq)$ be a semilattice. Then there is a one-to-one correspondence between invariant weighted quasi-metrics satisfying $\leq_d = \leq$ and strictly decreasing fading semi-co-valuations.*

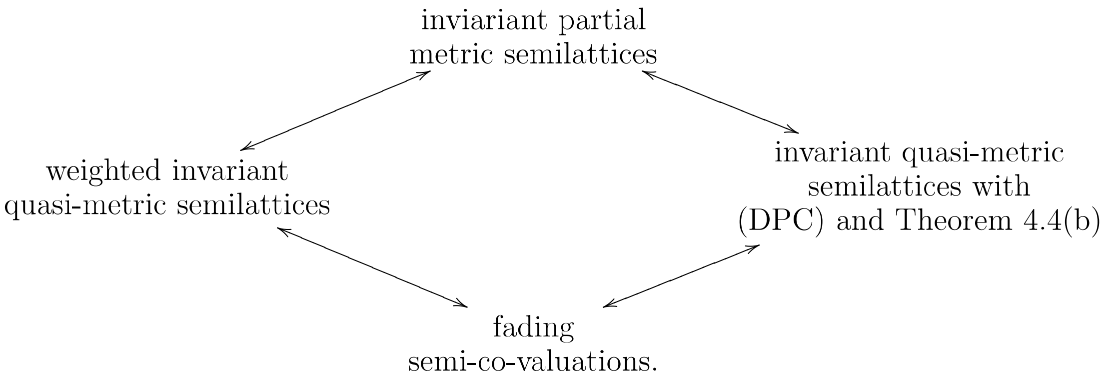
Let us represent the equivalences that we have proved so far. For a set X, we have the following one-to-one correspondence:

$$\text{equivalence classes of partial metrics} \longleftrightarrow \text{weighted quasi-metrics.}$$

As for semilattices, the situation is richer. Several notions that we have defined for quasi-metrics can be adapted to partial metrics. For every partial metric space (X, p), the partial order $\leq_p = \leq_{d_p}$ can be described as follows: for every $x, y \in X$, $x \leq_p y$ if and only if $p(x,x) = p(x,y)$. A partial metric space (X,p) is a *partial metric semilattice* if $(X, \leq_p)$ is a semilattice. Moreover, a partial metric semilattice (X,p) is *invariant* if, for every $x, y \in X$,

$$p(x \wedge y, x) = p(x, x \wedge y) = p(x,y).$$

Trivially, a partial metric space is an invariant partial metric semilattice if and only if the associated quasi-metric space (via Theorem 2.5) is an invariant quasi-metric semilattice. Thus, if X is a semilattice, then we have the following situation:



Those equivalences can be extended in some sense to the case of generalised quasi-metric semilattices. We refer to [7] for a complete discussion.

5 Applications to intrinsic entropy

We have mentioned in the introduction that generalised quasi-metric semilattices were used in dynamics in [4]. In this section we briefly sketch their role. Note that in the mentioned paper as in [9] the authors did not consider the specialisation order, but its dual: for a generalised quasi-metric space (X,d), $x \preceq_d y$ if and only if $d(y,x) = 0$ if and only if $y \leq_d x$.

5.1 Intrinsic semilattice entropy

A map $f\colon X \to Y$ between semilattices with top element is a *monoid homomorphism* if $f(\top_X) = \top_Y$ and $f(x \wedge y) = f(x) \wedge f(y)$, for every $x, y \in X$. The name is justified by the following observation: since X has a top element, $(X, \wedge)$ is a commutative monoid in which every element is idempotent and $\top$ is the neutral element. We define the category $\mathcal{L}_{qm}$ of invariant generalised quasi-metric semilattices with top element and contractive monoid homomorphism between them.

Let $f\colon X \to X$ be a morphism of $\mathcal{L}_{qm}$. An element $x \in X$ is said to be *f-inert* if $f(x) \wedge x \in \mathcal{Q}(x)$. For an f-inert element x we define its *n-trajectory*, where $n \in \mathbb{N} \setminus \{0\}$, as the subset

$$T_n(f,x) = x \wedge f(x) \wedge \cdots \wedge f^{n-1}(x).$$

Proposition 5.1. *Let $f\colon X \to X$ be a morphism of $\mathcal{L}_{qm}$ and $x \in X$ be an f-inert element. Then, for every $n \in \mathbb{N} \setminus \{0\}$, $T_n(f,x) \in \mathcal{Q}(x)$.*

Proof. Let us prove the result by induction. The base cases ($n = 1$ and $n = 2$) are trivial because of the definition. Assume that $T_n(f,x) \in \mathcal{Q}(x)$. We want to show that $T_{n+1}(f,x) \in \mathcal{Q}(x)$. First of all, $d(T_{n+1}(f,x),x) = 0$. Conversely, let us estimate $d(x, T_{n+1}(f,x))$. We have

$$\begin{aligned} d(x, T_{n+1}(f,x)) &= d(x, x \wedge f(x) \wedge \cdots \wedge f^{n-1}(x) \wedge f^n(x)) \leq \\ &\leq d(x, T_n(f,x)) + d(T_n(f,x), T_n(f,x) \wedge f^n(x)), \end{aligned}$$

and, using repetitively the invariance and the properties of f,

$$\begin{aligned} d(T_n(f,x), T_n(f,x) \wedge f^n(x)) &\leq d(f(x) \wedge \cdots \wedge f^{n-1}(x), f(x) \wedge \cdots \wedge f^{n-1}(x) \wedge f^n(x)) = \\ &= d(f(x \wedge \cdots \wedge f^{n-2}(x)), f(x \wedge f^{n-2}(x) \wedge f^{n-1}(x))) \leq \\ &\leq d(x \wedge \cdots \wedge f^{n-2}(x), x \wedge f^{n-2}(x) \wedge f^{n-1}(x)) \leq \\ &\leq \cdots \leq \\ &\leq d(x, x \wedge f(x)) < \infty. \end{aligned}$$

By combining the two previous chains, we obtain the claim. □

For a morphism $f\colon X \to X$ of $\mathcal{L}_{qm}$ and x an f-inert point of X, the sequence $\{d(x, T_n(f,x))\}_n$ of positive finite values is increasing as

$$x = T_1(f,x) \geq T_2(f,x) \geq \cdots \geq T_n(f,x) \geq \cdots,$$

taking into account Fact 3.1.

Definition 5.2. Let $f\colon X \to X$ be a morphism of $\mathcal{L}_{qm}$.

(a) For an f-inert element $x \in X$, we define the *intrinsic semilattice entropy of f relative to x* as the value

$$\widetilde{H}(f,x) = \lim_{n\to\infty} \frac{d(x, T_n(f,x))}{n}. \tag{6}$$

(b) We define the *intrinsic semilattice entropy of f* as the value

$$\widetilde{h}(f) = \sup\{\widetilde{H}(f,x) \mid x \text{ is } f\text{-inert}\}.$$

As for the proof of the existence of the limit in (6), we refer to [4], where the authors used Fekete Lemma.

5.2 Intrinsic algebraic entropy

In this subsection we show us the intrinsic algebraic entropy defined in [10] can be recovered using the intrinsic semilattice entropy.

Let $\phi\colon G \to G$ be an endomorphism of an abelian group G. A subgroup K of G is said to be *ϕ-inert* if the index of K in $K + \phi(K)$ ($|K + \phi(K) : K|$) is finite. For a ϕ-inert subgroup K and $n \in \mathbb{N} \setminus \{0\}$, we define the *$n$-algebraic trajectory* as the subgroup

$$T_n^{alg}(\phi, K) = K + \phi(K) + \cdots + \phi^{n-1}(K).$$

It can be shown that, in the previous notation, if K is f-inert, then, for every $n \in \mathbb{N} \setminus \{0\}$, $|T_n^{alg}(\phi, K) : K| < \infty$.

Definition 5.3 ([10])**.** Let ϕ be an endomorphism of an abelian group G.

(a) For a ϕ-inert subgroup K, we define the *intrinsic algebraic entropy of ϕ relative to K* as the value

$$\widetilde{ent}(\phi, K) = \lim_{n\to\infty} \frac{\log|T_n^{alg}(\phi, K) : K|}{n}. \tag{7}$$

(b) We define the *intrinsic algebraic entropy of ϕ* as the value

$$\widetilde{ent}(\phi) = \sup\{\widetilde{ent}(\phi, K) \mid K \text{ is a } \phi\text{-inert subgroup of } G\}.$$

Again, as for the proof of the existence of the limit in (7), we refer to [10].

Definitions 5.2 and 5.3 are similar in flavour. The final part of the paper is devoted to show that the intrinsic algebraic entropy can be indeed described using the intrinsic semilattice entropy.

Let **AbGrp** be the category of abelian groups and their homomorphisms. We define a functor $F\colon \mathbf{AbGrp} \to \mathcal{L}_{qm}$ as follows. If G is a group, $F(G)$ is the invariant generalised quasi-metric semilattice $L(G)$ described in Examples 2.6(e) and 3.12(e) which has $\{0\} \in L(G)$ as top element. It is easy to see that it is actually a functor. The functor F induces a functor between the corresponding flow categories.

Given a category $\mathcal{X}$, the *category of flows in $\mathcal{X}$* $\mathbf{Flow}_{\mathcal{X}}$ consists of pairs (X, f) called *flows* as objects, where $f\colon X \to X$ is a morphism of $\mathcal{X}$, and a morphism between two flows $(X, f), (Y, g)$ is a morphism $h\colon X \to Y$ of $\mathcal{X}$ making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & Y. \end{array}$$

A functor $F\colon \mathcal{X} \to \mathcal{Y}$ between categories induces a functor $\overline{F}\colon \mathbf{Flow}_{\mathcal{X}} \to \mathbf{Flow}_{\mathcal{Y}}$ between the corresponding flow categories in the obvious way: for every flow (X, f) in $\mathcal{X}$, $\overline{F}(X, f) = (F(X), Ff)$, and, for every morphism $h\colon (X, f) \to (Y, g)$, $\overline{F}(f) = F(f)$. In particular, we have a functor $\overline{F}\colon \mathbf{Flow}_{\mathbf{AbGrp}} \to \mathbf{Flow}_{\mathcal{L}_{qm}}$.

Adding the intrinsic semilattice and the intrinsic algebraic entropies to the picture, we obtain the following diagram:

$$\begin{array}{ccc} \mathbf{Flow}_{\mathbf{AbGrp}} & \xrightarrow{\quad\overline{F}\quad} & \mathbf{Flow}_{\mathcal{L}_{qm}} \\ & {\scriptstyle\widetilde{ent}}\searrow \quad \swarrow{\scriptstyle\widetilde{h}} & \\ & \mathbb{R}_{\geq 0} \cup \{\infty\}. & \end{array}$$

We claim that the diagram commutes, which implies that the intrinsic algebraic entropy can be recovered from the intrinsic semilattice entropy.

The following lemma can be easily derived.

Lemma 5.4. *Let $(G, \phi) \in \mathbf{Flow_{AbGrp}}$. Then a subgroup H of G is ϕ-inert if and only if $H \in L(G)$ is $F(\phi)$-inert.*

Theorem 5.5. *On* **AbGrp**, $\widetilde{ent} = \widetilde{h} \circ F$.

Proof. Let $(G, \phi) \in \mathbf{Flow_{AbGrp}}$, and H be a ϕ-inert subgroup of G. Lemma 5.4 implies that $H \in L(G)$ is $F(\phi)$-inert. Moreover, for every $n \in \mathbb{N} \setminus \{0\}$, $T_n^{alg}(\phi, H) = T_n(F(\phi), H)$ and

$$\begin{aligned}\widetilde{ent}(\phi, H) &= \lim_{n\to\infty} \frac{\log|T_n^{alg}(\phi, H) : H|}{n} = \\ &= \lim_{n\to\infty} \frac{d(H, T_n^{alg}(\phi, H))}{n} = \widetilde{h}(F(\phi), H).\end{aligned}$$

Another application of Lemma 5.4 implies that $\widetilde{ent}(\phi) = \widetilde{h}(F(\phi))$. □

References

[1] S. Abramsky, A. Jung, *Domain Theory*, 1–168 di S. Abramsky, D. M. Gabbay, T. S. E. Maibaum, *Handbook of Logic in Computer Science*, vol. 3 Oxford University Press (1994).

[2] R. L. Adler, A. G. Konheim, M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. 114 (1965) 309–319.

[3] G. Birkhoff, *Lattice Theory*, in AMS Colloquium Publications, Vol. 25, American Mathematical Society, Providence, RI (1984).

[4] I. Castellano, D. Dikranjan, D. Freni, A. Giordano Bruno, D. Toller, *Intrinsic entropy for generalized quasimetric semilattices*, to appear in J. Algebra Appl.

[5] I. Castellano, A. Giordano Bruno, *Algebraic entropy on locally linearly compact vector spaces*, Rings, Polynomials, and Modules, Springer (2017) 103–127.

[6] I. Castellano, A. Giordano Bruno, *Topological entropy on locally linearly compact vector spaces*, Topology Appl. 252 (2019) 112–144.

[7] I. Castellano, A. Giordano Bruno, N. Zava, *Weakly weighted generalised quasi-metric spaces and semilattices*, work in progress.

[8] D. Dikranjan, A. Giordano Bruno, *Entropy on normed semigroups (Towards a unifying approach to entropy)*, Dissertationes Mathematicae, 542 (2019), 1–90.

[9] D. Dikranjan, A. Giordano Bruno, H.-P. Künzi, N. Zava, D. Toller, *Generalized quasi-metric semilattices*, submitted.

[10] D. Dikranjan, A. Giordano Bruno, L. Salce, S. Virili, *Intrinsic algebraic entropy*, J. Pure Appl. Algebra 219 (2015), 2933–2961.

[11] D. Dikranjan, I. Protasov, N. Zava, *Hyperballeans of groups and G-spaces*, Topol. Appl. 263 (2019), 172–198.

[12] A. Giordano Bruno, M. Shlossberg, D. Toller, *Algebraic entropy in compactly covered locally compact groups*, Topology Appl. 263 (2019) 117–140.

[13] A. Giordano Bruno, S.Virili, *Topological entropy in totally disconnected locally compact groups*, Ergodic Theory Dynam. Systems 37 (2017) 2163–2186.

[14] J. Goubault-Larrecq, *Non-Hausdorff Topology and Domain Theory, Selected Topics in Point-Set Topology*, New Mathematical Monographs, Cambridge University Press (2013).

[15] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig (1914).

[16] A. N. Kolmogorov, *New metric invariants of transitive dynamical systems and automorphisms of Lebesgue spaces*, Doklady Akad. Nauk. SSSR 119 (1958) 861–864.

[17] H.-P. Künzi, *Nonsymmetric distances and their associated topologies: about the origins of basic ideals in the area of asymmetric topology*, in Handbook of the history of general topology, Vol. 3, 853–968, Hist. Topol., 3, Kluwer Acad. Publ., Dordrecht, 2001.

[18] H.-P. Künzi, V. Vajner, *Weighted Quasi-Metrics*, Annals of the New York Academy of Sciences 728 (1) (1994), 64–77.

[19] S. G. Matthews, *Partial metric topology.* Annals of the New York Academy of Sciences-Paper Edition 728 (1994), 183–197.

[20] M. Misiurewicz, *Coarse entropy*, abstract of the talk at the conference 'Dynamics of (Semi-)Group Actions', Łódź, Poland, July 9–12, 2019 (joint work with W. Geller).

[21] Y. Nakamura, *Entropy and semivaluations on semilattices*, Kodai Math. Sem. Rep., vol. 22, n. 4 (1970), 443-468.

[22] S. J. O'Neill, *Partial metrics, valuations, and domain theory*, Annals of the New York Academy of Sciences, 806(1) (1996) 304–315.

[23] J. Peters, *Entropy on discrete abelian groups*, Adv. Math. 33 (1979), 1–13.

[24] J. Peters, *Entropy of automorphisms on L.C.A. groups*, Pacific J. Math. 96 (2) (1981) 475–488.

[25] M. P. Schellekens, *Extendible spaces*, Appl. Gen. Top. 3, 2 (2002) 169–184.

[26] M. P. Schellekens, *The correspondence between partial metrics and semivaluations*, Theoretical Computer Science 315 (2004), 135–149.

[27] D. Scott, *A type theoretic alternative to ISWIM, CUCH, OWHY*, manuscript, University of Oxford (1969).

[28] D. Scott, *Outline of a mathematical theory of computation*, in 4^{th} *Annual Princeton Conference on Information Sciences and Systems* (1970), 169–176.

[29] C. E. Shannon, *A Mathematical Theory of Communication*, Bell System Technical Journal, 27 (3) (1948) 379–423.

[30] Y. G. Sinai, *On the concept of entropy of a dynamical system*, Doklady Akad. Nauk. SSSR 124 (1959) 789–781 (in Russian).

[31] M. D. Weiss, *Algebraic and other entropies of group endomorphisms*, Math. Systems Theory 8 (3) (1974/75) 243–248.

[32] W. A. Wilson, *On Quasi-Metric Spaces*, Amer. J. Math. 53 (1931), no. 3, 675–684.

[33] N. Zava, *Generalisations of coarse spaces*, Topology Appl. 263 (2019), 230–256.

[34] N. Zava, *On a notion of entropy in coarse geometry*, Topol. Algebra Appl. 7 (2019), 48–68.

University of Udine
Udine, 33100
ITALY
E-mail address: `nicolo.zava@gmail.com`