ALGEBRAIC STRUCTURE OF ABELIAN GROUPS INDUCED BY CONTINUOUS HOMOMORPHIC IMAGES TO COMPACT GROUPS

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As usual, $\mathbb{Z}$ and $\mathbb{N}$ denote the integer and natural numbers respectively, we also let $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. Given a group we shall denote by e its neutral element. We say that a group G is *torsion* if every element of G is of finite order. A group G is *bounded* if there exists a positive integer n satisfying $g^n = e$ for every $g \in G$.

We shall use the bold letters $\mathbf{P}$ and $\mathbf{Q}$ to denote properties of topological groups.

Finally, all topological groups are assumed to be Hausdorff.

1. Introduction

Definition 1.1. A topological group G is called:

(a) *maximally almost periodic* (MAP) if the continuous homomorphisms from G to compact groups separate its points, or
(b) *minimally almost periodic* (MinAP) if the only continuous homomorphism from G to any compact group is the trivial homomorphism.

These two concepts were introduced by von Neumann [10]. The MAP groups as introduced in this monograph, are those whose points may be separated by a set of complex-valued almost-periodic functions. These groups follow a classical research line of Bohr regarding almost periodic functions. On the other hand, the MinAP groups were introduced as their "opposite", that is, those whose only complex-valued almost periodic functions are constant. Some years later, von Neumann and Wigner published a joint paper [11] focusing only on MinAP topological groups.

Among the first examples of MinAP topological groups, the one given by Nienhuys [12] consisted of a connected and monothetic (= containing a dense copy of $\mathbb{Z}$) MinAP group of cardinality at most continuum. A consequence of his result shows that the *integers* may be equipped with a minimally almost periodic group topology. This fact sparked significant interest in the study of Abelian groups which are also minimally almost periodic, as a few years later Protasov asked if *every* Abelian group may be equipped with a minimally almost periodic group topology.

Remus gave an answer to Protasov's question by finding a bounded Abelian group which does not admit any MinAP group topology. Afterwards, Comfort proposed a modification of Protasov's original question, which in turn would become a major open problem in topological group theory:

Problem 1.2 (Protasov-Comfort (1990) [1, Question 521])**.** Does every Abelian group which is not of bounded order admit a minimally almost periodic group topology?

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In a more general setting, this problem prompted interest on describing *all Abelian groups which admit minimally almost periodic group topologies.* Being a problem of algebraic structure, solutions to this problem require the construction of MinAP group topologies, a feat which is known to be rather difficult.

Gabriyelyan obtained one of the first results in this direction. By means of the *Ulm-Kaplansky invariants*, he described the algebraic structure of the Abelian bounded groups admitting a MinAP group topology:

Theorem 1.3 ([7, Corollary 3]). *A bounded Abelian group admits a minimally almost periodic group topology if and only if all of its leading Ulm-Kaplansky invariants are infinite.*

Resolving the most general question (and including Problem 1.2), a complete characterization of Abelian groups which admit a MinAP group topology was achieved by Dikranjan and Shakhmatov [6]:

Theorem 1.4 ([6, Theorem 3.3]). *For an Abelian group G, the following conditions are equivalent:*

(i) *G admits a minimally almost periodic group topology;*
(ii) *G is connected with respect to its Markov-Zariski group topology* [3]*;*
(iii) *for every $n \in \mathbb{N}$, the subgroup $nG = \{ng : g \in G\}$ of G is either trivial or infinite.*

Surprisingly, this result makes a connection between minimal almost periodicity and connectedness through an ancient topology of Markov and Zariski (see also [3, 4, 8, 9]).

2. Groups which are almost minimally almost periodic

Definition 2.1. For every topological group G, there exists a compact group bG along with a continuous homomorphism $b_G : G \to bG$ having the following two properties:

(i) $b_G[G]$ is dense in bG, and
(ii) for each continuous homomorphism $f : G \to K$ from G to a compact group K, there exists a continuous homomorphism $f^+ : bG \to K$ satisfying the equality $f = f^+ \circ b_G$.

The group bG is called the *Bohr compactification of G with respect to b_G*, and the kernel of the homomorphism b_G (denoted by $\mathfrak{n}(G)$) is known as the *von Neumann kernel of G*.

When b_G is a monomorphism, the group topology inherited by $b_G(G)$ from bG is known as the *Bohr topology* of G. G equipped with its Bohr topology is denoted by G^+ (see [2]).

We introduced the following concept in [13]:

Definition 2.2 ([13, Definition 2.1]). Let $\mathbf{P}$ be a property of topological groups. We say that a topological group G is MinAP *modulo* $\mathbf{P}$ (MinAP *mod* $\mathbf{P}$) if for each continuous homomorphism $f : G \to K$ from G to a compact group K, the image $f[G]$ of G considered as a subgroup of K has property $\mathbf{P}$.

Naturally, one may compare MinAP modulo properties as follows:

Remark 2.3. If $\mathbf{P}$ and $\mathbf{Q}$ are properties of topological groups such that $\mathbf{P}$ implies $\mathbf{Q}$, then

$$\text{MinAP mod } \mathbf{P} \to \text{MinAP mod } \mathbf{Q}.$$

With this new terminology, if **P** is taken to be the property of being the "trivial group", then the class of MinAP modulo **P** topological groups coincides with the standard class of MinAP topological groups. Our motivation to introduce these classes of topological groups, is to determine the impact of diverse properties **P** on the corresponding algebraic structure of the Abelian MinAP modulo **P** groups.

First, we considered topological groups "very close" to being minimally almost periodic:

Question 2.4 ([13, Question 1.2 & 1.3])**.** What are topological groups every continuous homomorphic image of which in a compact group is *finite*?

To address this question we obtained a characterization of MinAP modulo **P** topological groups, for cases where **P** is preserved by continuous homomorphisms:

Theorem 2.5 ([13, Theorem 6.1, Corollary 6.2])**.** *Let* **P** *be a property of topological groups invariant under surjective continuous homomorphisms. For a topological group G the following are equivalent:*

(i) *G is* MinAP *modulo* **P***,*
(ii) *Its image $b_G(G)$ in the Bohr compactification bG of G has property* **P***, and*
(iii) *$(G/\mathfrak{n}(G))^+$ has property* **P***.*

When property **P** is considered as the property of a group being finite, torsion or bounded, then Theorem 2.5 shows the following:

Corollary 2.6 ([13, Corollary 6.5])**.** *A topological group G is* MinAP *modulo finite (bounded, torsion) if and only if the quotient $G/\mathfrak{n}(G)$ of G with respect to its von Neumann kernel $\mathfrak{n}(G)$ is finite (bounded, torsion, respectively).*

Thus, the groups in Question 2.4 are those with a *von Neumann kernel of finite index.*

As pointed out by Corollary 2.6, changing the algebraic strength of property **P** imposes different algebraic restrictions on the quotient with respect to $\mathfrak{n}(G)$. In [13] we compared the MinAP modulo **P** classes for five common properties **P**: finite, bounded, torsion, compact and connected. All of which are preserved by continuous homomorphisms (and therefore within the scope of Theorem 2.5). These properties are related as follows:

$$\begin{array}{ccccc} \text{MinAP} & \longrightarrow & \text{MinAP mod finite} & \longrightarrow & \text{MinAP mod compact} \\ \downarrow & & \downarrow & & \\ \text{MinAP mod connected} & & \text{MinAP mod bounded} & \longrightarrow & \text{MinAP mod torsion.} \end{array}$$

Figure 1. Diagram of implications

We determined that none of these implications are reversible, even for Abelian groups (see [13, Example 3.2 & Example 6.8]).

3. Algebraic structure of Abelian MinAP modulo finite topological groups

Theorem 2.5 allowed us to easily describe the properties displayed in Figure 1, and so our next goal was to obtain a description of their algebraic structure as a natural variation of Problem 1.2:

Question 3.1 ([13, Question 8.4])**.** For a "reasonable" property **P** of topological groups, can one describe the algebraic structure of (Abelian) groups which admit a MinAP mod **P** group topology?

To address this question, we applied Theorems 1.3 and 1.4 to obtain the following:

Theorem 3.2 ([13, Theorem 7.5])**.** *Every Abelian group G admits a decomposition $G = H \oplus F$, where H is a group which admits a minimally almost periodic group topology and F is a finite group.*

As a consequence, we answered Question 3.1 for almost all the properties in Figure 1:

Corollary 3.3 ([13, Corollary 7.6])**.** *Assume that* **P** *is a property of topological groups satisfied by all finite groups. Then, every Abelian group admits a* MinAP *mod* **P** *group topology.*

This result shows that all Abelian groups admit a MinAP modulo **P** group topology for **P** = *finite, compact, bounded, and torsion.* This motivated us to formulate the following question:

Question 3.4 ([13, Question 8.1])**.** Can one characterize all properties **P** of topological groups such that a topological group admits a MinAP modulo **P** group topology if and only if it admits a MinAP group topology?

Indeed, Corollary 3.3 shows that if a topological group has *finite* continuous homomorphic images to compact groups (instead of trivial ones), then it completely eliminates any need of the algebraic restrictions described in Theorem 1.4. As such, Question 3.4 aims to find *all* MinAP modulo **P** properties which can be described by Dikranjan and Shakhmatov's theorem. We also had one case remaining from the properties in Figure 1:

Question 3.5 ([13, Question 8.3])**.** Can one describe the algebraic structure of (Abelian) groups which admit a MinAP mod connected group topology?

Our latest results in [14] resolve Question 3.5.

4. Our current investigation regarding MinAP modulo connected groups

The algebraic structure of Abelian connected groups was completely described by Dikranjan and Shakhmatov in [5]. This description is as follows:

Theorem 4.1 ([5, Theorem 1.9, Corollary 1.10])**.** *For an Abelian group G, the following conditions are equivalent:*

- (i) *G admits a connected group topology, and*
- (ii) *for every $n \in \mathbb{N}$, the subgroup $nG = \{ng : g \in G\}$ of G is either trivial or its cardinality is at least the continuum.*

We note the clear similarity between this algebraic description and the one given in Theorem 1.4 for Abelian minimally almost periodic groups. While the only difference is the size requirement of the subgroups of the form nG for $n \in \mathbb{N}$, we also note that the techniques involved in proving both results are very different and highly complex. Every connected group is MinAP modulo connected, as *any* continuous homomorphic image of a connected group is again connected. With this in mind, our interest was to determine if item (ii) of Theorem 4.1 could also describe the algebraic structure of Abelian MinAP modulo connected topological groups. We proved that this is not the case.

First, our main result in [14] is a necessary condition for Abelian groups to admit a MinAP modulo connected group topology:

Theorem 4.2 ([14]). *Let G be an Abelian group G. If G admits a* MinAP *mod connected group topology, then for all $m \in \mathbb{N}$ the subgroup $mG = \{mg : g \in G\}$ is either trivial or infinite.*

Unlike the algebraic description given in Theorem 4.1, our Theorem 4.2 now matches the *weaker* requirements imposed by minimal almost periodicity in Theorem 1.4. If we recall the implications of Figure 1, every MinAP group is MinAP modulo connected. So, by applying Theorem 1.4 we obtain that the *converse also holds*:

Corollary 4.3 ([14]). *An Abelian topological group admits a* MinAP *modulo connected group topology if and only if it admits a minimally almost periodic group topology.*

This result solves our Question 3.5, and also completes the algebraic description of all properties featured in Figure 1 for Abelian groups.

5. FURTHER COMMENTS AND AN OPEN QUESTION

We note that Corollary 4.3 gives a partial answer to Question 3.4. Indeed, if one wishes to find the family of all properties **P** from Question 3.4, then such a family includes both the property of being the trivial group, and the property of being a *connected* group.

The following is a natural companion to Question 3.4.

Question 5.1. Can one find properties **P** such that a topological group admits a MinAP modulo **P** group topology if and only if it admits a *connected group topology*?

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