Two-microlocal estimates in wavelet theory and related function spaces

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1 Introduction

This report is based on the author's talk given at RIMS on November 7th, 2019. Two-microlocal ideas in wavelet analysis are considered. Sections 2 and 3 are taken from [JM] and [MY], respectively. Section 4 deals with some recent results obtained in [Mo].

2 What is "two-microlocal estimate" ?

We first give a brief survey of Jaffard-Meyer (1996). See [JM]. The determination of the pointwise regularity of a function f requires the use of some tools introduced by Bony (1986). See [Bo].

Let S_j be the "low-pass filter" which, after performing the Fourier transform, is the multiplication by $\widehat{\varphi}(2^{-j}\xi)$, where $\widehat{\varphi}(\xi) = 1$ if $|\xi| \leq 1/2$ and $\widehat{\varphi}(\xi) = 0$ if $|\xi| \geq 1$. Define $\Delta_j = S_{j+1} - S_j$. Thus we have the Littlewood-Paley decomposition:

$$Id = S_0 + \Delta_0 + \Delta_1 + \cdots .$$

The Fourier transform of $\Delta_j(f)$ is supported by the set $2^{j-1} \leq |\xi| \leq 2^{j+1}$.

Definition 2.1 (Jaffard-Meyer). *Let $s, s' \in \mathbb{R}$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to belong to $C^{s,s'}_{x_0}$ if*

$$|S_0(f)(x)| \leq C(1 + |x - x_0|)^{-s'}$$

and

$$|\Delta_j(f)(x)| \leq C 2^{-js}(1 + 2^j|x - x_0|)^{-s'}.$$

Definition 2.2 (Bony). *Let $s, s' \in \mathbb{R}$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to belong to $H^{s,s'}_{x_0}$ if*

$$\|2^{js}(1 + 2^j|x - x_0|)^{s'}\Delta_j(f)\|_{L^2} \leq c_j$$

with $\sum |c_j|^2 < \infty$.

Remark 2.3. We have the following fact: $u \in H^{s,-k}_{x_0}$, with k being a positive integer, if and only if $u = \sum_{|\alpha| \leq k}(x - x_0)^\alpha u_\alpha$, where $u_\alpha \in H^{s-|\alpha|}(\mathbb{R}^n)$.

Let us now consider an orthonomal wavelet basis on $\mathbb{R}^n$. Such a basis is composed by translations and dilations of $2^n - 1$ functions $\psi^{(i)}$. Recall the usual notation

$$\psi^{(i)}_{j,k}(x) = 2^{nj/2}\psi^{(i)}(2^j x - k), \quad j \in \mathbb{Z},\ k \in \mathbb{Z}^n.$$

The wavelet decomposition of a function f will be written

$$f = \sum_{i,j,k} C_{j,k} 2^{nj/2}\psi^{(i)}(2^j x - k).$$

We will usually forget the index i. The following result is easy to check:

Proposition 2.4. *$f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $C^{s,s'}_{x_0}$ if and only if*

$$|C_{j,k}| \leq C 2^{-(s+n/2)j}(1 + 2^j|k2^{-j} - x_0|)^{-s'}.$$

The following characterization also holds:

Proposition 2.5. *$f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $H^{s,s'}_{x_0}$ if and only if*

$$\sum_{j,k} 2^{2js}(1 + 2^j|k2^{-j} - x_0|)^{2s'}|C_{j,k}|^2 < \infty.$$

Our next purpose is to characterize the two-microlocal spaces in terms of local "Hölder type" conditions. In order to state these conditions, we need the Hölder-Zygmund spaces $\dot{C}^s(\mathbb{R}^n)$. If $0 < s < 1$, then $f \in \dot{C}^s(\mathbb{R}^n)$ is characterized by

$$|f(x) - f(y)| \leq C|x - y|^s.$$

If $s = 1$, then $f \in \dot{C}^s(\mathbb{R}^n)$ is characterized by

$$|f(x + h) - 2f(x) + f(x - h)| \leq C|h|.$$

The definition of the case where $s > 1$ needs higher order differences and is omitted.

It is easily checked that $f \in \dot{C}^s(\mathbb{R}^n)$ if and only if its wavelet coefficients satisfy the condition

$$|C_{j,k}| \leq C 2^{-(s+n/2)j}.$$

Let $A \subset \mathbb{R}^n$. By definition, a function f belongs to $C^s(A)$ if it is the restriction to A of a function F in $\dot{C}^s(\mathbb{R}^n)$. The norm of f is then the infimum of all possible norms of F in $\dot{C}^s(\mathbb{R}^n)$. Let B_ρ be the ball $|x - x_0| \leq \rho$, and Γ_ρ the annulus $\rho \leq |x - x_0| \leq 3\rho$. The following characterizations are the starting point of the talk:

Theorem 2.6. *If $s' < 0$, then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $C^{s,s'}_{x_0}$ if and only if*

$$\|f \mid C^{s+s'}(B_\rho)\| \leq C\rho^{-s'}. \tag{1}$$

If $s' > 0$, then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $C^{s,s'}_{x_0}$ if and only if $f \in \dot{C}^s(\mathbb{R}^n)$ and

$$\|f \mid C^{s+s'}(\Gamma_\rho)\| \leq C\rho^{-s'}. \tag{2}$$

Proof. We assume that the wavelet ψ is compactly supported and that $0 \in \operatorname{supp}\psi$. See [D]. We denote by C' the diameter of $\operatorname{supp}\psi$. We first suppose that f belongs to $C^{s,s'}_{x_0}$ so that its wavelet coefficients satisfy

$$|C_{j,k}| \le C2^{-(s+n/2)j}(1 + |2^j x_0 - k|)^{-s'}. \tag{3}$$

Note first that if $s' > 0$, then the inequality (3) implies that $|C_{j,k}| \le C2^{-(s+n/2)j}$, and so f belongs to $\dot{C}^s(\mathbb{R}^n)$. We split the wavelet decomposition

$$f = \sum C_{j,k}\psi_{j,k}$$

into three sums: $f = f_1 + f_2 + f_3$: The first, f_1, corresponds to the wavelets whose supports do not intersect the ball B_ρ (or the annulus Γ_ρ), and we can forget this sum.

Next we consider the sum f_2 whose coefficients satisfy $2^j\rho \le 10C'$; in that case, because $2^j|k2^{-j} - x_0|$ can be estimated from above by some constant comparable to $10C'$, the inequality (3) becomes

$$|C_{j,k}| \le C2^{-(s+n/2)j},$$

and so $\|f_2 \,|\, \dot{C}^s(\mathbb{R}^n)\| \le C$. The inequalities (1) and (2) for f_2 follow from this. (The details are omitted.)

Finally we consider the remaining sum f_3 whose coefficients satisfy $2^j\rho \ge 10C'$. If $s' > 0$, then the inequality (3) becomes

$$|C_{j,k}| \le C2^{-(s+s'+n/2)j}\rho^{-s'},$$

because the supports of the wavelets are inside the annulus Γ_ρ so that $|x_0 - k2^{-j}| \ge \rho$. The corresponding sum f_3 satisfies

$$\|f_3 \,|\, \dot{C}^{s+s'}(\mathbb{R}^n)\| \le C\rho^{-s'}.$$

If $s' < 0$, then the inequality (3) implies that

$$|C_{j,k}| \le C2^{-(s+n/2)j}(1 + 2^j\rho)^{-s'} \le C2^{-(s+s'+n/2)j}\rho^{-s'}.$$

We have the same conclusion as above.

Conversely let us assume that (1) or (2) holds. We consider a given wavelet $\psi_{j,k}$. If $s' < 0$, then we take for ρ the smallest number such that the support of $\psi_{j,k}$ is completely included in B_ρ so that any function extending f outside B_ρ has the same wavelet coefficient $C_{j,k}$, and the inequality (1) implies that

$$2^{(s+s'+n/2)j}|C_{j,k}| \le C\rho^{-s'}.$$

If $s' > 0$ and $|x_0 - k2^{-j}| > 2C'2^{-j}$, then the support of $\psi_{j,k}$ is completely included in Γ_ρ when $\rho = |x_0 - k2^{-j}|/2$ so that any function extending f outside Γ_ρ has the same wavelet coefficient $C_{j,k}$, and the inequality (2) implies that

$$2^{(s+s'+n/2)j}|C_{j,k}| \le C\rho^{-s'}.$$

If $s' > 0$ and $|x_0 - k2^{-j}| \le 2C'2^{-j}$, then we have to prove that $|C_{j,k}| \le C2^{-(s+n/2)j}$, which is implied by the assumption that $f \in \dot{C}^s(\mathbb{R}^n)$. $\square$

3 Two-microlocal Besov spaces and wavelets

Two-microlocal Besov spaces are considered by Moritoh-Yamada (2004), which is a natural extension of Jaffard-Meyer (1996). See [MY].

We first give the following definition and proposition. See [M] and [T].

Definition 3.1 (homogeneous Besov space)**.** *Let* $s > 0$ *and* $1 \leq p, q \leq \infty$. *Then the homogeneous Besov* $\dot{B}^s_{p,q}(\mathbb{R}^n)$ *is defined as the set of all tempered distributions* f *(modulo polynomials) satisfying*

$$\|f \mid \dot{B}^s_{p,q}(\mathbb{R}^n)\| = \left(\sum_{j \in \mathbb{Z}} 2^{jsq} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \mid L_p(\mathbb{R}^n)\|^q \right)^{1/q} < \infty.$$

Here, $\mathcal{F}f(\xi)$ denotes the Fourier transform of $f(x)$, and $\{\varphi_j\}_{j\in\mathbb{Z}}$ is a smooth resolution of unity.

Proposition 3.2. $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ *if and only if*

$$\sum_{j\in\mathbb{Z}} 2^{j\tilde{s}q} \left(\sum_{k\in\mathbb{Z}^n} |C_{j,k}|^p \right)^{q/p} < \infty,$$

where $\tilde{s} = s + n/2 - n/p$.

We can define the local Besov spaces $B^s_{p,q}(U)$ by restriction (see the previous section), and we now give the definition of the two-microlocal Besov spaces $B^{s,s'}_{p,q}(U)$, where U is an open subset of $\mathbb{R}^n$.

Definition 3.3 (two-microlocal Besov space)**.** *Let* $s > 0$, $s' \in \mathbb{R}$ *and* $1 \leq p, q \leq \infty$. *Then* $f \in \mathcal{S}'(\mathbb{R}^n)$ *is said to belong to the two-microlocal Besov space* $B^{s,s'}_{p,q}(U)$ *if the following two-microlocal estimate holds:*

$$\|f \mid B^{s,s'}_{p,q}(U)\| = \left[\sum_{j\in\mathbb{Z}} 2^{j\tilde{s}q} \left\{ \sum_{k\in\mathbb{Z}^n} \left| \left(1 + 2^j d(k2^{-j}, U)\right)^{s'} C_{j,k} \right|^p \right\}^{\frac{q}{p}} \right]^{\frac{1}{q}} < \infty,$$

where $d(k2^{-j}, U)$ *denotes the distance from* $k2^{-j}$ *to* U.

In order to state the local Besov type conditions in our theorem below, we shall use the following notation as an analogue of Hörmander's notation [H]: If $g(\rho)$ is a function of the real variable ρ, defined for all positive ρ, we write $g(\rho) = \mathcal{O}^{(p)}(\rho^{-s})$ if and only if

$$\int_0^R (g(\rho)\rho^s)^p \frac{d\rho}{\rho} = \int_0^R g(\rho)^p \rho^{sp-1} d\rho < \infty \quad \text{for every } R > 0.$$

Theorem 3.4. *Let $s > 0$, $s' < 0$ and $1 \le p \le \infty$. Let U be an open subset in $\mathbb{R}^n$ and $A_\rho = \{x \in \mathbb{R}^n;\, d(x,U) < \rho,\, x \notin U\}$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^{s,s'}_{p,p}(U)$ if and only if there exists a decomposition $f = f_1 + f_2$ such that*

$$f_1 \in \dot{B}^s_{p,p}(\mathbb{R}^n),$$

and

$$\|f_2 \,|\, B^{s+s'}_{p,p}(A_\rho)\| = \mathcal{O}^{(p)}(\rho^{-s'}).$$

Proof. We assume that the wavelet ψ is compactly supported and that $0 \in \operatorname{supp}\psi$. We denote by C' the diameter of the support of the wavelet ψ. Let $f \in B^{s,s'}_{p,p}(U)$. Then its wavelet coefficients satisfy

$$\sum_{j\in\mathbb{Z}} 2^{j\tilde{s}p} \sum_{k\in\mathbb{Z}^n} \left| \left(1 + 2^j d(k2^{-j}, U)\right)^{s'} C_{j,k} \right|^p < \infty. \tag{4}$$

We write f as

$$f = \sum_{\operatorname{supp}\psi_{j,k}\cap U \neq \emptyset} C_{j,k}\psi_{j,k} + \sum_{\operatorname{supp}\psi_{j,k}\cap U = \emptyset} C_{j,k}\psi_{j,k} =: f_1 + f_2.$$

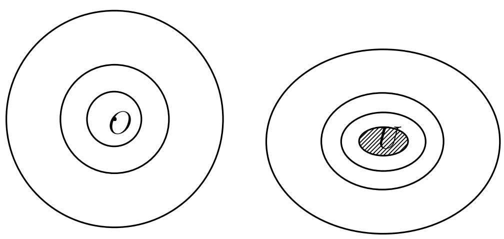
If $\operatorname{supp}\psi_{j,k} \cap U \neq \emptyset$, then $2^j d(2^{-j}k, U)$ is estimated from above by some constant comparable to C'. Therefore $f_1 \in \dot{B}^s_{p,p}(\mathbb{R}^n)$.

Next we split the wavelet decomposition of f_2 into three sums $f_2 = \sum_1 + \sum_2 + \sum_3$: Let $R > 0$ be fixed. The first, $\sum_1$, corresponds to the wavelets whose supports do not intersect A_R, and we can forget this sum.

Next we consider the sum $\sum_2$ whose coefficients satisfy $2^j R \le 10C'$; in that case, because $2^j d(2^{-j}k, U)$ can be estimated from above by some constant comparable to $10C'$, we have that $\sum_2 \in \dot{B}^s_{p,p}(\mathbb{R}^n)$.

Finally we consider the remaining sum $\sum_3$ whose coefficients satisfy $2^j R \ge 10C'$. We decompse A_R into the "curved annuli" as follows:

$$A_R = \bigcup_{m\in\mathbb{Z};\, 2^{-m}\le R} \{x \in \mathbb{R}^n;\, 2^{-m-1} \le d(x,U) \le 2^{-m}\} = \bigcup_{m;\, 2^m R \ge 1} D_m. \tag{5}$$



By using this decomposition (5), we can write (4) as follows:

$$\sum_{j;\, 2^j R \ge 10C'} 2^{j\tilde{s}p} \sum_{m;\, 2^m R \ge 1} (1 + 2^{j-m})^{s'p} \sum_{k;\, k2^{-j}\in D_m} |C_{j,k}|^p < \infty. \tag{6}$$

The case where $m > j + L(C')$, $L(C')$ being an integer dependent only on C', is negligible because $\operatorname{supp}\psi_{j,k} \cap U = \emptyset$. Therefore we obtain from (6) that

$$\begin{aligned}&\sum_{j;\,2^jR\geq 10C'} \sum_{\substack{m;\,2^mR\geq 1\\ m\leq j+L(C')}} 2^{j\tilde{s}p} 2^{(j-m)s'p} \sum_{k;\,k2^{-j}\in D_m} |C_{j,k}|^p = \\ &= \sum_{m;\,2^mR\geq 1} 2^{-ms'p} \sum_{\substack{j;\,2^jR\geq 10C'\\ j\geq m-L(C')}} 2^{jp(\tilde{s}+s')} \sum_{k;\,k2^{-j}\in D_m} |C_{j,k}|^p < \infty.\end{aligned} \tag{7}$$

On the other hand, the $\mathcal{O}^{(p)}$-condition that for every $R > 0$,

$$\int_0^R \left(\rho^{s'} \| f_2 \,|\, B^{s+s'}_{p,p}(A_\rho)\|\right)^p \frac{d\rho}{\rho} < \infty$$

follows from the condition that

$$\sum_{u\in\mathbb{Z};\,2^{-u}\leq R} 2^{-us'p} \sum_{j;\,2^jR\geq 10C'} 2^{jp(\tilde{s}+s')} \sum_{v\in\mathbb{Z};\,v\geq u} \sum_{k;\,k2^{-j}\in D_v} |C_{j,k}|^p < \infty. \tag{8}$$

Because $\operatorname{supp}\psi_{j,k} \cap U = \emptyset$, and the geometric series $\sum_{u;u\leq v} 2^{-us'p}$ is estimated from above by some constant comparble to $2^{-vs'p}$ (note that $s' < 0$), this last condition (8) follows from that

$$\sum_{v;\,2^vR\geq 1} 2^{-vs'p} \sum_{\substack{j;\,2^jR\geq 10C'\\ j\geq v-L(C')}} 2^{jp(\tilde{s}+s')} \sum_{k;\,k2^{-j}\in D_v} |C_{j,k}|^p < \infty. \tag{9}$$

It follows from (7) and (9) that the remaining sum $\sum_3$ satisfies the local Besov $\mathcal{O}^{(p)}$-condition, as desired.

Conversely let us assume that $f = f_1 + f_2$ satisfies the following conditions:

$$f_1 \in \dot{B}^s_{p,p}(\mathbb{R}^n), \tag{10}$$

and

$$\| f_2 \,|\, B^{s+s'}_{p,p}(A_\rho)\| = \mathcal{O}^{(p)}(\rho^{-s'}). \tag{11}$$

We note that if the support of the wavelet $\psi_{j,k}$ is completely included in A_ρ, then any function extending f_2 outside A_ρ has the same wavelet coefficient $C_{j,k}$. From this remark and (11), we have that for any $R > 0$,

$$\sum_{u;\,2^uR\geq 1} 2^{-us'p} \sum_{j\in\mathbb{Z}} 2^{jp(\tilde{s}+s')} \sum_{k;\,k2^{-j}\in A_{2^{-u}}} |C_{j,k}|^p < \infty. \tag{12}$$

The condition (12) is equivalent to that

$$\sum_{j\in\mathbb{Z}} 2^{jp(\tilde{s}+s')} \sum_{k\in\mathbb{Z}^n} |C_{j,k}|^p \sum_{\substack{u;\,2^uR\geq 1\\ 2^u d(k2^{-j},U)\leq 1}} 2^{-us'p} < \infty.$$

After the calculation of the geometric sum, we arrive at the following:

$$\sum_{j\in\mathbb{Z}} 2^{jp(\tilde{s}+s')} \sum_{k\in\mathbb{Z}^n} |C_{j,k}|^p \left(d(k2^{-j},U)^{s'p} - R^{s'p}\right) < \infty. \tag{13}$$

Note that $s' < 0$. Then as $R \to \infty$ in (13), we obtain that

$$\sum_{j\in\mathbb{Z}} 2^{j\tilde{s}p} \sum_{k\in\mathbb{Z}^n} \left|\left(1+2^j d(k2^{-j},U)\right)^{s'} C_{j,k}\right|^p < \infty,$$

that is $f_2 \in B^{s,s'}_{p,p}(U)$. Taking into account the assumption (10) that $f_1 \in \dot{B}^s_{p,p}(\mathbb{R}^n)$, we conclude that $f = f_1 + f_2 \in B^{s,s'}_{p,p}(U)$. □

4 Two-microlocal Besov spaces with dominating mixed smoothness

Moritoh (2016) considers "two-microlocal Besov spaces with dominating mixed smoothness" as a natural extension of Jaffard-Meyer (1996) and Moritoh-Yamada (2004) by taking account of uncertainty functions given by Weyl-Hörmander calculus (Bony-Lerner, 1989). See [Mo] and [BL].

We treat only the case where $n = 2$. Let us now consider an orthonormal wavelet basis on $\mathbb{R}^2$ composed by translations and dilations of $\psi(x_1)\psi(x_2)$, where $\psi(x)$ is a one-dimensional compactly supported smooth wavelet. Let $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$ for $j \in \mathbb{Z}, k \in \mathbb{Z}$. Then every $f \in \mathcal{S}'(\mathbb{R}^2)$ will be written

$$f(x) = \sum_{\boldsymbol{j}\in\mathbb{Z}^2} \sum_{\boldsymbol{k}\in\mathbb{Z}^2} C_{\boldsymbol{j},\boldsymbol{k}}\, \psi_{j_1,k_1}(x_1)\psi_{j_2,k_2}(x_2),$$

where $\boldsymbol{j} = (j_1, j_2)$ and $\boldsymbol{k} = (k_1, k_2)$.

Let $s_1, s_2 > 0$ and $1 \le p_1, p_2, q_1, q_2 \le \infty$. Then the homogeneous Besov space with dominating mixed smoothness $S\dot{B}^{\boldsymbol{s}}_{\boldsymbol{p},\boldsymbol{q}}(\mathbb{R}^2)$ is defined as the set of all tempered distributions f (modulo polynomials) satisfying

$$\|f|S\dot{B}^{\boldsymbol{s}}_{\boldsymbol{p},\boldsymbol{q}}(\mathbb{R}^2)\|$$
$$= \left[\sum_{j_2\in\mathbb{Z}}\left(\int_{\mathbb{R}}\left(\sum_{j_1\in\mathbb{Z}}\left(\int_{\mathbb{R}}\left|2^{j_1 s_1 + j_2 s_2}\left(\varphi_{j_1}\varphi_{j_2}\hat{f}\right)^{\vee}(x_1,x_2)\right|^{p_1} dx_1\right)^{\frac{q_1}{p_1}}\right)^{\frac{p_2}{q_1}} dx_2\right)^{\frac{q_2}{p_2}}\right]^{\frac{1}{q_2}} < \infty,$$

where $\boldsymbol{s} = (s_1, s_2), \boldsymbol{p} = (p_1, p_2), \boldsymbol{q} = (q_1, q_2)$, and

$$(\varphi_{j_1}\varphi_{j_2}\hat{f})^{\vee}(x_1,x_2) = (\varphi_{j_1}(\xi_1)\varphi_{j_2}(\xi_2)\hat{f}(\xi_1,\xi_2))^{\vee}(x_1,x_2).$$

See Schmeisser-Triebel [ST].

Let us recall the fact that $f \in S\dot{B}^{\boldsymbol{s}}_{\boldsymbol{p},\boldsymbol{q}}(\mathbb{R}^2)$ if and only if

$$\left(\sum_{j_2 \in \mathbb{Z}} \left(\sum_{k_2 \in \mathbb{Z}} \left(\sum_{j_1 \in \mathbb{Z}} \left(\sum_{k_1 \in \mathbb{Z}} |2^{j_1 \tilde{s}_1 + j_2 \tilde{s}_2} C_{\boldsymbol{j},\boldsymbol{k}}|^{p_1} \right)^{\frac{q_1}{p_1}} \right)^{\frac{p_2}{q_1}} \right)^{\frac{q_2}{p_2}} \right)^{\frac{1}{q_2}} < \infty,$$

where $\tilde{s}_i = s_i + 1/2 - 1/p_i$ $(i = 1, 2)$. See [B] and [V]. We treat only the case where $\boldsymbol{p} = \boldsymbol{q} = (p, p)$, $1 \le p \le \infty$. We can define the local Besov space $SB^{\boldsymbol{s}}_{p,p}(\mathbb{R}_{x_1} \times A_\rho)$ as usual, where $\mathbb{R}_{x_1} \times A_\rho$ denotes the horizontal strip $\{(x_1, x_2)\,;\, x_1 \in \mathbb{R}, |x_2| < \rho\}$ for $\rho > 0$. We can also give the definition of the two-microlocal Besov space with dominating mixed smoothness $SB^{(s_1,s_2),s_3}_{p,p}(\mathbb{R}_{x_1} \times \{0\})$ as follows:

Definition 4.1. *Let $s_1, s_2 > 0$, $s_3 \in \mathbb{R}$, and $1 \le p \le \infty$. Then $f \in \mathcal{S}'(\mathbb{R}^2)$ is said to belong to the two-microlocal Besov space with dominating mixed smoothness $SB^{(s_1,s_2),s_3}_{p,p}(\mathbb{R}_{x_1} \times \{0\})$ if the following two-microlocal estimate holds:*

$$\begin{aligned} &\left\| f \,\middle|\, SB^{(s_1,s_2),s_3}_{p,p}(\mathbb{R}_{x_1} \times \{0\}) \right\| \\ &:= \left[\sum_{\boldsymbol{j} \in \mathbb{Z}^2} \sum_{\boldsymbol{k} \in \mathbb{Z}^2} 2^{(j_1 \tilde{s}_1 + j_2 \tilde{s}_2)p} (1 + 2^{j_1} + (|k_2| + 1) 2^{-j_2} 2^{j_1 \vee j_2})^{s_3 p} |C_{\boldsymbol{j},\boldsymbol{k}}|^p \right]^{\frac{1}{p}} < \infty, \end{aligned}$$

where $j_1 \vee j_2 = \max\{j_1, j_2\}$.

Our main theorem of this section is the following:

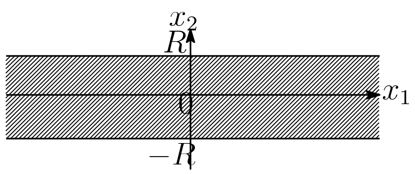
Theorem 4.2. *Let $s_i > 0$, $s_3 < 0$, $s_i + s_3 > 0$ $(i = 1, 2)$, and $1 \le p \le \infty$. Then $f \in \mathcal{S}'(\mathbb{R}^2)$ belongs to $SB^{(s_1,s_2),s_3}_{p,p}(\mathbb{R}_{x_1} \times \{0\})$ if and only if there exists a decomposition $f = f_1 + f_2 + f_3 + f_4$ such that*

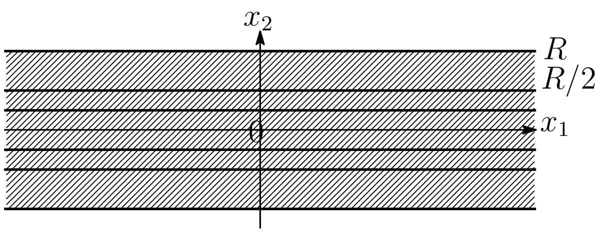
$$f_1 \in S\dot{B}^{(s_1,s_2)}_{p,p}(\mathbb{R}^2), \quad f_2 \in S\dot{B}^{(s_1+s_3,s_2)}_{p,p}(\mathbb{R}^2),$$

$$f_3 \in S\dot{B}^{(s_1+s_3,s_2-s_3)}_{p,p}(\mathbb{R}^2),$$

and

$$\left\| f_4 \,\middle|\, SB^{(s_1,s_2+s_3)}_{p,p}(\mathbb{R}_{x_1} \times A_\rho) \right\| = \mathcal{O}^{(p)}(\rho^{-s_3}).$$





Skech of the proof: We employ the method used in the proof of Theorem 4.2. Let $f \in SB^{(s_1,s_2),s_3}_{p,p}(\mathbb{R}_{x_1} \times \{0\})$. Then its wavelet coefficients satisfy

$$\sum_{\boldsymbol{j}\in\mathbb{Z}^2} \sum_{\boldsymbol{k}\in\mathbb{Z}^2} 2^{(j_1\tilde{s}_1+j_2\tilde{s}_2)p}(1+2^{j_1}+(|k_2|+1)2^{-j_2}2^{j_1\vee j_2})^{s_3 p}|C_{\boldsymbol{j},\boldsymbol{k}}|^p < \infty. \tag{14}$$

We decompose f as follows:

$$f = f_1 + f_2,$$

where f_1 and f_2 correspond to the cases where $0 \in \operatorname{supp} \psi_{j_2,k_2}$ and $0 \notin \operatorname{supp} \psi_{j_2,k_2}$, respectively.

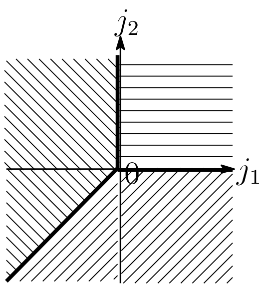
First: We decompose f_1 into three parts according to $\{j_1 > 0, j_2 > 0\}$, $\{j_2 < 0, j_1 > j_2\}$, and $\{j_1 < 0, j_2 > j_1\}$.

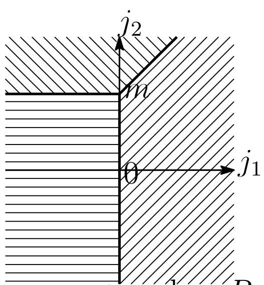
Second: We decompose f_2 into three parts, among which the case where $2^{j_2}R \geq 10C'$ is the most important.

Third: We decompose this important term into three parts according to $\{j_1 < 0, j_2 < m\}$, $\{j_1 > 0, j_2 < j_1 + m\}$, and $\{j_2 > m, j_2 > j_1 + m\}$. The last term yields the function f_4 characterized by the local Besov type condition with dominating mixed smoothness.

Summing up, the case where $2^{j_2}R \geq 10C'$ (R is a fixed positve number), $j_2 > m$, $j_2 > j_1 + m$ ($m > -\log_2 R$) yields the function f_4.

We finally remark that the case where $j_1 > j_2$ and $j_2 < 0$ in the wavelet decomposition of f_1 yields the function $f_3 \in S\dot{B}^{(s_1+s_3,s_2-s_3)}_{p,p}(\mathbb{R}^2)$.



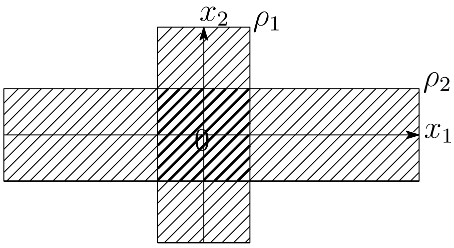


$m > -\log_2 R$

Remark 4.3. The idea of this theorem is that every f belonging to the generalized function space $SB^{(s_1,s_2),s_3}_{p,p}(\mathbb{R}_{x_1} \times \{0\})$ has a good decomposition $f = \sum_{i=1}^{4} f_i$, where the term f_4 represents the singularities of the function f along the line $\mathbb{R}_{x_1}$; they satisfy the local Besov type conditions in the neighborhood of the x_1-axis. (As we have seen in section 2, every $f \in B^{s,s'}_{p,p}(x_0)$ has a good decomposition $f = f_1 + f_2$, where the term f_2 represents the singularities of the function f at the point x_0.) Our future research is a more complete theory of two-microlocal spaces using Weyl-Hörmander calculus.

Remark 4.4. The typical examples considered by Jaffard-Meyer are an indefinitely oscillating function of the form $x^\alpha \sin(1/x^\beta)$, and Riemann's nondifferentiable function $\sigma(x) = \sum_{n=1}^{\infty}(1/n^2)\sin(\pi n^2 x)$, where the Hölder regularity at a point x_0 depends on the Diophintine approximation properties of x_0. Higher dimensional singularities will be studied in our future research.

Remark 4.5. The two-microlocal Besov spaces of product type are easily introduced and characterized. It is associated with the uncertainty functions $\lambda_i = 1 + |x_i||\xi_i|$ $(i = 1, 2)$; the norm of the wavelet coefficients $C_{\boldsymbol{j},\boldsymbol{k}}$ is defined by means of the weighted coefficients $2^{(j_1\tilde{s}_1+j_2\tilde{s}_2)}(1 + |k_1|)^{s'_1}(1 + |k_2|)^{s'_2}|C_{\boldsymbol{j},\boldsymbol{k}}|$.



$\mathbb{R}_{x_1} \times A_{\rho_2}$ and $A_{\rho_1} \times \mathbb{R}_{x_2}$

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