Aspects of Frame Analysis

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Abstract

In this note, we overview the basic theory of frame analysis in Hilbert spaces. We also introduce some important aspects of specific frames in $L^2(\mathbb{R})$: a Gabor frame and a wavelet frame.

1 Introduction

A collection of functions $\{f_n\}_{n\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ is said to be a *frame* if the inequalities

$$A \le \frac{\sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2}{\|f\|^2} \le B \tag{1}$$

hold for all $f \in \mathcal{H}$, where A and B are positive constants. If $A = B = 1$, then the condition (1) is written as

$$\sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2 = \|f\|^2. \tag{2}$$

If we find such functions, by the polarization identity,

$$\langle f, g\rangle = \frac{1}{4}\left\{\|f+g\|^2 - \|f-g\|^2 + i\left(\|f+ig\|^2 - \|f-ig\|^2\right)\right\}, \quad f, g \in \mathcal{H},$$

we can show that any $f \in \mathcal{H}$ can be represented as a linear combination of $\{f_n\}_{n\in\mathbb{Z}}$:

$$f = \sum_{n\in\mathbb{Z}} \langle f, f_n\rangle\, f_n. \tag{3}$$

This is called the *frame expansion* or *frame decomposition* of f.

At first glance, frames and bases have very similar concepts because the frame expansion (3) looks similar to an orthonormal expansion. In fact, a frame $\{f_n\}_{n\in\mathbb{Z}}$ satisfies the Parseval identity (2) that holds usually when $\{f_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis. However, unlike an orthonormal expansion, (3) is a nonunique decomposition of f, which is to say that there are several ways to represent f with respect to $\{f_n\}_{n\in\mathbb{Z}}$. Despite the convenient properties of frames, the condition (1) is weaker than that for orthonormal bases, which is what makes frames a universal tool in various scientific fields such as pure and applied mathematics and signal processing, both in theory and in applications.

A frame satisfying the inequality (1) was originally introduced by Duffin and Schaeffer in 1952 [22] in the context of nonharmonic Fourier analysis. According to [12], frames did not, however, receive much attention from mathematicians and the related scientific community at that time, until the 1980s. In 1980, Young re-introduced the concept of a frame in his book [35], though he still focused on it as a tool for nonharmonic Fourier series. The modern concept of a frame and its usefulness were established by Daubechies, Grossmann and Morlet in 1986 [16], when their breakthrough was to connect frames with Gabor systems and wavelets. Gabor frames and wavelet frames have specific structures in $L^2(\mathbb{R})$, which are generated by a Gabor system $\left\{e^{2\pi imb\cdot}g(\cdot - na)\right\}_{m,n\in\mathbb{Z}}, g \in L^2(\mathbb{R})$ and a wavelet system $\{a^{m/2}g(a^m \cdot -nb)\}_{m,n\in\mathbb{Z}}$, respectively. They have played very important roles in both theory and applications. For more details on how frame theory and signal processing relate, see [1, 7, 15, 25, 26].

2 Frame and Basis

We begin by highlighting some important relations between a frame and a basis. Other quick introductions to frames are given in [11, 24]. Throughout this paper, we shall use the Fourier transform normalized as $\hat{f}(\xi) := \int_{\mathbb{R}} f(x)\, e^{-2\pi i\xi x}dx$, and $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ will denote respectively the inner product and norm on either $\mathcal{H}$ or $L^2(\mathbb{R})$. Let us now give a formal definition of a frame.

Definition 2.1. *Let I be a countable index set. A sequence of elements $\{f_n\}_{n\in I}$ in a separable Hilbert space $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist two constants $A, B > 0$ such that*

$$A\|f\|^2 \leq \sum_{n\in I} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{4}$$

The constants A and B are called an upper frame bound and a lower frame bound, respectively. When $A = B$, $\{f_n\}_{n\in I}$ is called a *tight frame*; when $A = B = 1$, $\{f_n\}_{n\in I}$ is called a *Parseval frame*. The inequality (4) is called the frame condition or the frame inequality.

From the frame inequality (4), we observe some important relations between frames and bases as follows.

(i) A sequence $\{f_n\}_{n\in I}$ in $\mathcal{H}$ is a *Bessel sequence* if there exists a constant $B > 0$ such that

$$\sum_{n\in I} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

(ii) A Bessel sequence $\{f_n\}_{n\in I}$ is a frame for $\mathcal{H}$ if there exists a constant $A > 0$ such that

$$A\|f\|^2 \leq \sum_{n\in I} |\langle f, f_n\rangle|^2, \quad \forall f \in \mathcal{H}.$$

(iii) A sequence $\{f_n\}_{n\in I}$ is a basis or *Schauder basis* for $\mathcal{H}$ if there exist unique coefficients $\{c_n(f)\}_{n\in I}$ such that

$$f = \sum_{n\in I} c_n(f)\, f_n, \quad \forall f \in \mathcal{H}.$$

(iv) A basis $\{f_n\}_{n\in I}$ is an *orthonormal basis* for $\mathcal{H}$ if it is orthonormal, i.e.,

$$\langle f_m, f_n\rangle = \delta_{m,n},$$

for every $m, n \in I$. In this case, the coefficients become $c_n(f) = \langle f, f_n\rangle$. As a consequence, we have

$$f = \sum_{n\in I} \langle f, f_n\rangle f_n,$$

for all $f \in \mathcal{H}$, which is an orthonormal expansion in terms of the orthonormal basis $\{f_n\}_{n\in I}$.

(v) A sequence $\{f_n\}_{n\in I}$ in $\mathcal{H}$ is a *Riesz basis* if it is a complete sequence for $\mathcal{H}$, i.e., $\overline{\text{span}}\{f_n\}_{n\in I} = \mathcal{H}$, and there exist two constants $A, B > 0$ such that, for all square summable sequences $\{c_n\}_{n\in I}$ in $\mathcal{H}$,

$$A \sum_{n\in I} |c_n|^2 \le \left\| \sum_{n\in I} c_n f_n \right\|^2 \le B \sum_{n\in I} |c_n|^2,$$

where constants A and B are called Riesz bounds. If $\{f_n\}_{n\in I}$ is a Riesz basis, it is called an *exact frame* and there exists a unique sequence $\{g_n\}_{n\in I}$ in $\mathcal{H}$ such that

$$f = \sum_{n\in I} \langle f, g_n \rangle f_n,$$

for all $f \in \mathcal{H}$. The sequence $\{g_n\}_{n\in I}$ is also a Riesz basis for $\mathcal{H}$ and is called a dual Riesz basis. When $\{g_n\}_{n\in I} = \{f_n\}_{n\in I}$, it is an orthonormal basis. A frame that is not a Riesz basis is called an *overcomplete frame* or *redundant frame*.

From these observations, all orthonormal bases, Riesz bases, and frames are Bessel sequences. We also see that every orthonormal basis is a Riesz basis and a frame. However, every frame is not a Riesz basis (or orthonormal basis). A frame $\{f_n\}_{n\in I}$ is a Riesz basis if and only if, for all square summable sequences $\{c_n\}_{n\in I}$,

$$\sum_{n\in I} c_n f_n = 0 \quad \Rightarrow \quad c_n = 0, \quad \forall n \in I.$$

This means that the frame elements $\{f_n\}_{n\in I}$ might be linearly dependent. This is equivalent to the statement that frame coefficients $\{c_n\}_{n\in I} = \{\langle f, f_n \rangle\}_{n\in I}$ are not necessarily uniquely determined and, as we mentioned before, the representation

$$f = \sum_{n\in I} c_n f_n$$

is not necessarily unique for every $f \in \mathcal{H}$. They are unique if and only if $\{f_n\}_{n\in I}$ is a Riesz basis.

Below, we show simple examples of frames and bases, including a finite-dimensional case.

Example 2.2. *Let $b > 0$. Consider a set of functions $\{f_n(x) = e^{2\pi ibnx}\}_{n\in\mathbb{Z}}$.*

(i) If $b = 1$, $\{f_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2([0,1])$.

(ii) If $b < 1$, $\{f_n\}_{n\in\mathbb{Z}}$ is not a Riesz basis for $L^2([0,1])$.

(iii) If $b \leq 1$, $\{f_n\}_{n\in\mathbb{Z}}$ is a tight frame for $L^2([0,1])$.

Example 2.3 (Harmonic frame in finite dimensions)**.** *Let $m \geq n$. Consider a set of vectors $\{u_k\}_{k=0}^{m-1} \subset \mathbb{C}^n$ defined by*

$$u_k = \frac{1}{\sqrt{m}} \left(1, \ e^{2\pi ik/m}, \ \cdot \ \cdot \ \cdot, \ e^{2\pi ik(n-1)/m} \right)^T.$$

(i) If $m > n$, $\{u_k\}_{k=0}^{m-1}$ is a tight frame for $\mathbb{C}^n$.

(ii) If $m = n$, $\{u_k\}_{k=0}^{m-1}$ is an orthonormal basis for $\mathbb{C}^n$.

A frame $\{f_n\}_{n\in I}$ comprising elements equal in norm, $\|f_n\| = c$ for all $n \in I$, is called an *equal norm frame*. If $c = 1$, it is called a *unit norm frame*. Thus, the above examples are equal norm frames, and every orthonormal basis is a unit norm frame.

3 Frame Operator

In order to develop more insight on frame theory, we revisit the frame inequality (4) from the point of view of operator theory in $\mathcal{H}$. Let

$$\ell^2(I) := \left\{ x = \{x_n\}_{n\in I} \ : \ x_n \in \mathbb{C}, \sum_{n\in I} |x_n|^2 < \infty \right\}$$

be a space for square summable sequences in $\mathcal{H}$. For $x = \{x_n\}_{n\in I}, y = \{y_n\}_{n\in I} \in \ell^2(I)$, we define an inner product and its norm by

$$\langle x, y\rangle_{\ell^2(I)} := \sum_{n\in I} x_n \, \overline{y_n}, \qquad \|x\|_{\ell^2(I)} := \sqrt{\langle x, x\rangle_{\ell^2(I)}}.$$

Let $\{f_n\}_{n\in I}$ be a frame for $\mathcal{H}$. Consider the *analysis* or *decomposition operator*

$$C : \mathcal{H} \to \ell^2(I), \qquad Cf := \{\langle f, f_n\rangle\}_{n\in I},$$

which is also called the *pre-frame operator*. We can say that a sequence $\{f_n\}_{n\in I}$ is a Bessel sequence if the analysis operator is bounded, i.e., there exists a constant $B > 0$ such that

$$\|Cf\|_{\ell^2(I)} \le B\|f\|,$$

for all $f \in \mathcal{H}$. This leads to the second inequality of the frame condition (4):

$$\sum_{n\in I} |\langle f, f_n\rangle|^2 \le B\|f\|^2.$$

Consequently, a Bessel sequence $\{f_n\}_{n\in I}$ is a frame when the analysis operator is bounded below as in the first inequality of (4):

$$A\|f\|^2 \le \sum_{n\in I} |\langle f, f_n\rangle|^2.$$

Then we define the *synthesis operator* or *reconstruction operator* as

$$C^* : \ell^2(I) \to \mathcal{H}, \qquad C^*\{c_n\}_{n\in I} := \sum_{n\in I} c_n f_n.$$

This is an adjoint operator for $C : \mathcal{H} \to \ell^2(I)$, because for $x = \{c_n\}_{n\in I} \in \ell^2(I)$ and $y \in \mathcal{H}$, we have

$$\begin{aligned}
\langle C^*x, y\rangle &= \langle x, Cy\rangle_{\ell^2(I)} \\
&= \big\langle \{c_n\}_{n\in I}, \{\langle y, f_n\rangle\}_{n\in I}\big\rangle_{\ell^2(I)} \\
&= \sum_{n\in I} c_n \overline{\langle y, f_n\rangle} = \left\langle \sum_{n\in I} c_n f_n, y \right\rangle.
\end{aligned}$$

The composition of these two operators $S = C^*C : \mathcal{H} \to \mathcal{H}$ is called the *frame operator* of a given frame $\{f_n\}_{n\in I}$ for $\mathcal{H}$ and is defined by

$$Sf := C^*Cf = \sum_{n\in I} \langle f, f_n\rangle\, f_n.$$

The frame operator $S : \mathcal{H} \to \mathcal{H}$ has the following properties:

(i) S is bounded.

(ii) S is positive, as for a given $x \in \mathcal{H}$,

$$\langle Sx, x\rangle = \langle C^*Cx, x\rangle = \langle Cx, Cx\rangle_{\ell^2(I)} \geq 0.$$

(iii) S is self-adjoint, i.e., $S = S^*$. From the definition of a frame operator, we have

$$S^* = (C^*C)^* = C^*(C^*)^* = C^*C = S.$$

In general, if S is positive, then we have $\langle Sx, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$, which yields

$$\langle Sx, x\rangle = \overline{\langle Sx, x\rangle} = \overline{\langle x, S^*x\rangle} = \langle S^*x, x\rangle.$$

(iv) $S : \mathcal{H} \to \mathcal{H}$ is injective ($\ker(S) = \{x \in \mathcal{H} : Sx = 0\} = \{0\}$) and also surjective ($R(S) = \{Sx : x \in \mathcal{H}\} = \mathcal{H}$). Thus, $S : \mathcal{H} \to \mathcal{H}$ is bijective and there exists an inverse operator S^{-1}.

Using the frame operator S, we have

$$\langle Sf, f\rangle = \sum_{n\in I} |\langle f, f_n\rangle|^2.$$

Thus, we can rewrite the frame condition (4) as

$$A\|f\|^2 \leq \langle Sf, f\rangle \leq B\|f\|^2,$$

or

$$AI \leq S \leq BI,$$

where I denotes the identity operator on $\mathcal{H}$. Since S is self-adjoint and invertible, we obtain

$$f = S(S^{-1}f) = \sum_{n\in I} \left\langle S^{-1}f, f_n\right\rangle f_n = \sum_{n\in I} \left\langle f, S^{-1}f_n\right\rangle f_n.$$

Similarly, using $f = S^{-1}Sf$, we obtain

$$f = S^{-1}(Sf) = S^{-1}\left(\sum_{n\in I} \langle f, f_n\rangle f_n\right) = \sum_{n\in I} \langle f, f_n\rangle S^{-1}f_n.$$

Thus, we have the frame expansion

$$f = \sum_{n \in I} \left\langle f, S^{-1} f_n \right\rangle f_n, \tag{5}$$

or

$$f = \sum_{n \in I} \langle f, f_n \rangle \, S^{-1} f_n, \tag{6}$$

for all $f \in \mathcal{H}$. A frame $\{S^{-1} f_n\}_{n \in I}$ that satisfies (5) and (6) is called a *canonical dual frame* for $\mathcal{H}$ with frame bounds $1/B$ and $1/A$.

If $\{f_n\}_{n \in I}$ is a frame but not a Riesz basis, i.e., an overcomplete frame, then there exist other frames $\{\tilde{f}_n\}_{n \in I} \neq \left\{S^{-1} f_n\right\}_{n \in I}$ such that

$$f = \sum_{n \in I} \langle f, \tilde{f}_n \rangle f_n = \sum_{n \in I} \langle f, f_n \rangle \tilde{f}_n, \tag{7}$$

for all $f \in \mathcal{H}$. In this case, $\{\tilde{f}_n\}_{n \in I}$ is called a *dual frame* of $\{f_n\}_{n \in I}$. In general, it is known that there exist many dual frames for a given frame $\{f_n\}_{n \in I}$ in $\mathcal{H}$ besides its canonical dual frame. For a canonical dual frame, it is not always easy to find S^{-1}, but S^{-1} has a unique property that its frame coefficients $\{\langle f, S^{-1} f_n \rangle\}_{n \in I}$ in the representation of (5) have the least ℓ^2 norm.

For the expansion formulas as in (5), (6) and (7), the frame coefficients are calculated by the inner product on $\mathcal{H}$. This property is useful, but inverting the frame operator is not easy in the infinite-dimensional case; this difficulty of inverting the frame operator is the fundamental problem with frames. One way to avoid this problem is to find a dual frame $\{\tilde{f}_n\}_{n \in I} \neq \left\{S^{-1} f_n\right\}_{n \in I}$ that satisfies (7). Another way is to consider a tight frame. If a frame is tight, then $\tilde{f}_n = A^{-1} f_n$, which gives the frame expansion

$$f = \frac{1}{A} \sum_{n \in I} \langle f, f_n \rangle f_n, \quad \forall f \in \mathcal{H}.$$

4 Gabor System

In the rest of this paper, we introduce two applications of frames in $L^2(\mathbb{R})$: a Gabor frame and a wavelet frame. Let us first define three types of unitary operators on $L^2(\mathbb{R})$ that are closely related to these frames.

Definition 4.1. *The following are three types of bounded linear operators on $L^2(\mathbb{R})$:*

(i) For $b \in \mathbb{R}$, a translation operator is defined by

$$(T_b f)(x) := f(x - b).$$

(ii) For $\xi \in \mathbb{R}$, a modulation operator is defined by

$$(M_\xi f)(x) := e^{2\pi i \xi x} f(x).$$

(iii) For $a \in \mathbb{R}_+$, a dilation operator is defined by

$$(D_a f)(x) := a^{1/2} f(ax).$$

We now define a set of functions, called a *Gabor system*, that is generated by translations and modulations of a fixed function $g \in L^2(\mathbb{R})$:

$$\mathcal{G}(g, a, b) := \left\{ M_{mb} T_{na} g(x) = e^{2\pi i m b x} g(x - na) \right\}_{m,n \in \mathbb{Z}},$$

where $a, b > 0$. The set $\{(na, mb)\}_{m,n\in\mathbb{Z}} \subset \mathbb{R}^2$ is called the *time-frequency lattice* and $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ are called the *Gabor atoms*.

Definition 4.2. *Let $a, b > 0$ and $g \in L^2(\mathbb{R})$ be given. A Gabor system $\mathcal{G}(g, a, b)$ is called a Gabor frame for $L^2(\mathbb{R})$ if a frame condition holds. Namely, there exist constants $A, B > 0$ such that*

$$A\|f\|^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, M_{mb}T_{na}g\rangle|^2 \leq B\|f\|^2, \quad \forall f \in L^2(\mathbb{R}).$$

A Gabor frame is also called a *Weyl–Heisenberg frame.* As in the general frame theory, the analysis operator $C_g : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z}^2)$ associated with the Gabor system $\mathcal{G}(g, a, b)$ is defined by

$$C_g f := \{\langle f, M_{mb}T_{na}g\rangle\}_{m,n\in\mathbb{Z}}.$$

Similarly, we define the synthesis operator $C_g^* : \ell^2(\mathbb{Z}^2) \to L^2(\mathbb{R})$ by

$$C_g^*\{c_{m,n}\}_{m,n\in\mathbb{Z}} := \sum_{m,n\in\mathbb{Z}} c_{m,n} M_{mb} T_{na} g.$$

The frame operator $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is then defined by

$$Sf := C_g^* C_g f = \sum_{n,m\in\mathbb{Z}} \langle f, M_{mb}T_{na}g\rangle M_{mb}T_{na}g.$$

A system $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbb{R})$ if and only if S is a bounded, self-adjoint, bijective linear map of $L^2(\mathbb{R})$ onto itself.

The frame expansion associated with a Gabor frame can be written as

$$\begin{aligned} f = S\left(S^{-1}f\right) &= \sum_{m,n\in\mathbb{Z}} \left\langle S^{-1}f, M_{mb}T_{na}g\right\rangle M_{mb}T_{na}g \\ &= \sum_{m,n\in\mathbb{Z}} \left\langle f, S^{-1}M_{mb}T_{na}g\right\rangle M_{mb}T_{na}g, \end{aligned} \tag{8}$$

or

$$\begin{aligned} f = S^{-1}(Sf) = S^{-1}\left(\sum_{m,n\in\mathbb{Z}} \langle f, M_{mb}T_{na}g\rangle M_{mb}T_{na}g\right) \\ = \sum_{m,n\in\mathbb{Z}} \langle f, M_{mb}T_{na}g\rangle S^{-1}M_{mb}T_{na}g. \end{aligned} \tag{9}$$

Note that the operators S and $M_{mb}T_{na}$ commute for all $m, n \in \mathbb{Z}$, i.e.,

$$S\left(M_{mb}T_{na}f\right) = M_{mb}T_{na}(Sf). \tag{10}$$

We see that the operators S^{-1} and $M_{mb}T_{na}$ commute as well in a similar manner. Thus, we have the following theorem.

Theorem 4.3 (Reconstruction formula for Gabor frame)**.** *Let $a, b > 0$. Suppose that $\mathcal{G}(g, a, b)$ is a Gabor frame for $L^2(\mathbb{R})$ with frame bounds A, B, and $\mathcal{G}(S^{-1}g, a, b)$ is the canonical dual frame for $L^2(\mathbb{R})$ with frame bounds $1/B, 1/A$. Then,*

$$f = \sum_{m,n\in\mathbb{Z}} \left\langle f, M_{mb}T_{na}S^{-1}g\right\rangle M_{mb}T_{na}g \tag{11}$$

$$= \sum_{m,n\in\mathbb{Z}} \langle f, M_{mb}T_{na}g\rangle M_{mb}T_{na}S^{-1}g, \tag{12}$$

for all $f \in L^2(\mathbb{R})$*.*

We next make a few remarks on Theorem 4.3. We have two types of the canonical dual frames $\{M_{mb}T_{na}S^{-1}g\}_{m,n\in\mathbb{Z}}$ and $\{S^{-1}M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. While the canonical dual frame $\{M_{mb}T_{na}S^{-1}g\}_{m,n\in\mathbb{Z}}$ is obviously a Gabor frame, the canonical dual frame $\{S^{-1}M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ has also the Gabor structure due to the commutator relation (10), which is a nice property of a Gabor frame. The frame expansions with Gabor frames (11) and (12) are efficient compared to (8) and (9) because we only need to consider the inverse operator S^{-1} of g instead of a whole set $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. On the other hand, as in the general theory of frame analysis, it is convenient to use a dual Gabor frame $\mathcal{G}(\tilde{g}, a, b)$ with $\tilde{g} \in L^2(\mathbb{R})$ such that

$$\begin{aligned} f &= \sum_{m,n\in\mathbb{Z}} \langle f, M_{mb}T_{na}\tilde{g}\rangle\, M_{mb}T_{na}g \\ &= \sum_{m,n\in\mathbb{Z}} \langle f, M_{mb}T_{na}g\rangle\, M_{mb}T_{na}\tilde{g}, \end{aligned} \tag{13}$$

for all $f \in L^2(\mathbb{R})$.

Duality conditions for a pair of Gabor systems that guarantee that the dual frame has the Gabor structure were obtained by Ron and Shen [30,31].

Theorem 4.4 (Ron and Shen)**.** *Given* $a, b > 0$ *and* $g, \tilde{g} \in L^2(\mathbb{R})$*, two Bessel sequences* $\mathcal{G}(g, a, b)$ *and* $\mathcal{G}(\tilde{g}, a, b)$ *are a pair of dual Gabor frames if and only if, for all* $n \in \mathbb{Z}$*,*

$$\sum_{k\in\mathbb{Z}} \overline{g(x - ka - n/b)}\, \tilde{g}(x - ka) = b\delta_{n,0}, \quad a.e.\ x \in [0, a].$$

As is clear from the history of the development of Gabor systems, a Gabor frame is closely related to time-frequency analysis and signal processing. The Gabor atoms $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ are used for the *short-time Fourier transform* (STFT) defined by

$$V_g f(x, \xi) := \langle f, M_\xi T_x g\rangle = \int_{\mathbb{R}} f(t)\overline{g(t - x)}e^{-2\pi i t\xi}dt.$$

This gives a time-frequency representation of f into $V_g f(x,\xi)$ on $\mathbb{R}^2$. For $g, \tilde{g} \in L^2(\mathbb{R})$ such that $\langle g, \tilde{g}\rangle \neq 0$, we have the following inversion formula:

$$f(t) = \frac{1}{\langle g, \tilde{g}\rangle} \iint_{\mathbb{R}^2} V_g f(x,\xi) M_\xi T_x \tilde{g}(t) dx d\xi, \tag{14}$$

which is called the inverse STFT. By sampling the STFT on the lattice $a\mathbb{Z} \times b\mathbb{Z}$ and using its coefficients $\{V_g f(ma, nb)\}_{m,n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$, we see that the frame expansion in terms of a Gabor frame (13) is the discretization of the inverse STFT (14).

Let us now consider a specific function $g \in L^2(\mathbb{R})$ for a Gabor frame. Let $g(x) = \chi_{[0,1]}(x)$ and $0 < a, b \leq 1$; then the Gabor system

$$\left\{e^{2\pi imbx}\chi_{[0,1]}(x - na)\right\}_{m,n\in\mathbb{Z}}$$

forms a tight frame for $L^2(\mathbb{R})$. If $a = b = 1$, the system becomes an orthonormal basis for $L^2(\mathbb{R})$. However, for the more general case where $g \in L^2(\mathbb{R})$ is a continuous function with compact support, the associated Gabor system $\mathcal{G}(g, a, b)$ cannot form an orthonormal basis and Riesz basis. The condition determining whether Gabor systems are Gabor frames involves the product ab.

Theorem 4.5 (Density Theorem)**.** *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Then the following statements hold:*

(i) If $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbb{R})$, then $0 < ab \leq 1$.

(ii) If $ab > 1$, then $\mathcal{G}(g, a, b)$ is incomplete in $L^2(\mathbb{R})$.

(iii) If $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbb{R})$ with $ab = 1$, then $\mathcal{G}(g, a, b)$ is a Riesz basis.

(iv) If $\mathcal{G}(g, a, b)$ is a tight frame for $L^2(\mathbb{R})$ with $ab = 1$ and $\|g\| = 1$, then $\mathcal{G}(g, a, b)$ is an orthonormal basis.

The value $1/(ab)$ is called the *redundancy* of a Gabor frame. Theorem 4.5 shows that $0 < ab \leq 1$ is a necessary condition for $\mathcal{G}(g, a, b)$ to be a frame for $L^2(\mathbb{R})$. For example, the Gabor system $\mathcal{G}(g, a, b)$ with

$g(x) = e^{-x^2}$ is a frame if and only if $0 < ab < 1$. If $ab \leq 1$, Theorem 4.5 (iii) indicates that $\mathcal{G}(e^{-x^2}, a, b)$ should form a Riesz basis. However, this is not possible due to the following restriction on a Gabor frame.

Theorem 4.6 (The Balian–Low Theorem)**.** *Let $g \in L^2(\mathbb{R})$ and $a > 0$. If $\mathcal{G}(g, a, 1/a)$ is an orthonormal basis for $L^2(\mathbb{R})$, then*

$$\left(\int_{\mathbb{R}} |x|^2 \, |g(x)|^2 dx \right) \left(\int_{\mathbb{R}} |\xi|^2 \, |\hat{g}(\xi)|^2 d\xi \right) = \infty.$$

The Balian–Low Theorem claims that if a Gabor frame is a Riesz basis for $L^2(\mathbb{R})$, then the Heisenberg product becomes infinite. This means that the function $g \in L^2(\mathbb{R})$ cannot be well localized in both the time and the frequency domains. To overcome the Balian–Low theorem, a *Wilson basis* was proposed [19].

We mention that it is not always possible to generate a Gabor frame with $0 < ab \leq 1$. The combination of $(a, b) \in \mathbb{R}_+^2$ is different depending on the functions. This is known as the frame set problem for Gabor frames. The frame set of a function $g \in L^2(\mathbb{R})$ is defined by

$$\mathcal{F}(g) := \left\{ (a, b) \in \mathbb{R}_+^2 : \mathcal{G}(g, a, b) \text{ is a frame for } L^2(\mathbb{R}) \right\}.$$

Example 4.7. *Examples of the frame set for Gabor frames.*

(i) $g \in \{e^{-x^2}, \, 1/\cosh(\pi x), \, e^{-|x|}\} \Rightarrow \mathcal{F}(g) = \left\{ (a, b) \in \mathbb{R}_+^2 : ab < 1 \right\}$ *[23].*

(ii) $g(x) = e^{-x} \chi_{[0, +\infty)}(x) \Rightarrow \mathcal{F}(g) = \left\{ (a, b) \in \mathbb{R}_+^2 : ab \leq 1 \right\}$.

(iii) $g(x) = \chi_{[0,\, c)}(x)$, $c > 0 \Rightarrow$ *see [14] (abc problem).*

(iv) $g(x) = \max(1 - |x|, 0) \Rightarrow$ *see [29].*

Figure 4 shows the frame set for a Gabor system $\mathcal{G}(g, a, b)$ with $g(x) = e^{-x^2}$. This frame set is very simple, but those for (iii) and (iv) in Example 4.7 have very complicated structures. More details on these results, including a well-written tutorial for Gabor analysis, are given in [29].

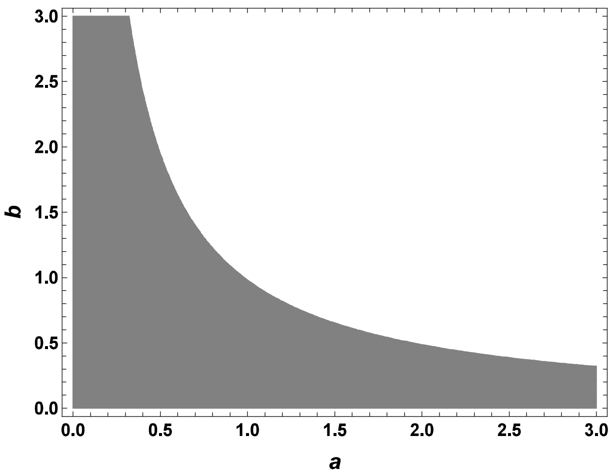

Figure 1: $\mathcal{F}(g) = \left\{(a,b) \in \mathbb{R}_+^2 : ab < 1\right\}$ with $g(x) = e^{-x^2}$.

5 Wavelet System

A wavelet system is very similar to a Gabor system. A wavelet is a function $\psi \in L^2(\mathbb{R})$ and it has dilation and translation parameters $a > 0, b \in \mathbb{R}$ defined by

$$\psi_{a,b}(x) = T_b D_{a^{-1}} \psi(x) = \frac{1}{\sqrt{a}} \psi\left(\frac{x-b}{a}\right).$$

After discretization of the point (a, b) to $\{(a^{-j}, kba^{-j})\}_{j,k\in\mathbb{Z}}$, we define a *wavelet system* that has the form

$$\mathcal{W}(\psi, a, b) := \left\{T_{kba^{-j}} D_{a^j} \psi(x) = D_{a^j} T_{kb} \psi(x) = a^{j/2} \psi\left(a^j x - kb\right)\right\}_{j,k\in\mathbb{Z}},$$

where $a > 1, b > 0$. Given $\psi \in L^2(\mathbb{R})$, if a wavelet system $\mathcal{W}(\psi, a, b)$ is a frame for $L^2(\mathbb{R})$, then we call it a *wavelet frame*. In this note, we consider the case of the *dyadic wavelet system* defined by

$$\{D_{2^j} T_k \psi\}_{j,k\in\mathbb{Z}} = \left\{2^{j/2} \psi\left(2^j x - k\right)\right\}_{j,k\in\mathbb{Z}},$$

which has been well investigated in the study of discrete wavelet theory.

If a wavelet frame for $L^2(\mathbb{R})$ with a frame operator $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is given, we can represent the frame expansions associated with a wavelet frame as

$$
\begin{aligned}
f &= \sum_{j,k\in\mathbb{Z}} \langle f, S^{-1}D_{2^j}T_k\psi\rangle D_{2^j}T_k\psi \\
&= \sum_{j,k\in\mathbb{Z}} \langle f, D_{2^j}T_k\psi\rangle S^{-1}D_{2^j}T_k\psi,
\end{aligned}
$$

for all $f \in L^2(\mathbb{R})$. Unlike Gabor frames, it is possible for a wavelet frame to be a Riesz basis and even an orthonormal basis, both of which are smooth, continuous functions and have compact support. However, the nice feature of Gabor frames that the canonical dual frame always has a Gabor structure is not maintained for wavelet frames. This is because a wavelet frame operator S does not commute with both the dilation and translation operators, i.e., for the canonical dual wavelet frame, we have

$$
S^{-1}\left(D_{2^j}T_k\psi\right) = D_{2^j}(S^{-1}T_k\psi) \neq D_{2^j}T_k(S^{-1}\psi).
$$

As a result, the canonical dual frame is not necessarily a wavelet system [9, 17]. This means that the canonical dual frame for a wavelet frame may not be generated by dilations and translations of a single function. Moreover, some of the important properties of wavelets such as smoothness and compact support might be loosened for the canonical dual wavelet frame.

Therefore, we are led to consider a tight frame or a pair of dual frames. We introduce some conditions, obtained by Chui and Shi [10], for constructing these from wavelet systems in the case of multiwavelet frames.

Definition 5.1. *Consider a collection of functions* $\{\psi_\ell\}_{\ell=1}^n \subset L^2(\mathbb{R})$*. If a system* $\{D_{2^j}T_k\psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,n}$ *forms a wavelet frame for* $L^2(\mathbb{R})$*, we call it a multiwavelet frame.*

Note that a multiwavelet frame is often called simply a wavelet frame. We therefore will not distinguish between them here.

Theorem 5.2 (Chui and Shi)**.** *Let* $\psi_1, \ldots, \psi_n \in L^2(\mathbb{R})$*. A wavelet system* $\{D_{2^j}T_k\psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,n}$ *forms a tight wavelet frame for* $L^2(\mathbb{R})$ *with a frame* *bound A if and only if*

$$\sum_{\ell=1}^{n}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_\ell(2^j\xi)\right|^2 = A, \ \ a.e.\ \xi\in\mathbb{R},$$

$$\sum_{\ell=1}^{n}\sum_{j=0}^{\infty}\overline{\hat{\psi}_\ell(2^j\xi)}\,\hat{\psi}_\ell\left(2^j(\xi+2k+1)\right) = 0, \ \ k\in\mathbb{Z}, \ \ a.e.\ \xi\in\mathbb{R}.$$

For a fixed ℓ, if $\{D_{2^j}T_k\psi_\ell\}_{j,k\in\mathbb{Z}}$ is a tight wavelet frame with a frame bound $A=1$ and $\|\psi_\ell\|=1$, then $\{D_{2^j}T_k\psi_\ell\}_{j,k\in\mathbb{Z}}$ is an orthonormal wavelet that forms an orthonormal basis for $L^2(\mathbb{R})$. Wavelet frames for higher dimensions, such as curvelets [4–6], shearlets [27] and contourlets [20], are tight frames for $L^2(\mathbb{R}^2)$ with frame bounds $A=B=1$. We remark that there have been significant developments in wavelet frames for $L^2(\mathbb{R}^2)$ over the last two decades (see [21] for more details).

A result for a dual frame similar to Theorem 5.2 is given below, namely the duality condition for a wavelet frame that corresponds to Theorem 4.4 for a Gabor frame.

Theorem 5.3 (Chui and Shi)**.** *Let $\psi_1,\ldots,\psi_n\in L^2(\mathbb{R})$ and $\tilde{\psi}_1,\ldots,\tilde{\psi}_n\in L^2(\mathbb{R})$. Two wavelet systems $\{D_{2^j}T_k\psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,n}$ and $\{D_{2^j}T_k\tilde{\psi}_\ell\}_{j,k\in\mathbb{Z},\,\ell=1,\ldots,n}$ are a pair of dual wavelet frames for $L^2(\mathbb{R})$ if and only if*

$$\sum_{\ell=1}^{n}\sum_{j\in\mathbb{Z}}\overline{\hat{\psi}_\ell\left(2^j\xi\right)}\,\hat{\tilde{\psi}}_\ell\left(2^j\xi\right) = 1, \ \ a.e.\ \xi\in\mathbb{R},$$

$$\sum_{\ell=1}^{n}\sum_{j=0}^{\infty}\overline{\hat{\psi}_\ell\left(2^j\xi\right)}\,\hat{\tilde{\psi}}_\ell\left(2^j(\xi+2k+1)\right) = 0, \ \ k\in\mathbb{Z}, \ \ a.e.\ \xi\in\mathbb{R}.$$

Dual wavelet frames are also called *sibling frames* or *bi-frames*. It is possible to construct a pair of dual wavelet frames directly from Gabor frames [13].

In the case of a wavelet frame, there is no analog to the Balian–Low theorem. For systematic constructions of a wavelet frame, two standard methods are available. One way is to use a *frame multiresolution analysis* (FMRA), as introduced by Benedetto and Li [3].

Definition 5.4. *A frame multiresolution analysis consists of nested sequences of a closed subspace* $\{V_j\}_{j\in\mathbb{Z}}$ *of* $L^2(\mathbb{R})$*, for which the following hold.*

(i) $V_j \subset V_{j+1}$, $\forall j \in \mathbb{Z}$.

(ii) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ *and* $\cap_{j\in\mathbb{Z}} V_j = \{0\}$.

(iii) $V_j = D_{2^j} V_0$, $\forall j \in \mathbb{Z}$.

(iv) $f \in V_0 \Rightarrow T_k f \in V_0$, $\forall k \in \mathbb{Z}$.

(v) There exists a function $\phi \in V_0$ *such that* $\{T_k\phi\}_{k\in\mathbb{Z}}$ *is a frame for* V_0.

A MRA as originally defined for constructing orthonormal wavelets requires the existence of a function $\phi \in V_0$ such that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0. The FMRA provides a more relaxed condition, needing $\phi \in V_0$ to be a frame for V_0 instead of an orthonormal basis. Once we find such a frame, we can then construct a wavelet frame by using a strategy similar to that for a MRA.

We next introduce the second way for constructing a wavelet frame, called the *unitary extension principle*, which was proposed by Ron and Shen [32, 33].

Theorem 5.5 (Unitary Extension Principle)**.** *Let* $\psi_0 \in L^2(\mathbb{R})$*. Assume that the following two conditions hold.*

1. *There exists a bounded measurable 1-periodic function* $H_0 \in L^\infty(\mathbb{T})$ *such that*

$$\hat{\psi}_0(\xi) = H_0(\xi/2)\, \hat{\psi}_0(\xi/2).$$

2. $\lim_{\xi\to 0} \hat{\psi}_0(\xi) = 1$.

Let $H_1, \ldots, H_n \in L^\infty(\mathbb{T})$ *and define* $\psi_1, \ldots, \psi_n \in L^2(\mathbb{R})$ *by*

$$\hat{\psi}_\ell(\xi) = H_\ell(\xi/2)\, \hat{\psi}_0(\xi/2), \quad \ell = 1, \ldots, n.$$

The wavelet system $\{D_{2^j}T_k\psi_\ell\}_{j\in\mathbb{Z},k\in\mathbb{Z},\ell=1,\dots,n}$ *forms a tight wavelet frame for* $L^2(\mathbb{R})$ *with a frame bound* $A = 1$ *if and only if*

$$\sum_{\ell=1}^{n} |H_\ell(\xi)|^2 = 1,$$

$$\sum_{\ell=1}^{n} H_\ell(\xi)\,\overline{H_\ell(\xi+1/2)} = 0,$$

for a.e. $\xi \in \mathbb{T}$.

The unitary extension principle can be extended to multiple dimensions. For example, in the case of $L^2(\mathbb{R}^2)$, spline-based wavelet frames [2,28,32,34] have been proposed. There is also a more flexible concept for constructing wavelet frames, called the *oblique extension principle*, available that was proposed by Chui et al. [8] and Daubechies et al. [18] independently.

6 Concluding Remarks

In this note, we surveyed the general frame theory and two examples of frames in $L^2(\mathbb{R})$: a Gabor frame and a wavelet frame. As we have seen, a frame is considered to be a generalization of a basis, and therefore is used in both theory and applications in various scientific fields, including applied harmonic analysis and signal processing. A Gabor frame and a wavelet frame have been studied in detail not only for $L^2(\mathbb{R})$ but also for more general function spaces, such as modulation spaces in the Gabor case and Sobolev spaces, Besov spaces, and Triebel–Lizorkin spaces in the wavelet case.

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References

[1] H. Arai, Wavelets, Kyoritsu Shuppan, Tokyo, 2010 (Japanese).

[2] H. Arai and S. Arai, 2D tight framelets with orientation selectivity suggested by vision science, JSIAM Letters, **1** (2009), 9–12.

[3] J. Benedetto and S. Li, The theory of multiresolution analysis frames and applications to filter banks, Appl. Comput. Harmon. Anal., **5** (1998), 389–427.

[4] E. Candès and D. Donoho, Curvelets – A surprisingly effective non-adaptive representation for objects with edges, in Curves and Surface Fitting: Saint-Malo 1999, A. Cohen, C. Rabut, and L. Schumaker, Eds. Nashville: Vanderbilt Univ. Press, 105–120, 2000.

[5] E. Candès and D. Donoho, Continuous curvelet transform: I. resolution of the wavefront set, Appl. Comput. Harmon. Anal., **19** (2005), 162–197.

[6] E. Candès and D. Donoho, Continuous curvelet transform: II. discretization and frames, Appl. Comput. Harmon. Anal., **19** (2005), 198–222.

[7] P. G. Casazza and G. Kutyniok (eds), Finite Frames: Theory and Applications, Birkhäuser, Boston, 2012.

[8] C. Chui, W. He and J. Stöckler, Compactly supported tight and sibling frames with maximum vanishing moments, Appl. Comput. Harmon. Anal., **13** (2002), 224–262.

[9] C. Chui and X. Shi, Inequalities of Littlewood-Paley type for frames and wavelets, SIAM J. Math. Anal., **24** (1993), 263–277.

[10] C. Chui and X. Shi, Orthonormal wavelets and tight frames with arbitrary real dilations, Appl. Comput. Harmon. Anal., **9** (2000), 243–264.

[11] O. Christensen, A short introduction to frames, Gabor systems, and wavelet systems, Azerbaijan Journal of Mathematics, **4** (2014), 25–39.

[12] O. Christensen, An Introduction to Frames and Riesz Bases, 2nd ed., Applied and Numerical Harmonic Analysis, Birkhäuser Basel, 2016.

[13] O. Christensen and S. Goh, Pairs of dual periodic frames, Appl. Comput. Harmon. Anal., **33** (2012), 315–329.

[14] X. Dai and Q. Sun, The *abc*-problem for Gabor systems, Mem. Amer. Math. Soc., **244** (2016).

[15] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia, 1992.

[16] I. Daubechies, A. Grossmann and Y. Meyer, Painless nonorthogonal expansions, J. Math. Phys., **27** (1986), 1271–1283.

[17] I. Daubechies and B. Han, The canonical dual frame of a wavelet frame, Appl. Comput. Harmon. Anal., **12** (2002), 269–285.

[18] I. Daubechies, B. Han, A. Ron and Z. Shen, Framelets: MRA-based constructions of wavelet frames, Appl. Comput. Harmon. Anal., **14** (2003), 1–46.

[19] I. Daubechies, S. Jaffard and J. L. Journé, A simple Wilson orthonormal basis with exponential decay, SIAM J. Math. Anal., **22** (1991), 554–572.

[20] M. Do and M. Vetterli, The contourlet transform: an efficient directional multiresolution image representation, IEEE Trans. Image Process., **14** (2005), 2091–2106.

[21] K. Fujinoki, A note on curvelets and multiscale directional transforms, RIMS Kôkyûroku, **2102** (2019), 8–31.

[22] R. Duffin and A. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc., **72** (1952), 341–366.

[23] K, Gröchenig, The mystery of Gabor frames, J. Fourier Anal. Appl., **20** (2014), 865–895.

[24] C. Heil, WHAT IS... a frame? Notices Amer. Math. Soc., **60** (2013), 1–3.

[25] J. Kovačević and A. Chebira, Life beyond bases: the advent of frames (part I), IEEE Signal Process. Mag., **24** (2007), 86–104.

[26] J. Kovačević and A. Chebira, Life beyond bases: the advent of frames (part II), IEEE Signal Process. Mag., **24** (2007), 115–125.

[27] D. Labate, W.-Q Lim, G. Kutyniok and G. Weiss, Sparse multidimensional representation using shearlets, in Wavelets XI (San Diego, CA, 2005), 254–262, SPIE Proc. 5914, Bellingham, WA, 2005.

[28] M. J. Lai and J. Stöckler, Construction of multivariate compactly supported tight wavelet frames, Appl. Comput. Harmon. Anal., **21** (2006), 324–348.

[29] K. A. Okoudjou, An invitation to Gabor analysis. Notices Amer. Math. Soc., **66** (2019), 1–12.

[30] A. Ron and Z. Shen, Frames and stable bases for shift-invariant subspaces of $L_2(\mathbb{R}^d)$, Can. J. Math., **47** (1995), 1051–1094.

[31] A. Ron and Z. Shen, Weyl-Heisenberg frames and Riesz bases in $L_2(\mathbb{R}^d)$. Duke Math. J., **89** (1997), 237–282.

[32] A. Ron and Z. Shen, Affine system in $L_2(\mathbb{R}^d)$: the analysis of the analysis operator, J. Funct. Anal., **148** (1997), 408–447.

[33] A. Ron and Z. Shen, Compactly supported tight affine spline frames in $L_2(\mathbb{R}^d)$, Math. Comp., **67** (1998), 191–207.

[34] Z. Shen and Z. Xu, On B-spline framelets derived from the unitary extension principle, SIAM J. Math. Anal., **45** (2013), 127–151.

[35] R. Young, An Introduction to Nonharmonic Fourier Series, Academic Press, New York, 1980.

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