WELL-POSEDNESS FOR THE ENERGY CRITICAL HARDY-SOBOLEV PARABOLIC EQUATION

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1. Introduction

This is a summary of the well-posedness results in our paper [3]. We study the Cauchy problem of the energy critical Hardy-Sobolev parabolic equation

$$\begin{cases} \partial_t u - \Delta u = |x|^{-\gamma}|u|^{2^*(\gamma)-2}u, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(0) = u_0 \end{cases} \tag{1.1}$$

in spatial dimensions $d \geq 3$ with initial data u_0 in the energy space $\dot{H}^1(\mathbb{R}^d)$, defined by

$$\dot{H}^1(\mathbb{R}^d) := \left\{ f \in L^{q_c}(\mathbb{R}^d) \ ; \ \|f\|_{\dot{H}^1} = \left(\int_{\mathbb{R}^d} |\nabla f(x)|^2 \, dx \right)^{\frac{1}{2}} < \infty \right\}, \quad q_c := \frac{2d}{d-2},$$

where $T > 0$, $\gamma \in [0, 2)$, and $2^*(\gamma)$ is the critical Hardy-Sobolev exponent, i.e.,

$$2^*(\gamma) := \frac{2(d-\gamma)}{d-2}.$$

Here, $\partial_t := \partial/\partial t$ is the time derivative, $\nabla := (\partial/\partial x_1, \ldots, \partial/\partial x_d)$ is the vector differential operator, $\Delta := \sum_{j=1}^d \partial^2/\partial x_j^2$ is the Laplace operator on $\mathbb{R}^d$, $u = u(t,x)$ is an unknown complex-valued function on $(0,T) \times \mathbb{R}^d$, and $u_0 = u_0(x)$ is a prescribed complex-valued function on $\mathbb{R}^d$. The *total energy* (or simply *energy*) functional E_γ is defined by

$$E_\gamma(f) := \frac{1}{2}\|f\|_{\dot{H}^1}^2 - \frac{1}{2^*(\gamma)} \int_{\mathbb{R}^d} \frac{|f(x)|^{2^*(\gamma)}}{|x|^\gamma} dx, \quad f \in \dot{H}^1(\mathbb{R}^d),$$

where the first and second terms correspond to the *kinetic* and *potential energies*, respectively. The energy of solution is (formally) dissipated:

$$\frac{d}{dt} E_\gamma(u(t)) = -\int_{\mathbb{R}^d} |\partial_t u(t,x)|^2 \, dx \leq 0. \tag{1.2}$$

Moreover, the equation (1.1), and the total energies, kinetic energies, and potential energies of its solutions are invariant under the scaling transformation $u \mapsto u_\lambda$ for $\lambda > 0$, which is defined by

$$u_\lambda(t,x) := \lambda^{\frac{2-\gamma}{2^*(\gamma)-2}} u(\lambda^2 t, \lambda x) = \lambda^{\frac{d-2}{2}} u(\lambda^2 t, \lambda x).$$

Thus, the problem (1.1) is called *energy critical*, and the space $\dot{H}^1(\mathbb{R}^d)$ (as well as $L^{q_c}(\mathbb{R}^d)$) is often called a *scaling critical* space. We say that the problem is *energy* *subcritical* (*energy supercritical* resp.) if the power p of the nonlinearity $|x|^{-\gamma}|u|^{p-2}u$ is strictly less than (strictly greater than resp.) the critical exponent $2^*(\gamma)$. The case $\gamma = 0$ corresponds to a heat equation with a standard power-type nonlinearity, often called the *Fujita equation*, which has been extensively studied in various directions. In the case $\gamma \neq 0$, the equation (1.1) is not invariant under the translation with respect to space variables, owing to the existence of the space-dependent potential.

Our interest is to study the problem on local well-posedness, i.e., existence of local in time solution, uniqueness, and continuous dependence on initial data, for (1.1). The problem has been studied in the space $L^q(\mathbb{R}^d)$ and the space of continuous bounded functions on $\mathbb{R}^d$ (see [1, 2, 8, 9]). In particular, Ben Slimene, Tayachi, and Weissler proved local well-posedness, except the uniqueness, for (1.1) in the scaling critical space $L^{q_c}(\mathbb{R}^d)$ (see [1]). Recently, the unconditional uniqueness for (1.1) in $C([0,T]; L^{q_c}(\mathbb{R}^d))$ has been proved by Tayachi [8], and local well-posedness has been studied in scaling critical Besov spaces by Chikami [2]. This paper is devoted to studying well-posedness for (1.1) and more general nonlinear heat equations in the energy framework. Similarly to these previous works, we can obtain the local well-posedness in the energy space $\dot{H}^1(\mathbb{R}^d)$, but a more detailed argument is required to justify the energy identity (1.2).

2. Statement of results

Let Ω be a domain of $\mathbb{R}^d$ which contains the origin 0, and $\partial\Omega$ denote the boundary of Ω. We consider the Dirichlet problem of more general nonlinear heat equation

$$\begin{cases} \partial_t u - \Delta u = F(x,u), & (t,x) \in (0,T) \times \Omega, \\ u|_{\partial\Omega} = 0, \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where $T > 0$ and $F : \Omega \times \mathbb{C} \to \mathbb{C}$. We regard $\mathbb{C}$ as the two-dimensional vector space $\mathbb{R}^2$, and assume that $F(x,\cdot) \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ with $F(x,0) = 0$ and

$$|F(x,z_1) - F(x,z_2)| \le C|x|^{-\gamma}(|z_1| + |z_2|)^{2^*(\gamma)-2}|z_1 - z_2| \tag{2.2}$$

for almost everywhere $x \in \Omega$ and any $z_1, z_2 \in \mathbb{C}$. We write the problem (2.1) in the integral form

$$u(t,x) = (e^{t\Delta_\Omega}u_0)(x) + \int_0^t e^{(t-\tau)\Delta_\Omega} F(\cdot, u(\tau,\cdot))(x)\, d\tau \tag{2.3}$$

for any $t \in [0,T)$ and almost everywhere $x \in \Omega$, where $\{e^{t\Delta_\Omega}\}_{t>0}$ is the semigroup generated by the Dirichlet Laplacian $-\Delta_\Omega$ with domain

$$D(-\Delta_\Omega) = \left\{ f \in H_0^1(\Omega) : \Delta f \in L^2(\Omega) \text{ in the distribution sense} \right\}.$$

The space $C_0^\infty(\Omega)$ is the set of all C^∞-functions on Ω having compact support in Ω, and the space $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the Sobolev norm $\|\cdot\|_{H^1(\Omega)}$. We discuss the local well-posedness, small-data global existence, and dissipation of global solutions for (2.1) in the scaling critical spaces $L^{q_c}(\Omega)$, $H^1(\Delta_\Omega)$ and $\dot{H}^1(\Delta_\Omega)$. Here, the Dirichlet Laplacian $-\Delta_\Omega$ is a non-negative and self-adjoint operator on $L^2(\Omega)$, and $H^1(\Delta_\Omega)$ and $\dot{H}^1(\Delta_\Omega)$ are Sobolev spaces associated with $-\Delta_\Omega$ and their norms are given by

$$\|f\|_{H^1(\Delta_\Omega)} := \|(I-\Delta_\Omega)^{\frac{1}{2}} f\|_{L^2(\Omega)} \quad \text{and} \quad \|f\|_{\dot{H}^1(\Delta_\Omega)} := \|(-\Delta_\Omega)^{\frac{1}{2}} f\|_{L^2(\Omega)},$$

respectively, where I is the identity operator on $L^2(\Omega)$. For these precise definitions, we refer to Definition 1.1 in [4]. Note that $H^1(\Delta_\Omega) = H^1_0(\Omega)$, $H^1(\Delta_{\mathbb{R}^d}) = H^1(\mathbb{R}^d)$, and $\dot{H}^1(\Delta_{\mathbb{R}^d}) = \dot{H}^1(\mathbb{R}^d)$. For convenience, we set

$$X = L^{q_c}(\Omega), H^1(\Delta_\Omega) \text{ or } \dot{H}^1(\Delta_\Omega).$$

To state our results, let us introduce some notations.

Definition 2.1. Let $T \in (0,\infty]$, $q \in [1,\infty]$, and $\alpha \in \mathbb{R}$. The space $\mathcal{K}^{q,\alpha}(T,\Omega)$ is defined by

$$\mathcal{K}^{q,\alpha}(T,\Omega) := \left\{u \in \mathscr{D}'([0,T)\times\Omega)\ ;\ \|u\|_{\mathcal{K}^{q,\alpha}(T',\Omega)} < \infty \text{ for any } T' \in (0,T)\right\}$$

endowed with

$$\|u\|_{\mathcal{K}^{q,\alpha}(T,\Omega)} := \sup_{t\in[0,T]} t^{\frac{d}{2}(\frac{1}{q_c}-\frac{1}{q})+\alpha} \|u\|_{L^q(\Omega)},$$

where $\mathscr{D}'([0,T)\times\Omega)$ is the space of distributions on $[0,T)\times\Omega$. We simply write $\mathcal{K}^q(T,\Omega) = \mathcal{K}^{q,0}(T,\Omega)$ when $\alpha = 0$, and $\mathcal{K}^{q,\alpha}(\Omega) = \mathcal{K}^{q,\alpha}(\infty,\Omega)$ and $\mathcal{K}^q(\Omega) = \mathcal{K}^q(\infty,\Omega)$ when $T=\infty$ if they do not cause a confusion.

Hereafter, we assume that $q \in (1,\infty)$ satisfies

$$\frac{1}{q_c} - \frac{2}{d(2^*(\gamma)-1)} < \frac{1}{q} < \frac{1}{q_c} \quad \text{if } X = L^{q_c}(\Omega), \tag{2.4}$$

and

$$\frac{1}{q_c} - \frac{1}{d(2^*(\gamma)-1)} < \frac{1}{q} < \frac{1}{q_c} \quad \text{if } X = H^1(\Delta_\Omega) \text{ or } \dot{H}^1(\Delta_\Omega). \tag{2.5}$$

Let us give the notion of a mild solution.

Definition 2.2. Let $T \in (0,\infty]$ and $u_0 \in X$. A function $u : [0,T)\times\mathbb{R}^d \to \mathbb{C}$ is called an X-mild solution to (2.1) with initial data $u(0) = u_0$ if it satisfies $u \in C([0,T);X)\cap\mathcal{K}^q(T,\Omega)$ and the integral equation (2.3) for any $t \in [0,T)$ and almost everywhere $x \in \mathbb{R}^d$. The time T is said to be the maximal existence time, which is denoted by $T_{\max} = T_{\max}(u_0)$, if the solution cannot be extended beyond $[0,T)$. More precisely,

$$T_{\max}(u_0) := \sup\left\{T>0\ ;\ \begin{array}{l}\text{There exists a unique solution } u \text{ to (2.1)}\\ \text{in } C([0,T];X)\cap\mathcal{K}^q(T,\Omega) \text{ with initial data } u_0\end{array}\right\}. \tag{2.6}$$

We say that u is global in time if $T_{\max} = +\infty$ and that u blows up in finite time otherwise.

Then we have the following:

Theorem 2.3. *Let $d \geq 3$ and $0 \leq \gamma < 2$. Then the following statements hold:*

(i) (*Existence*) *For any* $u_0 \in X$, *there exists a maximal existence time* $T_{\max} = T_{\max}(u_0) \in (0,\infty]$ *such that there exists a unique mild solution*

$$u \in C([0,T_{\max}); X) \cap \mathcal{K}^q(T_{\max},\Omega)$$

to (2.1) *with* $u(0) = u_0$.

(ii) (*Uniqueness in* $\mathcal{K}^q(T,\Omega)$) *Let* $T > 0$. *If* $u_1, u_2 \in \mathcal{K}^q(T,\Omega)$ *satisfy the integral equation* (2.3) *with* $u_1(0) = u_2(0) = u_0$, *then* $u_1 = u_2$ *on* $[0,T]$.

(iii) (*Continuous dependence on initial data*) *The map* $T_{\max} : X \to (0,\infty]$ *is lower semicontinuous. Furthermore, for any* $u_0, v_0 \in X$ *and for any* $T < \min\{T_{\max}(u_0), T_{\max}(v_0)\}$, *there exists a constant* $C > 0$, *depending on* u_0, v_0 *and* T, *such that*

$$\sup_{t\in[0,T]} \|u(t) - v(t)\|_X + \|u - v\|_{\mathcal{K}^q(T,\Omega)} \le C\|u_0 - v_0\|_X.$$

(iv) (*Blow-up criterion*) *If* $T_{\max} < +\infty$, *then* $\|u\|_{\mathcal{K}^q(T_{\max},\Omega)} = \infty$.

(v) (*Small-data global existence and dissipation*) *There exists* $\rho > 0$ *such that if* $u_0 \in X$ *satisfies*

$$\|e^{t\Delta_\Omega} u_0\|_{\mathcal{K}^q(\Omega)} \le \rho,$$

then $T_{\max} = +\infty$ *and*

$$\|u\|_{\mathcal{K}^q(\Omega)} \le 2\rho \quad \text{and} \quad \lim_{t\to\infty} \|u(t)\|_X = 0.$$

(vi) *The following statements are equivalent*:
  (a) $T_{\max} = +\infty$ *and* $\|u\|_{\mathcal{K}^q(\Omega)} < \infty$.
  (b) $\lim_{t\to T_{\max}} \|u(t)\|_X = 0$.
  (c) $\lim_{t\to T_{\max}} t^{\frac{d}{2}(\frac{1}{q_c}-\frac{1}{q})} \|u(t)\|_{L^q(\Omega)} = 0$.

(vii) *Let* $d = 3$ *and* $X = H^1(\Delta_\Omega)$ *or* $\dot{H}^1(\Delta_\Omega)$. *Suppose additionally that* q *satisfies*

$$\frac{1}{q_c} - \frac{1}{12(2-\gamma)} < \frac{1}{q}, \tag{2.7}$$

and that F *satisfies*

$$|\partial_z F(x,z)| \le C|x|^{-\gamma}|z|^{2^*(\gamma)-2} \tag{2.8}$$

for almost everywhere $x \in \Omega$ *and any* $z \in \mathbb{C}$. *Then, for any* $u_0 \in X$, *there exists a maximal existence time* $T_{\max} = T_{\max}(u_0) \in (0,\infty]$ *such that there exists a unique mild solution*

$$u \in C([0,T_{\max}); X) \cap \mathcal{K}^q(T_{\max},\Omega) \quad \text{and} \quad \partial_t u \in \mathcal{K}^{3,1}(T_{\max},\Omega)$$

to (2.1) *with* $u(0) = u_0$. *Furthermore, the solution* u *satisfies*

$$\partial_t u \in \mathcal{K}^{2,1}(T_{\max},\Omega).$$

The statements (i)–(vi) are known, but the last statement (vii) is a new ingredient, which is utilized to justify the energy identity in Theorem 2.6 below. The proof is given in Section 3.

Remark 2.4. The statement (vii) in Theorem 2.3 implies that the maximal existence time is written as

$$T_{\max}(u_0) = \sup \left\{ T > 0\,;\ \begin{array}{c} \text{There exists a unique solution } u \text{ to (2.1)} \\ \text{in } C([0,T];X) \cap \mathcal{K}^q(T,\Omega) \text{ and } \partial_t u \in \mathcal{K}^{3,1}(T,\Omega) \\ \text{with initial data } u_0 \end{array} \right\}.$$

Remark 2.5. It is generally impossible to obtain classical solutions for (1.1). However, mild solutions u to (1.1) given in Theorem 2.3 are continuous and bounded on $\mathbb{R}^d$ for each $t \in (0, T_{\max})$, and belong to

$$u \in C^{1,2}_{\mathrm{loc}}((0,T_{\max}) \times (\mathbb{R}^d \setminus \{0\})) \cap C^{\frac{\alpha}{2},\alpha}_{\mathrm{loc}}((0,T_{\max}) \times \mathbb{R}^d)$$

for $\alpha \in (0, 2-\gamma)$ by the regularity theory for parabolic equations. Here, $C^{\alpha,\beta}_{\mathrm{loc}}(I \times \mathbb{R}^d)$ is the space of functions that are locally Hölder continuous with exponent $\alpha \geq 0$ in $t \in I$ and exponent $\beta \geq 0$ in $x \in \mathbb{R}^d$ for an interval $I \subset (0,\infty)$. See Remark 1.1 and Proposition 3.2 in [1] (see also the remark after Definition 2.1 in [9] on page 563).

Moreover, we introduce the energy functional $E_{\gamma,\Omega} : H^1_0(\Omega) \to \mathbb{R}$ associated with (2.1) with nonlinearity $F(x,u) = |x|^{-\gamma}|u|^{2^*(\gamma)-2}u$ as follows:

$$E_{\gamma,\Omega}(\phi) := \frac{1}{2}\|\phi\|^2_{\dot{H}^1(\Omega)} - \frac{1}{2^*(\gamma)}\int_\Omega \frac{|\phi(x)|^{2^*(\gamma)}}{|x|^\gamma}dx.$$

The energy identity (1.2) plays a crucial role in studying (2.1) in the energy framework, and is formally obtained by multiplying the equation (1.1) by $\overline{\partial_t u}$ and integrating it over $\mathbb{R}^d$. However, the validity of (1.2) is non trivial. We have the result on the validity.

Theorem 2.6. *Let $u_0 \in \dot{H}^1(\Delta_\Omega)$ and $t_0 \in (0, T_{\max})$. Then, the mild solution u to* (1.1) *with $u(0) = u_0$ satisfies the energy identity*

$$E_{\gamma,\Omega}(u(t)) + \int_{t_0}^t \int_\Omega |\partial_t u(\tau,x)|^2\,dxd\tau = E_{\gamma,\Omega}(u(t_0)) \tag{2.9}$$

for any $t \in [t_0, T_{\max})$. Furthermore, the energy inequality

$$E_{\gamma,\Omega}(u(t)) \leq E_{\gamma,\Omega}(u_0) \tag{2.10}$$

holds for any $t \in [0, T_{\max})$.

The proof is given in Section 4.

3. Proof of Theorem 2.3

3.1. **Key estimates.** To prove Theorem 2.3, let us prepare some estimates for $\{e^{t\Delta_\Omega}\}_{t>0}$. We recall the linear estimates with weights in the case $\Omega = \mathbb{R}^d$ (see Proposition 2.1 in [1]). By combining these linear estimates with the pointwise estimates for the integral kernel $G_\Omega(t,x,y)$ of $e^{t\Delta_\Omega}$:

$$0 \leq G_\Omega(t,x,y) \leq (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right), \quad t > 0,\ \text{a.e.}\, x,y \in \Omega \tag{3.1}$$

(see, e.g., Ouhabaz [7]), we have smoothing and decay estimates for $\{e^{t\Delta_\Omega}\}_{t>0}$.

Lemma 3.1 (Lemma A.4 in [3])**.** *Let $d \geq 1$, $0 < \gamma < d$ and $s \geq 0$. Then, the following statements hold*:

(i) *For any $1 \leq p_1 \leq p_2 \leq \infty$, there exists $C > 0$ such that*

$$\|(-\Delta_\Omega)^{\frac{s}{2}} e^{t\Delta_\Omega} f\|_{L^{p_2}(\Omega)} \leq C t^{-\frac{d}{2}(\frac{1}{p_1}-\frac{1}{p_2})-\frac{s}{2}} \|f\|_{L^{p_1}(\Omega)}$$

for any $t > 0$ and $f \in L^{p_1}(\Omega)$.

(ii) *Suppose*

$$0 \leq \frac{1}{p_2} < \frac{\gamma}{d} + \frac{1}{p_1} < 1.$$

Then, there exists $C > 0$ such that

$$\|(-\Delta_\Omega)^{\frac{s}{2}} e^{t\Delta_\Omega}(|\cdot|^{-\gamma} f)\|_{L^{p_2}(\Omega)} \leq C t^{-\frac{d}{2}(\frac{1}{p_1}-\frac{1}{p_2})-\frac{s+\gamma}{2}} \|f\|_{L^{p_1}(\Omega)}$$

for any $t > 0$ and $f \in L^{p_1}(\Omega)$.

Based on this lemma, we have the following:

Lemma 3.2 (Lemma A.5 in [3])**.** *Let $d \geq 3$, $0 \leq \gamma < 2$, and $T > 0$. Then, the following statements hold*:

(i) *Assume q satisfies* (2.4)*. Then, there exists a positive constant C_1 depending only on d, γ and q such that*

$$\begin{aligned}\Big\| \int_0^t e^{(t-\tau)\Delta_\Omega}\{F(\cdot,u(\tau)) - F(\cdot,v(\tau))\}\,d\tau \Big\|_{\mathcal{K}^q(T,\Omega)\cap L^\infty([0,T];L^{q_c}(\Omega))} \\ \leq C_1 \max\{\|u\|_{\mathcal{K}^q(T,\Omega)}, \|v\|_{\mathcal{K}^q(T,\Omega)}\}^{2^*(\gamma)-2}\|u-v\|_{\mathcal{K}^q(T,\Omega)}\end{aligned}$$

holds for any $u, v \in \mathcal{K}^q(T,\Omega)$.

(ii) *Assume q satisfies* (2.5)*. Then, there exists a positive constant C_2 depending only on d, γ and q such that*

$$\begin{aligned}\Big\| \int_0^t e^{(t-\tau)\Delta_\Omega}\{F(\cdot,u(\tau)) - F(\cdot,v(\tau))\}\,d\tau \Big\|_{L^\infty([0,T];\dot{H}^1(-\Delta_\Omega))\cap\mathcal{K}^2(T,\Omega)} \\ \leq C_2 \max\{\|u\|_{\mathcal{K}^q(T,\Omega)}, \|v\|_{\mathcal{K}^q(T,\Omega)}\}^{2^*(\gamma)-2}\|u-v\|_{\mathcal{K}^q(T,\Omega)}\end{aligned}$$

holds for any $u, v \in \mathcal{K}^q(T,\Omega)$.

(iii) *Assume q satisfies* (2.4) *and*

$$\frac{1}{q_c} - \frac{4-d}{2d(2^*(\gamma)-2)} < \frac{1}{q}, \tag{3.2}$$

and F satisfies the additional assumption (2.8)*. Then, there exists a positive constant C_3 depending only on d, γ and q such that*

$$\begin{aligned}\Big\| \partial_t \int_0^t e^{(t-\tau)\Delta_\Omega} F(\cdot,u(\tau))\,d\tau \Big\|_{\mathcal{K}^{d,1}(T,\Omega)\cap\mathcal{K}^{2,1}(T,\Omega)} \\ \leq C_3 \left(\|u_0\|_{L^{q_c}(\Omega)}^{2^*(\gamma)-1} + \|u\|_{\mathcal{K}^q(T,\Omega)}^{2^*(\gamma)-2}\|\partial_t u\|_{\mathcal{K}^{d,1}(T,\Omega)} \right)\end{aligned} \tag{3.3}$$

holds for any $u_0 \in L^{q_c}(\Omega)$ and for any $u \in \mathcal{K}^q(T,\Omega)$ satisfying the integral equation (2.3) *and $\partial_t u \in \mathcal{K}^{d,1}(T,\Omega)$.*

Remark 3.3. Note that the statement (iii) in Lemma 3.2 holds only if $d = 3$, as it is possible to take q satisfying both (2.4) and (3.2) only if $d = 3$. This statement (iii) is a key tool in the proof of (vii) in Theorem 2.3, and (3.2) yields the additional assumption (2.7) of (vii) in Theorem 2.3.

3.2. **Proof of Theorem 2.3.** The proofs of (i)–(vi) are obtained by combining Lemma 3.2 and the standard fixed-point argument (see [1] and also [3]). So, we may omit the proofs. We give only a sketch of proof of (vii) when $X = \dot{H}^1(\Delta_\Omega)$. Take $\rho > 0$ and $M > 0$ such that

$$\rho + C_1 M^{2^*(\gamma)-1} \le M \quad \text{and} \quad \max\{C_1, C_3\} M^{2^*(\gamma)-2} \le \frac{1}{2}, \tag{3.4}$$

where C_1 and C_3 are the same constants as those in (i) and (iii) of Lemma 3.2, respectively. Let $A > 0$. Suppose that $u_0 \in \dot{H}^1(\Delta_\Omega)$ and $T > 0$ satisfy

$$\|u_0\|_{\dot{H}^1(\Delta_\Omega)} \le A \quad \text{and} \quad \|e^{t\Delta_\Omega} u_0\|_{\mathcal{K}^q(T,\Omega)} \le \rho. \tag{3.5}$$

Define the map Φ_{u_0} by

$$\Phi_{u_0}[u](t) := e^{t\Delta_\Omega} u_0 + \int_0^t e^{(t-\tau)\Delta_\Omega} F(x, u(\tau))\, d\tau$$

for $t \in [0, T]$. Given $B > 0$, we define

$$Y := \{u \ ; \ \|u\|_{\mathcal{K}^q(T,\Omega)} \le M, \|\partial_t u\|_{\mathcal{K}^{3,1}(T,\Omega)} \le B\},$$

equipped with the metric $\mathrm{d}(u, v) := \|u - v\|_{\mathcal{K}^q(T,\Omega)}$. Then, (Y, d) is a complete metric space. By (i) in Lemma 3.2, (3.4) and (3.5), we have

$$\|\Phi_{u_0}[u]\|_{\mathcal{K}^q(T,\Omega)} \le \|e^{t\Delta_\Omega} u_0\|_{\mathcal{K}^q(T,\Omega)} + C_1 \|u\|_{\mathcal{K}^q(T,\Omega)}^{2^*(\gamma)-1} \le \rho + C_1 M^{2^*(\gamma)-1} \le M$$

for any $u \in Y$, and

$$\begin{aligned}
\|\Phi_{u_0}[u] - \Phi_{u_0}[v]\|_{\mathcal{K}^q(T,\Omega)} &\le C_1 \max\{\|u\|_{\mathcal{K}^q(T,\Omega)}, \|v\|_{\mathcal{K}^q(T,\Omega)}\}^{2^*(\gamma)-2} \|u - v\|_{\mathcal{K}^q(T,\Omega)} \\
&\le C_1 M^{2^*(\gamma)-2} \|u - v\|_{\mathcal{K}^q(T,\Omega)} \\
&\le \frac{1}{2} \|u - v\|_{\mathcal{K}^q(T,\Omega)}
\end{aligned}$$

for any $u, v \in Y$. On the other hand, by (iii) in Lemma 3.2, (3.4), and (3.5), we estimate

$$\begin{aligned}
\|\partial_t \Phi_{u_0}[u]\|_{\mathcal{K}^{3,1}(T,\Omega)} &\le \|\partial_t e^{t\Delta_\Omega} u_0\|_{\mathcal{K}^{3,1}(T,\Omega)} + C_3 \big(\|u_0\|_{L^{q_c}(\Omega)}^{2^*(\gamma)-1} + \|u\|_{\mathcal{K}^q(T,\Omega)}^{2^*(\gamma)-2} \|\partial_t u\|_{\mathcal{K}^{d,1}(T,\Omega)} \big) \\
&\le C_4 \big(\|u_0\|_{\dot{H}^1(\Delta_\Omega)} + \|u_0\|_{\dot{H}^1(\Delta_\Omega)}^{2^*(\gamma)-1} \big) + C_3 \|u\|_{\mathcal{K}^q(T,\Omega)}^{2^*(\gamma)-2} \|\partial_t u\|_{\mathcal{K}^{d,1}(T,\Omega)} \\
&\le C_4 (A + A^{2^*(\gamma)-1}) + C_3 M^{2^*(\gamma)-2} B \\
&\le \frac{B}{2} + \frac{B}{2} = B
\end{aligned}$$

for any $u \in Y$, where we take $B = 2C_4(A + A^{2^*(\gamma)-1})$. Summarizing the estimates obtained so far, we see that Φ_{u_0} is contractive from Y into itself. Therefore, Banach's fixed-point theorem allows us to prove that there exists a function $u \in Y$ such that $u = \Phi_{u_0}[u]$. Finally, it follows from (ii) and (iii) in Lemma 3.2 that

$u \in C([0,T); \dot{H}^1(\Delta_\Omega))$ and $\partial_t u \in \mathcal{K}^{2,1}(T,\Omega)$. The proof of Theorem 2.3 is finished. □

4. Proof of Theorem 2.6

In this section, we give a sketch of proof of Theorem 2.6. Let $u_0 \in \dot{H}^1(\Delta_\Omega)$ and u be a mild solution to (2.1) with $u(0) = u_0$. To prove the validity of (2.9), we need to know the integrability of $\partial_t u$, $\Delta_\Omega u$, and $|x|^{-\gamma}|u|^{2^*(\gamma)-2}u$. To begin with, we check the integrability of the nonlinear term. Let $\Omega_1 := \Omega \cap \{|x| < 1\}$ and $\Omega_2 := \Omega \cap \{|x| \geq 1\}$. It is easily seen from the argument of proof of (i) of Proposition 3.2 in [1] that

$$u(t) \in L^\infty(\Omega) \quad \text{for any } t \in (0, T_{\max}),$$

which implies that

$$|x|^{-\gamma}|u|^{2^*(\gamma)-2}u \in L^\infty_{\mathrm{loc}}((0,T_{\max}); L^{\sigma_1}(\Omega_1)) \tag{4.1}$$

for any $1 \leq \sigma_1 < d/\gamma$. Since $u \in L^\infty(0,T; L^{q_c}(\Omega))$, Hölder's inequality implies

$$|x|^{-\gamma}|u|^{2^*(\gamma)-2}u \in L^\infty((0,T); L^{\sigma_2}(\Omega_2)) \tag{4.2}$$

for any $\sigma_2 > 2d/(d+2)$. Let us divide the proof into two cases:

(a) $d \geq 4$ or $d = 3$ and $0 \leq \gamma < 3/2$;
(b) $d = 3$ and $3/2 \leq \gamma < 2$.

Case (a): Let $t_0 \in (0, T_{\max})$. Then, we have $u(t_0) \in \dot{H}^1(\Delta_\Omega)$ by Theorem 2.3, and

$$|x|^{-\gamma}|u|^{2^*(\gamma)-2}u \in L^2_{\mathrm{loc}}([t_0,T_{\max}); L^2(\Omega)) \tag{4.3}$$

by (4.1) and (4.2) provided that $d \geq 4$ or $d = 3$ and $0 \leq \gamma < 3/2$. Hence, we can apply the maximal regularity for parabolic equations (see Theorem 1.4 in [5]) to obtain

$$\partial_t u, \Delta_\Omega u \in L^2_{\mathrm{loc}}([t_0,T_{\max}); L^2(\Omega)). \tag{4.4}$$

Then, (4.3) and (4.4) ensure the energy identity (2.9) for any $t \in [t_0, T_{\max})$.

Case (b): It follows from (vii) in Theorem 2.3 and (4.2) that

$$\partial_t u, \Delta u, |x|^{-\gamma}|u|^{2^*(\gamma)-2}u \in L^2_{\mathrm{loc}}((0,T_{\max}); L^2(\Omega_2)).$$

Hence, multiplying the equation (1.1) by $\overline{\partial_t u}$ and integrating it over $[t_0,t] \times \Omega_2$ are justified. On the other hand, we see from (vii) in Theorem 2.3 and (4.1) that

$$\partial_t u \in L^2_{\mathrm{loc}}((0,T_{\max}); L^3(\Omega_2)) \subset L^2_{\mathrm{loc}}((0,T_{\max}); L^2(\Omega_2)),$$
$$|x|^{-\gamma}|u|^{2^*(\gamma)-2}u \in L^2_{\mathrm{loc}}((0,T_{\max}); L^{\frac{3}{2}}(\Omega_2)).$$

Then, we also have

$$\Delta u \in L^2_{\mathrm{loc}}((0,T_{\max}); L^{\frac{3}{2}}(\Omega_2)),$$

as u satisfies the differential equation (1.1) by Remark 2.5. Hence, multiplying (1.1) by $\overline{\partial_t u}$ and integrating it over $[t_0,t] \times \Omega_2$ are also justified. The above argument ensures the energy identity (2.9) for any $t \in [t_0, T_{\max})$.

Finally, it follows from (2.9) that

$$E_{\gamma,\Omega}(u(t)) \leq E_{\gamma,\Omega}(u(t_0)) \tag{4.5}$$

for any $t_0 \in (0, T_{\max})$. Since the energy $E_{\gamma,\Omega}(u(t))$ is continuous in $t \in [0, T_{\max})$, we have

$$E_{\gamma,\Omega}(u(t)) \leq E_{\gamma,\Omega}(u_0)$$

by taking the limit of (4.5) as $t_0 \to 0$. Thus, we conclude Proposition 2.6. □

Remark 4.1. The proof of case (a) cannot be applied to the case (b) as the nonlinear term does not necessarily satisfy (4.3) in the case (b). For example, we consider the case $\Omega = \mathbb{R}^d$. Then, the ground state W_γ (i.e, the minimal energy non trivial solution to the corresponding stationary problem) given by

$$W_\gamma(x) := ((d-\gamma)(d-2))^{\frac{d-2}{2(2-\gamma)}} (1 + |x|^{2-\gamma})^{-\frac{d-2}{2-\gamma}} \tag{4.6}$$

(see [6]), which is also a mild solution to (1.1), does not satisfy (4.3), as

$$\Delta W_\gamma = |x|^{-\gamma} W_\gamma^{2^*(\gamma)-1} \in L^2(\mathbb{R}^d) \text{ if and only if } d \geq 4 \text{ or } d = 3 \text{ and } 0 \leq \gamma < 3/2.$$

In contrast, we can perform the argument in the proof of case (b) only if $d = 3$, as it relies on (vii) in Theorem 2.3 (see Remark 3.3).

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