σ-finite acims for weakly expanding random maps with uniformly contractive branches

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1 Introduction

For a given measurable dynamical system or measurable random dynamical system over a probability space, the existence of an absolutely continuous finite or σ-finite infinite invariant measure (acim, for abbreviation) with respect to the reference measure is important and fundamental to analyze statistical properties of the system. See [A97, ANV15, BB16, F69, F80, S95] and references therein. In [T1], the author gave some necessary and sufficient conditions for the existence of a finite or σ-finite acim for a given system over a probability space (in terms of Markov operators). In this note, based on [T2], we introduce a certain model of (annealed type) random dynamical systems and estimate the density functions of their acim.

We recall definitions and notations which are needed in this note. Let $(X, \mathscr{B}, \lambda)$ be a probability space and $T : X \to X$ be a measurable and non-singular transformation (i.e., $T^{-1}\mathscr{B} \subset \mathscr{B}$ and the pushforward measure $\lambda \circ T^{-1}$ is absolutely continuous with respect to λ). Then a measure μ on $(X, \mathscr{B})$ is called an *acim* for T provided that μ is absolutely continuous with respect to λ and $\mu \circ T^{-1} = \mu$. μ is called a *finite* (resp. *σ-finite*) *acim* when μ is a finite (resp. σ-finite) measure. Analogously, for an annealed type random map we define an acim as follows. Let I be a non-empty and at most countable set and T_i be a measurable and non-singular transformation from X into itself for each $i \in I$. For a given probability vector $\{p_i\}_{i\in I}$, a *random iteration* (of $\{T_i\}_{i\in I}$ with probabilities $\{p_i\}_{i\in I}$) $\{T_i, p_i : i \in I\}$ is the Markov process given by the following transition probability:

$$\mathbb{P}(x, A) = \sum_{i\in I} p_i 1_A(T_i x)$$

for λ-almost every $x \in X$ and any $A \in \mathscr{B}$. Then a λ-absolutely continuous finite (resp. σ-finite) measure μ is called a *finite* (resp. *σ-finite*) *acim* if μ satisfies

$$\mu = \sum_{i\in I} p_i \mu \circ T_i^{-1}.$$

We also recall that a measure μ is called *ergodic* if for any measurable set with $\mathbb{P}(x, E) = 1_E(x)$ for μ-almost every $x \in X$ we have either $\mu(E) = 0$ or $\mu(X \setminus E) = 0$. For more general cases of random dynamics including the case when I is uncountable or position dependent cases (i.e., p_i is also a function of $x \in X$), we refer to [GB03, I12, I20] and references therein.

Now we are addressed to define the model of our random dynamical systems on the unit interval. Set arbitrary $\alpha > 0$. We define the partition of $X = [0, 1]$ into $\mathscr{Q}_\alpha = \{X_n = ((n+1)^{-1/\alpha}, n^{-1/\alpha}]\}_{n \geq 1}$. Then we have $\lambda(X_n)/\lambda(X_{n+1})$ monotonically decrease to 1 as $n \to \infty$ where λ is Lebesgue measure on X. Let I be a non-empty countable set and for each $i \in I$, J_i be a non-empty subset of $\mathbb{N}$. We consider a family of transformations $\{T_i : i \in I\}$ on X (see Figure 1 below for an example) which are piecewise monotone and piecewise linear on the partition $\mathscr{Q}_\alpha$ satisfying

(a-1) For each $i \in I$, $T_i\,|_{X_n}: X_n \to X_{n-1}$, for any $n \geq 2$, monotonically increasing given by

$$T_i\,|_{X_n}(x) = \frac{(n-1)^{-1/\alpha} - n^{-1/\alpha}}{n^{-1/\alpha} - (n+1)^{-1/\alpha}} x - \frac{(n+1)^{-1/\alpha}(n-1)^{-1/\alpha} - n^{-2/\alpha}}{n^{-1/\alpha} - (n+1)^{-1/\alpha}},$$

namely, all T_i are identical on $X \setminus X_1$;

(a-2) For each $i \in I$, $T_i\,|_{X_1}: X_1 \to \bigcup_{k \in J_i} X_k$, monotonically increasing and surjective which is piecewise linear in the sense

$$T_i'\,|_{X_1} = \frac{\sum_{k \in J_i} \lambda(X_k)}{\lambda(X_1)}$$

whenever the derivative can be defined.

We call a family of transformations $\{T_i : i \in I\}$ with the above conditions (a-1) and (a-2) *piecewise linear intermittent Markov maps* (with the index $\{J_i\}_{i \in I}$).

Then, for a given probability vector p_i on I, we consider a *random iteration of piecewise linear intermittent Markov maps with uniformly contractive part* such that

(b-1) $\{T_i : i \in I\}$ is piecewise linear intermittent Markov maps;

(b-2) On X_1, the random iteration $\{T_i, p_i : i \in I\}$ is uniformly contractive on average:

$$\sum_{i \in I} \frac{p_i}{|T_i'\,|_{X_1}|} = \sum_{i \in I} \frac{p_i \lambda(X_1)}{\sum_{k \in J_i} \lambda(X_k)} > 1.$$

Remark 1.1. *The requirement that I is countable is just for simplicity of notation. In fact, we can define the model even when I is an uncountable set and p is a probability measure on I. If I is uncountable then we integrate over I instead of summation over I. See [T2] for more detail.*

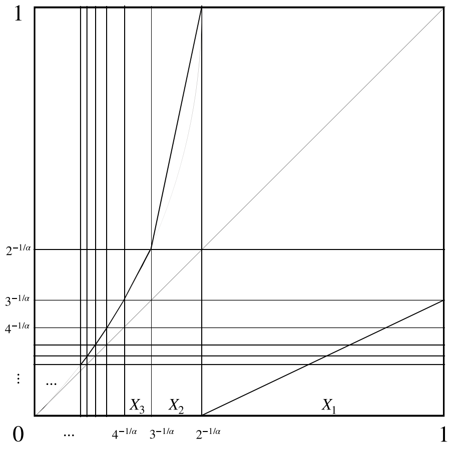

Figure 1: A graph of T_i for some $i \in I$ of piecewise linear intermittent Markov maps. For this T_i, we can see $J_i = \{n \in \mathbb{N} : n \geq 3\}$ since $T_i X_1 = \bigcup_{n \geq 3} X_n$.

By the definition, random iterations of piecewise linear intermittent Markov maps with uniformly contractive part do not satisfy expanding on average even if we remove a small neighborhood of a common indifferent fixed point 0 in the sense [I20]. In the next section we will see that this model always admits a finite/σ-finite acim and show the criterion whether an acim is finite or infinite. Moreover, under some mild condition, the acim is conservative, ergodic and hence unique up to multiplicative constants.

2 Main Result

In this section, we show the existence of a σ-finite acim for any random iteration of piecewise linear intermittent Markov maps with uniformly contractive part defined in the last section. For weakly expanding case (excluding the case like (b-2)), sufficient conditions for the existence of a σ-finite acim was already shown in [I20] via the inducing scheme. The main result of this note is as follows.

Theorem 2.1. *Any random iteration $\{T_i, p_i : i \in I\}$ of piecewise linear intermittent Markov maps with uniformly contractive part, which satisfies (b-1) and (b-2), admits a σ-finite acim. The invariant density $d\mu/d\lambda$ is given by the following formula:*

$$\frac{d\mu}{d\lambda} = \sum_{i \in I} \frac{p_i}{\sum_{j \in J_i} \lambda(X_j)} \sum_{n \geq 0} \sum_{\substack{j \in J_i \\ j > n}} \frac{\lambda(X_j)}{\lambda(X_{j-n})} 1_{X_{j-n}}. \tag{2.1}$$

Consequently, $\mu(X) < \infty$ if and only if

$$\sum_{i \in I} \frac{p_i}{\sum_{j \in J_i} \lambda(X_j)} \sum_{n \geq 0} \sum_{\substack{j \in J_i \\ j > n}} \lambda(X_j) < \infty.$$

Remark 2.2. *(i) The statements of Theorem 2.1 are still true when I is an uncountable set and p is a probability on I, by replacing the summation by integration over I. See Theorem 3.1 and Corollary 3.1 in [T2].*

(ii) For the case of nonlinear random maps like LSV map or Manneville–Pomeau map with uniformly contractive part, we also may be able to show the existence of σ-finite acims and to estimate its densities if the branches on X_1 is linear in the sense (a-2). Otherwise, it would be possible to estimate the invariant density under certain distortion property on X_1. That will be studied in the other paper.

The following corollary shows the uniqueness of the acim given in Theorem 2.1 under mild condition.

Corollary 2.3. *Suppose that $\#J_i \geq 2$ for each $i \in I$ with $p_i > 0$. Then the acim μ for a random iteration $\{T_i, p_i : i \in I\}$ given by (2.1) is an ergodic measure. Consequently, the acim μ is unique up to multiplicative constant.*

In the rest of this note, we give several examples of random iterations of piecewise linear intermittent Markov maps with uniformly contractive part.

The first example is a random iteration of piecewise linear intermittent Markov maps with "thin branches" which return to the indifferent fixed point. The σ-finite acim of the following example varies at $\alpha = 1$ from finite to infinite, which is same as the deterministic case [Th80].

Example 2.4. *Let $\alpha > 0$, $I = \{1, 2\}$ and $J_i = \{(10i)n : n \geq 1\}$ for $i \in I$. Set $p_1 = p$ (and $p_2 = 1 - p$). (See Figure 2 below for the corresponding map.) Then the random iteration of piecewise linear intermittent Markov maps with uniformly contractive part $\{T_i; p_i : i \in I\}$ has the unique ergodic σ-finite acim μ by Theorem 2.1. We also calculate that $\mu(X) < \infty$ if and only if $\alpha < 1$, same as deterministic transformation case.*

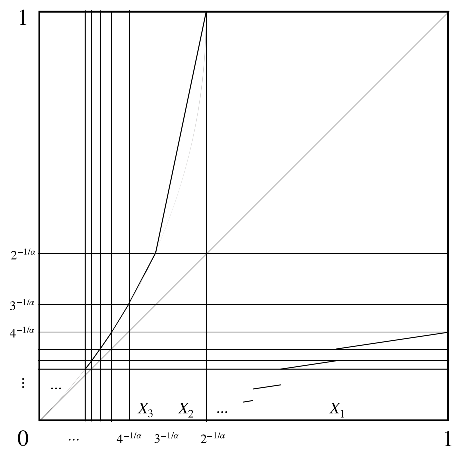

Figure 2: A graph of T_1 in Example 2.4

The next example is a random iteration which always admits a finite acim for any α of the order of tangency at the indifferent fixed point, because all points will never return to a neighborhood of the indifferent fixed point.

Example 2.5. *Let $\alpha > 0$, $I \subset \mathbb{N}$ and J_i satisfy that $\bigcup_{i\in I} J_i$ is a finite set. Then, from Theorem 2.1, for any probability vector $\{p_i\}_{i\in I}$, the random iteration of $\{T_i; p_i : i \in I\}$ admits a finite acim. Further, the invariant density for this random iteration is bounded above even around the point 0.*

The following example also admits a finite acim, although many points of positive measure will return to an enough small neighborhood of the indifferent fixed point with positive probability.

Example 2.6. *Let $\alpha > 0$, $I = \mathbb{N}$ and $J_i = \{2, 3, \ldots, i+1\}$ for $i \in I$. See Figure 3 for the maps $\{T_i : i \in I\}$. If we put $p_i = 1/2^i$ for $i \in I$, then the random iteration $\{T_i, p_i : i \in I\}$ admits the ergodic σ-finite acim μ by Theorem 2.1. We can also see $\mu(X) < \infty$ for any $\alpha > 0$. That is, the invariant measure μ is always finite.*

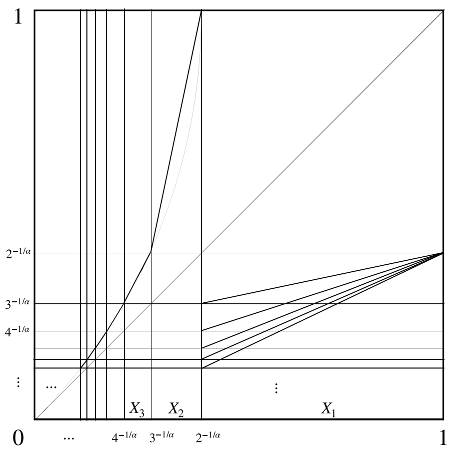

Figure 3: A graph of transformations in Example 2.6 and 2.7 (infinitely many branches on X_1 correspond to $T_i|_{X_1}$, $i \in I$, respectively)

We modify Example 2.6 and we show that the modified random dynamics admit both finite and σ-finite infinite acim depending on the parameter $\alpha > 0$. The critical point of α is different from the deterministic case.

Example 2.7. *Let $\alpha > 0$ and $k \geq 2$ a natural number. Set $I = \mathbb{N}$ and $p_i = 1/2^i$. Then we put $J_i = \{2, 3, \ldots, 2^{k(i+1)}\}$. As the same way in Example 2.6, we have the ergodic σ-finite acim μ for this random iteration. Furthermore we can calculate that μ is finite if and only if $\alpha < k/(k-1)$.*

Remark 2.8. *For random iterations of non-uniformly expanding maps as Example 2.7 and 2.9, the critical value of α where the invariant measure varies from finite to infinite* *can be different from deterministic case ([Th80]). Similar example which is expanding on average except a small neighborhood of an indifferent fixed point can be seen in Example 6.2 of [I20].*

In the last two examples, most of points will rarely return to the indifferent fixed point 0 so that the acim hard to become infinite. Then, conversely, we will see that the following example makes the acim tend to become an infinite measure.

Example 2.9. *Let $\alpha > 0$, $I = \mathbb{N}$ and $k(i) \geq 2$ increasing natural numbers of $i \in I$. We set $J_i = \{j \in \mathbb{N} : j \geq k(i)\}$. Then for any probability vector $\{p_i\}_{i \in I}$, the ergodic σ-finite acim μ for this random iteration satisfies that μ is finite if and only if $\alpha < 1$ and $\sum_i p_i k(i) < \infty$. Thus, for example, if $p_i = 6/(i^2\pi^2)$ and $k(i) = 2 + [i^\gamma]$ for some $\gamma > 0$, then μ is finite if and only if $\alpha < 1$ and $0 < \gamma < 1$.*

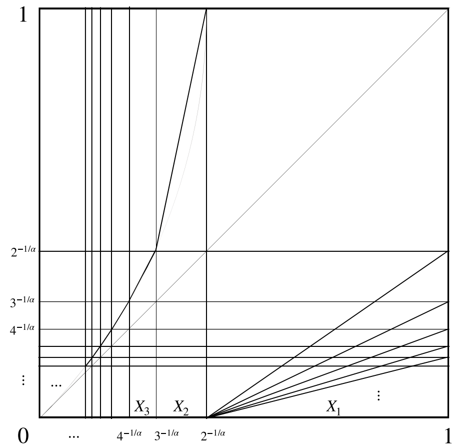

Figure 4: A graph of transformations in Example 2.9 (infinitely many branches on X_1 correspond to $T_i\,|_{X_1}$, $i \in I$, respectively)

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