Kazama-Suzuki Coset Vertex Superalgebras at Admissible Levels

Ryo SATO (joint work with Shinji Koshida)

Institute of Mathematics, Academia Sinica

1 Introduction

1.1 Verlinde Formula for Irrational Theories

One of the most fundamental problems in the study of two-dimensional conformal field theories (CFTs) is to compute *fusion rules* among primary fields. Each model of CFT specifies a vertex operator superalgebra as its chiral symmetry algebra. Mathematically, primary fields in a given model are identified with highest weight modules of the vertex operator superalgebra and their fusion rules are defined by the corresponding 3-point conformal blocks on the genus zero curve $\mathbf{P}^1(\mathbb{C})$ (see e.g. [Zhu94]). When the vertex operator superalgebra satisfies certain good conditions (*rationality* and C_2*-cofiniteness*), fusion rules can be computed in terms of the *modular* S*-matrix* via the *Verlinde formula*, which is originally proposed by E. Verlinde in [Ver88] and proved by Y.-Z. Huang in [Hua08]. Such models are referred to as *rational* CFTs and have been extensively studied in various contexts.

Recently, a certain class of irrational chiral CFTs turns out to play an important role in the conjectural 2d/4d correspondence firstly investigated by C. Beem et. al. in [BLL+15]. One of the best studied irrational chiral CFTs is the Wess–Zumino–Novikov–Witten (WZNW) model associated with the simple affine vertex operator algebra $L_k(\mathfrak{sl}_2)$, where k is a special rational number known as a Kac–Wakimoto admissible level (see [KW88] for detail). D. Adamović and A. Milas proved in [AM95] that irreducible highest weight $L_k(\mathfrak{sl}_2)$-modules coincide with the Kac–Wakimoto admissible highest weight modules of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ of level k. Based on the celebrated modular invariance property of the Kac–Wakimoto characters, one can define the corresponding modular S-matrix. However, there are no known general results about relationships between fusion rules and the modular S-matrix. In fact, a naive application of the Verlinde formula computes negative fusion rules from the modular S-matrix. To fix this failure, T. Creutzig and D. Ridout in [CR13] computed an extended *modular* S*-data* and proposed a conjectural Verlinde formula. We note that this conjectural Verlinde formula is verified only for ordinary ($=\mathfrak{sl}_2$-integrable) highest weight $L_k(\mathfrak{sl}_2)$-modules by T. Creutzig, Y.-Z. Huang, and J. Yang in [CHY18].

1.2 Our Result

In this note we consider a super-analog of Creutzig–Ridout's conjecture. More precisely, we construct the $\mathcal{N}=2$ superconformal vertex operator superalgebra $\mathbb{L}_{c(k)}$ from the affine vertex operator algebra $L_k(\mathfrak{sl}_2)$ by the Kazama–Suzuki supersymmetric coset construction and formulate an analogous Verlinde formula for $\mathbb{L}_{c(k)}$. Based on the modular S-data associated with irreducible highest weight $\mathbb{L}_{c(k)}$-modules at a general Kac–Wakimoto admissible level

$$k=-2+\frac{p}{p'} \quad \big((p,p')\in\mathbb{Z}_{\geq 2}\times\mathbb{Z}_{\geq 2}\colon \text{coprime}\big)$$

computed by the author in [Sat19], we obtain the following result:

Theorem 1.1. Fusion rules among irreducible highest weight $\mathbb{L}_{c(-\frac{1}{2})}$-modules are correctly computed from the modular S-data in [Sat19] via the conjectural Verlinde formula.

We divide our computation of fusion rules into two steps. First, we estimate the fusion rules from above by using the *Frenkel–Zhu theory*. Roughly speaking, the Zhu algebra associated to a vertex operator superalgebra V is an associative superalgebra $\mathsf{Zhu}(V)$ generated by the degree preserving adjoint actions of V. More generally, in [FZ92], I. Frenkel and Y. Zhu constructed $\mathsf{Zhu}(V)$-bimodules from highest weight V-modules. Frenkel–Zhu's bimodules have nice applications to the computation of fusion rules. For more details of the Frenkel–Zhu theory, we refer the reader to [KW94, DLM98a, DLM98b, Li99, DZ06].

Second, we estimate the fusion rules from below by using a certain *free field realization.* By using the oscillator representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_2=\widehat{\mathfrak{sp}}_2$ of level $k=-\frac{1}{2}$, we identify the simple vertex operator algebra $L_{-\frac{1}{2}}(\mathfrak{sl}_2)$ with a $\mathbb{Z}_2$-orbifold of the free bosonic $\beta\gamma$-system. Then, by the Kazama–Suzuki coset construction, we realize the simple vertex operator superalgebra $\mathbb{L}_{c(-\frac{1}{2})}$ as a Heisenberg coset of a $\mathbb{Z}_2$-orbifold of the tensor product of the $\beta\gamma$-system and the free fermionic bc-system. This construction enables us to find non-trivial intertwining operators (cf. [Zhu94, Proposition 7.4]) and completes our computation.

The detail of the proof will appear soon in a substantially revised version of our earlier draft [KS18].

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2 Superconformal Vertex Algebra

2.1 Notation

We first recall the vertex operator superalgebra associated with the two-dimensional $\mathcal{N}=2$ superconformal algebra of central charge $c\in\mathbb{C}$.

Proposition 2.1 ([Ada99])**.** There exists a unique vertex superalgebra $\mathbb{V}^c$ which is strongly generated by $\mathbb{Z}_2$-homogeneous fields $\mathbf{L}(z),\mathbf{J}(z),\mathbf{G}^+(z),\mathbf{G}^-(z)$ with parity

$$\mathsf{deg}\big(\mathbf{L}(z)\big)=\mathsf{deg}\big(\mathbf{J}(z)\big)=\bar{0},\quad \mathsf{deg}\big(\mathbf{G}^{\pm}(z)\big)=\bar{1}$$

only subject to the following operator product expansions:

$$\begin{aligned}
&\mathbf{L}(z)\mathbf{L}(w) \sim \frac{c/2}{(z-w)^4} + \frac{2\mathbf{L}(w)}{(z-w)^2} + \frac{\partial_w \mathbf{L}(w)}{z-w},\\
&\mathbf{L}(z)\mathbf{G}^{\pm}(w) \sim \frac{\frac{3}{2}\mathbf{G}^{\pm}(w)}{(z-w)^2} + \frac{\partial_w \mathbf{G}^{\pm}(w)}{z-w}, \quad \mathbf{L}(z)\mathbf{J}(w) \sim \frac{\mathbf{J}(w)}{(z-w)^2} + \frac{\partial_w \mathbf{J}(w)}{z-w},\\
&\mathbf{J}(z)\mathbf{J}(w) \sim \frac{c/3}{(z-w)^2}, \quad \mathbf{J}(z)\mathbf{G}^{\pm}(w) \sim \frac{\pm\mathbf{G}^{\pm}(w)}{z-w},\\
&\mathbf{G}^{\pm}(z)\mathbf{G}^{\pm}(w) \sim \frac{2c/3}{(z-w)^3} + \frac{2\mathbf{L}(w) + 2\mathbf{J}(w)}{(z-w)^2} + \frac{\partial_w \mathbf{J}(w)}{z-w}.
\end{aligned}$$

In addition, the Virasoro field $\mathbf{L}(z)$ gives rise to a vertex operator superalgebra structure of central charge c on $\mathbb{V}^c$.

Remark 2.2. Every $\mathbb{V}^c$-module admits a natural action of the Neveu-Schwarz sector of the $\mathcal{N}=2$ superconformal algebra generated by

$$L_n := \mathbf{L}_{(n+1)}, \quad G^{\pm}_r := \mathbf{G}^{\pm}_{(r+\frac{1}{2})}, \quad J_n := \mathbf{J}_{(n)},$$

where

$$A_{(n)} := \oint z^n A(z) \frac{\mathrm{d}z}{2\pi\sqrt{-1}}.$$

The Verma module $\mathcal{M}_{h,j,c}$ of highest weight $(h,j,c) \in \mathbb{C}^3$ is a $\mathbb{Z}_2$-graded $\mathbb{V}^c$-module freely generated by an even vector $|h,j,c\rangle$ subject to the relations

$$\begin{aligned}
&L_n\,|h,j,c\rangle = G^{\pm}_r\,|h,j,c\rangle = J_n\,|h,j,c\rangle := 0 \quad (n,r>0),\\
&L_0\,|h,j,c\rangle := h\,|h,j,c\rangle\,, \quad J_0\,|h,j,c\rangle := j\,|h,j,c\rangle\,, \quad C\,|h,j,c\rangle := c\,|h,j,c\rangle\,.
\end{aligned}$$

We write $\mathcal{L}_{h,j,c}$ for the irreducible quotient $\mathbb{V}^c$-module of $\mathcal{M}_{h,j,c}$.

2.2 Oscillator Realization

Let $V^k(\mathfrak{sl}_2)$ be the universal affine vertex algebra generated by $\mathbf{E},\mathbf{H},\mathbf{F}$ and $\pi^k_{\mathfrak{h}}$ its Heisenberg vertex subalgebra generated by $\mathbf{H}$. Let $\langle\beta\gamma\rangle$ and $\langle bc\rangle$ denote the free bosonic $\beta\gamma$-system and the free fermionic bc-system, respectively. By using the Kazama–Suzuki coset construction (see e.g. [Ada99])

$$\mathbb{V}^{\frac{3k}{k+2}} \otimes \pi^{k+2}_{\mathfrak{h}} \to V^k(\mathfrak{sl}_2) \otimes \langle bc\rangle,$$

which is uniquely determined by

$$\mathbf{G}^+ \otimes \mathbf{1} \mapsto \sqrt{\tfrac{2}{k+2}}\mathbf{E}\otimes b, \quad \mathbf{G}^- \otimes \mathbf{1} \mapsto \sqrt{\tfrac{2}{k+2}}\mathbf{F}\otimes c, \quad \mathbf{1}\otimes\mathbf{H} \mapsto \mathbf{H}\otimes\mathbf{1} + \mathbf{1}\otimes 2b_{(-1)}c,$$

and the oscillator representation (see e.g. [FF85])

$$V^{-\frac{1}{2}}(\mathfrak{sl}_2) \to \langle\beta\gamma\rangle; \quad \mathbf{E}\mapsto -\beta_{(-1)}\beta, \quad \mathbf{H}\mapsto -\beta_{(-1)}\gamma, \quad \mathbf{F}\mapsto \gamma_{(-1)}\gamma,$$

we construct a free field realization of $\mathbb{V}^{-1}$ as follows.

Lemma 2.3. There exist a unique vertex algebra homomorphism

$$\mathbb{V}^{-1} \otimes \pi_{\mathfrak{h}}^{\frac{3}{2}} \to \langle \beta\gamma \rangle \otimes \langle bc \rangle \tag{2.1}$$

such that

$$\begin{aligned} \mathbf{G}^{+} \otimes \mathbf{1} &\mapsto -\frac{1}{\sqrt{3}} \beta_{(-1)}\beta \otimes b, \\ \mathbf{G}^{-} \otimes \mathbf{1} &\mapsto \frac{1}{\sqrt{3}} \gamma_{(-1)}\gamma \otimes c, \\ \mathbf{1} \otimes \mathbf{H} &\mapsto \phi^{-} \otimes \mathbf{1} + \mathbf{1} \otimes 2\phi^{+}, \end{aligned}$$

where $\phi^{-} := -\beta_{(-1)}\gamma$ and $\phi^{+} := b_{(-1)}c$.

Let $\mathbb{L}_c$ denote the simple quotient vertex operator superalgebra of $\mathbb{V}^c$.

Corollary 2.4. The map (2.1) induces a vertex superalgebra isomorphism

$$\mathbb{L}_{-1} \simeq C := \mathsf{Com}\big(\pi_{\mathfrak{h}}^{\frac{3}{2}}, \langle \beta\gamma \rangle \otimes \langle bc \rangle\big).$$

Proof. One can verify that the induced homomorphism $\mathbb{V}^{-1} \to C$ is surjective by comparing their graded dimensions. Since $\pi_{\mathfrak{h}}^{\frac{3}{2}}$ and $\langle \beta\gamma \rangle \otimes \langle bc \rangle$ are simple, so is C (see e.g. [ACKL17, Lemma 2.1]). This completes the proof. □

2.3 Spectral Flow Twist

In this subsection V denotes $\langle \beta\gamma \rangle$ or $\mathbb{L}_c$ and $\mathbf{A}$ denotes ϕ^{-} or $\mathbf{J}$, respectively. Since $A_0 := \mathbf{A}_{(0)}$ acts diagonally on V with integer eigenvalues and fixes the conformal vector, $\sigma := \mathsf{exp}(\pi\sqrt{-1}A_0)$ gives a vertex operator superalgebra involution of V. In addition, one can consider the notion of strongly ($\mathbb{C}$-)graded V-modules in the sense of [HLZ14, Definition 2.25] with respect to the operator A_0. A strongly graded V-module M admits A_0-eigenspace decomposition

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$$

and each component M_λ further decomposes into finite-dimensional L_0-eigenspaces with the lower truncation condition. Such modules play an important role in a non-C_2-cofinite situation. In fact, neither $\langle \beta\gamma \rangle$ nor $\mathbb{L}_{-1}$ is C_2-cofinite.

Now we recall the following construction known as "spectral flow twist".

Lemma 2.5 ([Li97, Proposition 2,1])**.** Let $\varepsilon, \varepsilon' \in \{0, \frac{1}{2}\}$ and $\theta \in \mathbb{Z} + \varepsilon'$. For a $\sigma^{1-2\varepsilon}$-twisted strongly graded V-module (M, Y_M), we define

$$\Delta(\theta\mathbf{A}; z) := z^{\theta A_0} \mathsf{exp}\Big(\sum_{\ell=1}^{\infty} \frac{\theta A_\ell}{-\ell}(-z)^{-\ell}\Big) \in \mathsf{End}(V)[\![z^{\pm(1-\varepsilon')}]\!].$$

Then the even mapping

$$Y_{M^\theta}(?; z) := Y_M(\Delta(\theta\mathbf{A}; z)?; z) : V \to \mathsf{End}(M)[\![z^{\pm(\frac{1}{2}+|\varepsilon-\varepsilon'|)}]\!]$$

gives rise to a new $\sigma^{1-2|\varepsilon-\varepsilon'|}$-twisted strongly graded V-module $M^\theta := (M, Y_{M^\theta})$, called a spectral flow twisted module of M.

Fusion rules are compatible with the spectral flow twist in the following sense.

Lemma 2.6 ([Li97, Proposition 2.4])**.** There exists a natural isomorphism

$$\mathrm{IO}\binom{N}{L\ M} \simeq \mathrm{IO}\binom{N^{\theta+\theta'}}{L^{\theta}\ M^{\theta'}}$$

for any $\mathbb{Z}_2$-graded strongly graded V-modules L, M, N, and $\theta \in \frac{1}{2}\mathbb{Z}$, where $\mathrm{IO}\binom{N}{L\ M}$ denotes the space of $\mathbb{Z}_2$-graded intertwining operators of type $\binom{N}{L\ M}$.

Since all the irreducible strongly graded $\mathbb{L}_c$-modules are highest weight modules and they are closed under taking the spectral flow twist, we only need to consider a complete system of representatives with respect to the twist.

3 Fusion Rules at Central Charge $c = -1$

In this section, we determine all the fusion rules among ($\mathbb{Z}_2$-graded) irreducible highest weight $\mathbb{L}_{-1}$-modules. Our computation is divided into two steps:

1. estimation from above by using the Frenkel–Zhu theory;
2. estimation from below by using the free field realization.

The Zhu algebra $\mathsf{Zhu}(V)$ for a vertex operator superalgebra V is firstly introduced in [KW94] as a natural generalization of the original one in [Zhu96]. When $V = \mathbb{V}^c$ or $\mathbb{L}_c$, the following isomorphism is well-known (e.g. [Ada99, Remark 1.1]) and is proved in the same way as [KW94, Lemma 3.1].

Lemma 3.1. For $c \in \mathbb{C}$, there exists a unique algebra isomomorphism

$$\mathbb{C}[\mathsf{h}, \mathsf{q}] \xrightarrow{\simeq} \mathsf{Zhu}(\mathbb{V}_c); \quad \mathsf{h} \mapsto [\mathbf{L}], \quad \mathsf{q} \mapsto [\mathbf{J}],$$

where $[A]$ denotes the natural image of $A \in \mathbb{V}_c$ in the quotient space $\mathsf{Zhu}(\mathbb{V}_c)$. Moreover, $\mathsf{Zhu}(\mathbb{L}_c)$ is obtained as a quotient algebra of $\mathsf{Zhu}(\mathbb{V}_c) \simeq \mathbb{C}[\mathsf{h}, \mathsf{q}]$.

Let $\mathsf{Irr}(\mathbb{L}_c)$ denote the set of isomorphism classes of $\mathbb{Z}_2$-graded irreducible highest weight $\mathbb{L}_c$-modules and $\mathsf{Irr}(\mathbb{L}_c)/\mathbb{Z}$ denote the set of $\mathbb{Z}$-orbits with respect to the $\mathbb{Z}$-action induced by the spectral flow twist. From the classification result in [Ada99, Theorem 7.2] and direct calculation, we obtain the following.

Proposition 3.2. The disjoint union

$$\left\{\mathcal{L}(\epsilon) := \mathcal{L}_{\frac{4}{3}\epsilon^2, \frac{4}{3}\epsilon, -1}, \mathbf{\Pi}\mathcal{L}(\epsilon) \,\middle|\, \epsilon \in \{0, \pm\tfrac{1}{2}\}\right\} \sqcup \left\{\mathcal{L}_j := \mathcal{L}_{-\frac{1}{8}-\frac{2}{3}j^2, \frac{4}{3}j, -1}, \mathbf{\Pi}\mathcal{L}_j \,\middle|\, j \in \mathbb{C}\right\}$$

$$\left(\text{resp. } \left\{\mathcal{L}(\epsilon), \mathbf{\Pi}\mathcal{L}(\epsilon) \,\middle|\, \epsilon \in \{0, \tfrac{1}{2}\}\right\} \sqcup \left\{\mathcal{L}_j, \mathbf{\Pi}\mathcal{L}_j \,\middle|\, j \in S\right\}\right)$$

gives a complete representative set of $\mathsf{Irr}(\mathbb{L}_c)$ (resp. $\mathsf{Irr}(\mathbb{L}_c)/\mathbb{Z}$), where

$$S := \left\{x + y\sqrt{-1} \,\middle|\, -\tfrac{1}{2} \leq x < \tfrac{1}{2},\ x \notin \{\pm\tfrac{1}{4}\},\ y \in \mathbb{R}\right\}$$

and $\mathbf{\Pi}$ stands for the $\mathbb{Z}_2$-parity reversing functor.

The corresponding irreducible left $\mathsf{Zhu}(\mathbb{L}_{-1})$-modules are denoted by

$$\left\{\mathbb{C}(\epsilon), \mathbf{\Pi}\mathbb{C}(\epsilon) \,\middle|\, \epsilon \in \{0, \pm\tfrac{1}{2}\}\right\} \sqcup \left\{\mathbb{C}_j, \mathbf{\Pi}\mathbb{C}_j \,\middle|\, j \in \mathbb{C}\right\}.$$

3.1 Estimate from Above

We set a $\mathbb{Z}_2$-graded structure on

$$\mathsf{M} := \bigoplus_{a,b\in\{0,1\}} \mathbb{C}[x_\ell, x_r, y]\psi_{a,b},$$

by $\mathsf{deg}(f\psi_{a,b}) = \overline{a+b}$ for $f \in \mathbb{C}[x_\ell, x_r, y]$. For $j \in \mathbb{C}$, we define a $\mathbb{Z}_2$-graded $\mathbb{C}[\mathsf{h},\mathsf{q}]$-bimodule structure π_j on M by

$$\mathsf{h}.f\psi_{a,b} := x_\ell f\psi_{a,b},\ \ \mathsf{q}.f\psi_{a,b} := (y+j+a-b)f\psi_{a,b},$$
$$f\psi_{a,b}.\mathsf{h} := x_r f\psi_{a,b},\ \ f\psi_{a,b}.\mathsf{q} := yf\psi_{a,b}.$$

Then there exists a unique $\mathbb{Z}_2$-graded $\mathbb{C}[\mathsf{h},\mathsf{q}]$-bimodule isomorphism

$$\mathsf{M}_j := (\mathsf{M}, \pi_j) \xrightarrow{\simeq} \mathsf{Zhu}(\mathcal{M}_{h,j,c})$$

such that $\psi_{a,b} \mapsto \left[(G^+_{-\frac{1}{2}})^a(G^-_{-\frac{1}{2}})^b\,|h,j,c\rangle\,\right]$ for $a, b \in \{0,1\}$.

Lemma 3.3. The kernel of the natural surjective $\mathbb{C}[\mathsf{h},\mathsf{q}]$-bimodule homomorphism

$$\mathsf{M}_{\frac{2}{3}} \simeq \mathsf{Zhu}\big(\mathcal{M}_{\frac{1}{3},\frac{2}{3},-1}\big) \to \mathsf{Zhu}\big(\mathcal{L}(\tfrac{1}{2})\big)$$

has a set of explicit generators.

Lemma 3.4. Assume that $j \in S$ and set $s := \frac{4}{3}j$. Then the following vector

$$\chi_1 := \big(4(s-1)L_{-1} + 3(s-1)(s+1)J_{-1} + 4G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\big)\,|{-\tfrac{1}{8}}(1+3s^2), s, -1\rangle$$

gives an even singular vector in $\mathcal{M}_{-\frac{1}{8}(1+3s^2),s,-1}$. In addition, the kernel of the natural surjective $\mathbb{C}[\mathsf{h},\mathsf{q}]$-bimodule homomorphism

$$\mathsf{M}_s \simeq \mathsf{Zhu}(\mathcal{M}_{-\frac{1}{8}(1+3s^2),s,-1}) \to \mathsf{Zhu}\big(\mathcal{M}_{-\frac{1}{8}(1+3s^2),s,-1}/\langle\chi_1\rangle\big)$$

has a set of explicit generators.

By Lemma 3.3, Lemma 3.4, and [Li99, Proposition 2.10], we obtain the following.

Proposition 3.5 (Koshida–S.)**.** For $\epsilon, \epsilon' \in \{0, \pm\frac{1}{2}\}$, $j, j' \in S$, and $j'' \notin \{\frac{1}{4}\} + \frac{1}{2}\mathbb{Z}$, we have

$$\dim \mathrm{IO}\begin{pmatrix} \mathcal{L}(\epsilon')^\theta \\ \mathcal{L}(\frac{1}{2})\ \mathcal{L}(\epsilon)\end{pmatrix} = \begin{cases}(1|0) & \text{if } \epsilon' = \epsilon + \frac{1}{2} \text{ and } \theta = 0\\ (0|1) & \text{if } (\epsilon, \epsilon') = (\frac{1}{2}, 0) \text{ and } \theta = -1,\\ (0|0) & \text{otherwise,}\end{cases}$$

$$\dim \mathrm{IO}\begin{pmatrix} \mathcal{L}_{j''} \\ \mathcal{L}(\frac{1}{2})\ \mathcal{L}(\epsilon)\end{pmatrix} = (0|0),$$

$$\dim \mathrm{IO}\begin{pmatrix} \mathcal{L}_{j''} \\ \mathcal{L}(\frac{1}{2})\ \mathcal{L}_j\end{pmatrix} \le \begin{cases}(1|0) & \text{if } j'' = j + \frac{1}{2},\\ (0|0) & \text{otherwise,}\end{cases}$$

$$\dim \mathrm{IO}\begin{pmatrix} \mathcal{L}_{j''} \\ \mathcal{L}_j\ \mathcal{L}_{j'}\end{pmatrix} \le \begin{cases}(0|1) & \text{if } j'' = j + j' \pm \frac{3}{4},\\ (0|0) & \text{otherwise.}\end{cases}$$

Remark 3.6. Since we have isomorphisms

$$\mathcal{L}(\pm\tfrac{1}{2})^{\pm 1} \simeq \mathbf{\Pi}\mathcal{L}(\mp\tfrac{1}{2}), \quad \mathcal{L}(0)^\vee \simeq \mathcal{L}(0), \quad \mathcal{L}(\pm\tfrac{1}{2})^\vee \simeq \mathcal{L}(\mp\tfrac{1}{2}),$$

where $(?)^\vee$ stands for the contragredient dual, the symmetry of intertwining operators (see e.g. [HL95, §7]) and Lemma 2.6 imply

$$\mathrm{IO}\begin{pmatrix} \mathcal{L}(0) \\ \mathcal{L}(\frac{1}{2})\ \mathcal{L}(-\frac{1}{2})\end{pmatrix} \simeq \mathrm{IO}\begin{pmatrix} \mathcal{L}(\frac{1}{2}) \\ \mathcal{L}(\frac{1}{2})\ \mathcal{L}(0)\end{pmatrix} \simeq \mathrm{IO}\begin{pmatrix} \mathcal{L}(\frac{1}{2}) \\ \mathcal{L}(0)\ \mathcal{L}(\frac{1}{2})\end{pmatrix} \simeq \mathbb{C}^{1|0}$$

and

$$\mathrm{IO}\begin{pmatrix} \mathcal{L}(0)^{-1} \\ \mathcal{L}(\frac{1}{2})\ \mathcal{L}(\frac{1}{2})\end{pmatrix} \simeq \mathrm{IO}\begin{pmatrix} \mathcal{L}(0) \\ \mathcal{L}(\frac{1}{2})\ \mathbf{\Pi}\mathcal{L}(-\frac{1}{2})\end{pmatrix} \simeq \mathbb{C}^{0|1}.$$

3.2 Estimate from Below

Let $\Lambda = \mathbb{Z}a^+ + \mathbb{Z}a^-$ be a lattice equipped with the non-degenerate symmetric integer form defined by $\langle a^\pm, a^\pm\rangle = \pm 1$ and $\langle a^+, a^-\rangle = 0$. The Friedan–Martinec–Shenker (FMS) bosonization is an embedding of $\langle\beta\gamma\rangle$ to the lattice vertex algebra V_Λ defined by:

$$\langle\beta\gamma\rangle \to V_\Lambda; \quad \beta \mapsto |a^+ + a^-\rangle\,, \quad \gamma \mapsto -a^+_{-1}\,|-a^+ - a^-\rangle\,.$$

For $(\ell, \lambda) \in \mathbb{C}^2$, we define

$$\Pi_\ell(\lambda) := \Pi\big(\ell a^- + \lambda(a^+ + a^-) + \mathbb{Z}(a^+ + a^-)\big) = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}_{\Lambda, \ell a^- + (n+\lambda)(a^+ + a^-)},$$

where $\mathcal{F}_{\Lambda,\alpha}$ stands for the Heisenberg Fock $\mathcal{F}_\Lambda$-module of highest weight α. Then this carries a natural twisted $\langle\beta\gamma\rangle$-module structure via the FMS bosonization. More precisely, we obtain the next lemma.

Lemma 3.7. Let σ be the involution of $\langle\beta\gamma\rangle$ defined in §2.3. For $\ell \in \frac{1}{2}\mathbb{Z}$, the vector space $\Pi_\ell(\lambda)$ carries a natural $\sigma^{2\ell}$-twisted strongly graded $\langle\beta\gamma\rangle$-module structure. Moreover, the $\sigma^{2\ell}$-twisted $\langle\beta\gamma\rangle$-module $\Pi_\ell(\lambda)$ is irreducible if $\lambda \notin \mathbb{Z}$.

Now, by using Corollary 2.4, we obtain the following.

Proposition 3.8. For $j \in S$ and $b \in \mathbb{Z}$, we have isomorphisms

$$\langle\beta\gamma\rangle \otimes V_\mathbb{Z} \simeq \bigoplus_{\theta\in\mathbb{Z}} \mathbf{\Pi}^\theta \left(\mathcal{L}(0)^{-\theta} \otimes \mathcal{F}_{\mathbb{Z},\sqrt{\frac{4}{3}}\theta} \oplus \mathcal{L}(\tfrac{1}{2})^{-\theta} \otimes \mathcal{F}_{\mathbb{Z},\sqrt{\frac{4}{3}}(\theta+\frac{1}{2})}\right)$$

and

$$\Pi_{\frac{1-b}{2}}(2j + \tfrac{1}{2}) \otimes V_\mathbb{Z} \simeq \bigoplus_{\theta\in\frac{1}{2}\mathbb{Z}} \mathbf{\Pi}^b \mathcal{L}_{j+\theta} \otimes \mathcal{F}_{\mathbb{Z},\sqrt{\frac{4}{3}}(j+\theta+\frac{3b}{4})}$$

of $\mathbb{Z}_2$-graded strongly graded $\mathbb{L}_{-1} \otimes \mathcal{F}_\mathbb{Z}$-modules.

The next proposition follows from the result by D. Adamović and V. Pedić.

Proposition 3.9 ([AP19])**.** For any

$$(\ell_i, \lambda_i) \in \mathbb{Z} \times (\mathbb{C} \setminus \mathbb{Z}) \quad (i \in \{1,2,3\},\ \lambda_1 + \lambda_2 \notin \mathbb{Z}),$$

we obtain the following fusion rule among $\langle\beta\gamma\rangle$-modules:

$$\dim \mathrm{IO}\begin{pmatrix} \Pi_\ell(\lambda_3) \\ \Pi_{\frac{1}{2}}(\lambda_1)\ \ \Pi_{\frac{1}{2}}(\lambda_2) \end{pmatrix} = \begin{cases} 1 & \text{if } \ell \in \{0,1\} \text{ and } \lambda_3 = \lambda_1 + \lambda_2, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.10 (Koshida–S.)**.** Assume that

$$j, j_1, j_2 \in S, \quad j_3 := j_1 + j_2 \pm \frac{3}{4} \notin \left\{\frac{1}{4}\right\} + \frac{1}{2}\mathbb{Z}.$$

Then we have

$$\dim \mathrm{IO}\begin{pmatrix} \mathcal{L}_{j+\frac{1}{2}} \\ \mathcal{L}(\frac{1}{2})\ \mathcal{L}_j \end{pmatrix} = (1|0), \quad \dim \mathrm{IO}\begin{pmatrix} \mathcal{L}_{j_3} \\ \mathcal{L}_{j_1}\ \mathcal{L}_{j_2} \end{pmatrix} = (0|1).$$

Moreover, the Verlinde formula in the sense of [Sat19, Conjecture 5.1] correctly computes the above fusion rules.

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Institute of Mathematics
Academia Sinica
Taipei 106319
TAIWAN
E-mail address: `sato@gate.sinica.edu.tw`