FUSION RULES OF LATTICE COSETS WITH AN APPLICATION TO FEIGIN–SEMIKHATOV DUALITY

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1. Introduction

Let $\mathfrak{g}$ be a basic classical simple Lie superalgebra, k a complex number and f an even nilpotent element in $\mathfrak{g}$. Then one associates with the universal affine vertex superalgebra $V^k(\mathfrak{g})$ at level k the universal $\mathcal{W}$-superalgebra $\mathcal{W}^k(\mathfrak{g}, f)$ via the quantum Hamiltonian redcution [20]. In particular, the $\mathcal{W}$-superalgebras associated with a principal nilpotent element f_{prin} (which is unique up to conjugations), are conventionally denoted by $\mathcal{W}^k(\mathfrak{g})$. Affine vertex superalgebras and their $\mathcal{W}$-superalgebras are most important families of vertex superalgebras due to their essential role in various aspects of representation theory, geometry, and physics, including knot invariants [32], the geometric Langlands program [24] and invariants of low dimensional manifolds [13, 21], or invariants of three and four dimensional supersymmetric quantum field theories [4, 25].

The representation theory of the $\mathcal{W}$-superalgebras remains mysterious in general. One of the reasons is that the defining Operator Product Expansions of their strong generators are not known in general except for some cases, e.g., the Virasoro vertex algebra $\mathcal{W}^k(\mathfrak{sl}_2)$, the so-called W_3-algebra $\mathcal{W}^k(\mathfrak{sl}_3)$, for the even cases and the $\mathcal{N}=1$ super Virasoro algebra $\mathcal{W}^k(\mathfrak{osp}_{1|2})$ or the $\mathcal{N}=2$ super Virasoro algebra $\mathcal{W}^k(\mathfrak{sl}_{1|2})$ for the super cases. In the last decades, the representation theory of the *exceptional* $\mathcal{W}$-algebras (see [11] for the definition) has been developed by Arakawa and his collaborators [5, 6, 8, 10, 11, 14]. This is based on the analysis of the *associated variety* of a vertex algebra and the representation theory of the affine vertex algebras at *admissible* levels. For type A, the (simple) principal and subregular $\mathcal{W}$-algebras

$$\mathcal{W}_{-n+\frac{n+q}{n+p}}(\mathfrak{sl}_n), \quad \mathcal{W}_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{\mathrm{sub}}),$$

(p, q and r are non-negative integers satisfying $\gcd(n+p, n+q)=1$ and $\gcd(n+r, n-1)=1$, respectively) are rational and C_2-cofinite. (Here the notation $\mathcal{W}_k(\mathfrak{g}, f)$ means the unique simple quotient of $\mathcal{W}^k(\mathfrak{g}, f)$). Thus the categories of their ordinary modules, which are semisimple, admit a natural structure of braided tenor category. Then the Grothendieck groups of these categories have ring structures by the tensor product, called the fusion rings. The fusion rings of the principal $\mathcal{W}$-algebras for general (p, q) and the subregular $\mathcal{W}$-algebras for even n are described in terms of the fusion ring of the simple affine vertex algebra $L_k(\mathfrak{sl}_n)$

$$\mathcal{K}\left(\mathcal{W}_{-n+\frac{n+q}{n+p}}(\mathfrak{sl}_n)\right) \simeq \mathcal{K}(L_p(\mathfrak{sl}_n)) \bigotimes_{\mathbb{Z}[\mathbb{Z}_n]} \mathcal{K}(L_q(\mathfrak{sl}_n)), \quad \mathcal{K}\left(\mathcal{W}_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n)\right) \simeq \mathcal{K}(L_r(\mathfrak{sl}_n))$$

by [10, 14] and [11], respectively.

In contrast, the representation theory of $\mathcal{W}$-*super*algebras has not been developed yet except for the cases of the $\mathcal{N}=1,2$ super Virasoro algebra [1, 2, 3]. On of the reason seems that the representation theory of affine vertex superalgebras themselves are already quite difficult. Indeed, the representation theory of the $\mathcal{N}=2$ super Virasoro algebra, which is $\mathcal{W}^k(\mathfrak{sl}_{1|2})$, has been developed not based on the affine vertex superalgebra $V^k(\mathfrak{sl}_{1|2})$, but on the *duality*: the Kazama–Suzuki coset construction [28] and the Feigin–Semikhatov–Tipunin construction [23], which assert the isomorphisms of vertex superalgebras

$$\mathcal{W}^\ell(\mathfrak{sl}_{1|2}) \simeq \mathrm{Com}(\pi, V^k(\mathfrak{sl}_2) \otimes V_{\mathbb{Z}}), \quad V^k(\mathfrak{sl}_2) \simeq \mathrm{Com}(\pi, W^\ell(\mathfrak{sl}_{1|2}) \otimes V_{\sqrt{-1}\mathbb{Z}}), \tag{1.1}$$

where the levels $k, \ell \in \mathbb{C}$ are related by the equation $(k+2)(\ell+1)=1$ and π is a rank one Heisenberg vertex algebra embedded diagonally into the tensor products. These two construction imply a certain equivalence of their module categories.

In this article, we consider the representation theory of the principal $\mathcal{W}$-superalgebra $\mathcal{W}^\ell(\mathfrak{sl}_{1|n})$ and give a sufficient condition for the problem

(1) Determine the levels $\ell \in \mathbb{C}$ when the simple $\mathcal{W}$-superalgebra $\mathcal{W}_\ell(\mathfrak{sl}_{1|n})$ is rational and C_2-cofinite; and then give an answer to the following fundamental problem

(2) Classify the simple ordinary modules and describe the fusion ring at the above levels.

For the first problem, we start with identifying the partner of $\mathcal{W}^\ell(\mathfrak{sl}_{1|n})$, which plays the role of $V^k(\mathfrak{sl}_2)$ in the case $n = 2$. For the second problem, we use the theory of simple current extensions.

2. Feigin–Semikhatov conjecture

In [22], Feigin and Semikhatov introduced a family of vertex algebras, called the $\mathcal{W}_n^{(2)}$-algebras, which are proven later to be the subregular $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}})$ [26], and gave the following conjecture:

Conjecture. (Feigin and Semikhatov [22]) *Let π_{H_1} be the rank one Heisenberg subalgebra of $\mathcal{W}^{k_1}(\mathfrak{sl}_n, f_{\mathrm{sub}})$ and π_{H_2} the rank one Heisenberg subalgebra of $\mathcal{W}^{k_2}(\mathfrak{sl}_{1|n})$. Then we have the following.*

(1) *For generic levels $(k_1, k_2) \in \mathbb{C}^2$ satisfying $(k_1 + n)(k_2 + n - 1) = 1$,*

$$\mathrm{Com}\left(\pi_{H_1}, \mathcal{W}^{k_1}(\mathfrak{sl}_n, f_{\mathrm{sub}})\right) \simeq \mathrm{Com}\left(\pi_{H_2}, \mathcal{W}^{k_2}(\mathfrak{sl}_{1|n})\right).$$

(2) *For generic levels $(k_1, k_2) \in \mathbb{C}^2$ satisfying $\frac{1}{k_1+n} + \frac{1}{k_3+n} = 1$*[1]*,*

$$\mathrm{Com}\left(\pi_{H_1}, \mathcal{W}^{k_1}(\mathfrak{sl}_n, f_{\mathrm{sub}})\right) \simeq \mathrm{Com}\left(V^{k_3}(\mathfrak{gl}_n), V^{k_3}(\mathfrak{sl}_{1|n})\right).$$

There are some variants of this triality among affine cosets of $\mathcal{W}$-superalgebras (or affine vertex superalgebras). The most well-known one is the triality as follows:

$$\mathcal{W}^{\ell_1}(\mathfrak{sl}_n) \simeq \mathcal{W}^{\ell_2}(\mathfrak{sl}_n), \quad \text{if } (\ell_1 + n)(\ell_2 + n) = 1, \tag{2.1}$$

$$\mathcal{W}^{\ell_1}(\mathfrak{sl}_n) \simeq \mathrm{Com}\left(V^{\ell_3+1}(\mathfrak{sl}_n), V^{\ell_3}(\mathfrak{sl}_n) \otimes L_1(\mathfrak{sl}_n)\right), \quad \text{if } \frac{1}{\ell_1 + n} + \frac{1}{-(\ell_3 + n)} = 1. \tag{2.2}$$

The first duality (2.1) is the famous Feigin–Frenkel duality of the principal $\mathcal{W}$-algebras for $\mathfrak{sl}_n$ and the second one (2.2) is the higher rank generalization of the famous Goddard–Kent–Olive construction of the Virasoro vertex algebra proved by Arakawa, Creutzig, and Linshaw quite recently. See [9] for details on this triality. From this, one can see that Conjecture (1) is a variant of the Feigin–Frenkel duality for *non-principal* $\mathcal{W}$-algebras and that Conjecture (2) is a variant of the Goddard–Kent–Olive construction. We note that the appearance of the sign $-(\ell_3 + n)$ in (2.2) in contrast to Conjecture (2) is explained by the fact that the bare affine vertex algebra appearing in (2.2) is not $V^{\ell_3}(\mathfrak{sl}_n)$ itself, but $V^{\ell_3}(\mathfrak{sl}_{0|n})$ which is isomorphic to $V^{-\ell_3}(\mathfrak{sl}_{n|0}) = V^{-\ell_3}(\mathfrak{sl}_n)$. These two trialities are just the simplest examples of conjectural trialities made by Gaiotto and Rapčák in the relation to four dimensional super Yang–Mills theories [25].

Theorem 1 ([19, 15])**.** *For $(k, \ell) \in \mathbb{C}^2$ satisfying*

$$(k, \ell) \neq \left(-n + \frac{n}{n-1}, \frac{(n-1)^2}{n}\right), \quad (k_1 + n)(k_2 + n - 1) = 1,$$

we have isomorphisms of vertex algebras

(1) $\mathrm{Com}\left(\pi_{H_1}, \mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}})\right) \simeq \mathrm{Com}\left(\pi_{H_2}, \mathcal{W}^\ell(\mathfrak{sl}_{1|n})\right)$,

(2) $\mathrm{Com}\left(\pi_{H_1}, \mathcal{W}_k(\mathfrak{sl}_n, f_{\mathrm{sub}})\right) \simeq \mathrm{Com}\left(\pi_{H_2}, \mathcal{W}_\ell(\mathfrak{sl}_{1|n})\right)$.

The following gives a stronger version of Theorem 1, which generalizes (1.1):

Theorem 2 ([15])**.** *For $(k_1, k_2) \in \mathbb{C}^2$ satisfying $(k_1 + n)(k_2 + n - 1) = 1$, we have isomorphisms of vertex superalgebras*

(1) $\mathcal{W}^{k_2}(\mathfrak{sl}_{1|n}) \simeq \mathrm{Com}\left(\pi_{\widetilde{H}_1}, \mathcal{W}^{k_1}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}\right)$,

(2) $\mathcal{W}^{k_1}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \simeq \mathrm{Com}\left(\pi_{\widetilde{H}_2}, \mathcal{W}^{k_2}(\mathfrak{sl}_{1|n}) \otimes V_{\mathbb{Z}\sqrt{-1}}\right)$,

(3) $\mathcal{W}_{k_2}(\mathfrak{sl}_{1|n}) \simeq \mathrm{Com}\left(\pi_{\widetilde{H}_1}, \mathcal{W}_{k_1}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}\right)$,

[1] Although there are some conventions for describing levels of affine vertex superalgebras, a correct relation seems $\frac{1}{k_1+n} + \frac{1}{k_3+(1-n)} = 1$.

(4) $\mathcal{W}_{k_1}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \simeq \mathrm{Com}\left(\pi_{\widetilde{H}_2}, \mathcal{W}_{k_2}(\mathfrak{sl}_{1|n}) \otimes V_{\mathbb{Z}\sqrt{-1}}\right)$,

where $\pi_{\widetilde{H}_i}$, $(i = 1, 2)$, is a Heisenberg vertex subalgebra diagonally embedded into the tensor products.

The proofs of these two theorems use *free field realizations* of the relevant superalgebras. See [15] or Genra's article in this proceeding for details.

3. Rational cases

By [7, 11], the subregular $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{sl}_n, f_{\mathrm{sub}})$ is rational and C_2-cofinite if the level k is of the form

$$k = -n + \frac{n+r}{n-1}, \quad r \in \mathbb{Z}_{\geq 0},\ \gcd(n-1, n+r) = 1.$$

For $r \in \mathbb{Z}_{\geq 3}$, Creuztig and Linshaw [19] determines its Heisenberg coset, namely,

$$\mathrm{Com}\left(\pi, \mathcal{W}_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{\mathrm{sub}})\right) \simeq \mathcal{W}_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r), \tag{3.1}$$

where the right-hand side is the principal $\mathcal{W}$-algebra at a nondegenerate admissible level $-r + \frac{r+n}{r+1}$, which is rational and C_2-cofinite [7, 6]. The classification of the simple ordinary modules and description of the fusion rules among them are establsished in [6] and [10, 14], respectively. It is conveniently written in terms of the fusion rings:

$$\mathcal{K}\left(\mathcal{W}_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r)\right) \simeq \mathcal{K}(L_n(\mathfrak{sl}_r)). \tag{3.2}$$

Here $L_n(\mathfrak{sl}_r)$ is the simple affine vertex algebra of $\mathfrak{sl}_r$ at level r, whose simple ordinary modules are simple integrable highest weight modules $L_n(\Lambda)$ with

$$\Lambda = \sum_{i=0}^{r-1} a_i \Lambda_i, \quad \left(a_i \in \mathbb{Z}_{\geq 0},\ \sum_{i=0}^{r-1} a_i = n\right),$$

where Λ_i denotes the i-th fundamental weight. The modules $\{L_n(n\Lambda_i)\}_{i\in\mathbb{Z}_r}$ are known as *simple currents*, that is, invertible objects under the fusion product (in the language of braided tensor cagtegory under the assumption that the unit object is simple). Moreover, they satisfy

$$L_n(n\Lambda_i) \boxtimes L_n(\lambda) \simeq L_n(\sigma^i.\lambda),$$

where σ acts on the weight lattice $\widehat{P} = \oplus_{j\in\mathbb{Z}_r}\mathbb{Z}\Lambda_j$ by

$$\sigma. \sum_{j\in\mathbb{Z}_r} a_i\Lambda_i = \sum_{i\in\mathbb{Z}_r} a_i \Lambda_{i+1}.$$

Let $\mathbb{L}_{\mathcal{W}}(\lambda)$ denote the $\mathcal{W}_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r)$-module corresponding to $L_n(\lambda)$ under the isomorphism (3.2). Then $\{\mathbb{L}_{\mathcal{W}}(n\Lambda_i)\}_{i\in\mathbb{Z}_r}$ are simple currents. By using these modules and the coset (3.1), one can decompose the subregular $\mathcal{W}$-algebra into

$$\mathcal{W}_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \simeq \bigoplus_{a\in\mathbb{Z}_r} \mathbb{L}_{\mathrm{W}}(n\Lambda_a) \otimes V_{\frac{na}{\sqrt{nr}}+\sqrt{nr}\mathbb{Z}} \tag{3.3}$$

as modules of $\mathcal{W}_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r) \otimes V_{\sqrt{nr}\mathbb{Z}}$, which extends $\mathcal{W}_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r) \otimes \pi$. Then the Kazama–Suzuki type coset construction Theorem 2 (3) implies the following similar decomposition for the principal $\mathcal{W}$-superalgebra for $r \in \mathbb{Z}_{\geq 3}$:

$$\mathcal{W}_{-n+\frac{n-1}{n+r}}(\mathfrak{sl}_{1|n}) \simeq \bigoplus_{a\in\mathbb{Z}_r} \mathbb{L}_{\mathrm{W}}(n\Lambda_a) \otimes V_{\frac{(n+r)a}{\sqrt{(n+r)n}}+\sqrt{(n+r)n}\mathbb{Z}} \tag{3.4}$$

as modules of $\mathcal{W}_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r) \otimes V_{\sqrt{(n+r)r}\mathbb{Z}}$, which extends $\mathcal{W}_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r) \otimes \pi$. Therefore, the principal $\mathcal{W}$-superalgebra $\mathcal{W}_{-n+\frac{n-1}{n+r}}(\mathfrak{sl}_{1|n})$ is a simple current extension of the vertex operator superalgebra $\mathcal{W}_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r) \otimes V_{\sqrt{(n+r)r}\mathbb{Z}}$.

4. Categorical aspects of Simple currents

Let V be a $\mathbb{Z}_{\geq 0}$-graded, simple, rational and C_2-cofinite vertex operator algebra of CFT type. Let $\{S_g\}_{g\in G}$ be a set of (non-isomorphic) simple currents of V-modules, parametrized by a finite abelian group G, that is,

$$S_e = V, \quad S_g \boxtimes S_h \simeq S_{gh}.$$

Then the V-module structure on the direct sum $\mathcal{E} = \oplus_{g\in G} S_g$ may be extended to a simple vertex operator algebra structure which is compatible with the fusion product. Such a condition is studied by [12, 27] and turns out equivalent to the condition that $\mathcal{E}$ is a commutative algebra object in the category V-mod of ordinary V-modules as a braided tensor category with some additional conditions. Furthermore, properties like semisimplicity of the module category of $\mathcal{E}$ (as a vertex operator algebra) are controlled by V-mod and written in the language of braided tensor category. Hence, we start with a purely categorical treatment of this theory, which enables us to treat the super case as well. See [16] for details. We note that the results in this and the next sections have been more or less studied within the theory of vertex operator algebras and its relation to the theory of braided tensor category, e.g., [12, 17, 18, 27, 29, 34, 33] and thus there are no essentially new results. Nevertheless, the author hopes that the abstraction of the simple currents in the braided tensor (super)category helps some clarification of the theory.

Let $\mathcal{C}$ be an (essentially small, $\mathbb{C}$-linear) monoidal supercategory such that all the even morphisms admit kernels and cokernels. In this article, we call $\mathcal{C}$ a braided tensor supercategory. We note that we do not assume that parity-inhomogeneous morphisms admit kernels and cokernels since it seldom holds for natural module categories of superalgebras. Actually, we do not need this stronger assumption, but we stress that the standard notion like, kernels, cokernels, subobjects or simplicity still makes sense when one is working over the even morphisms, that is, the structure morphisms of kernels, cokernels or subobjects are *even* morphisms and the simplicity of an object is defined in terms of this class of subobjects.

Assume that the unit object $1_{\mathcal{C}}$ is simple. It follows that an arbitrary invertible object in $\mathcal{C}$ is simple. In this case, we call an invertible object an *simple current*, following the theory of vertex algebras. Given a simple current S and an arbitrary object $M \in Ob(\mathcal{C})$, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(M, M) \simeq \mathrm{Hom}_{\mathcal{C}}(S \boxtimes M, S \boxtimes M), \quad f \mapsto \mathrm{id} \boxtimes f. \tag{4.1}$$

Recall that we have the braiding (isomorphism) $\mathcal{R}_{A,B} \colon A \boxtimes B \to B \boxtimes A$ for each pair of objects $A, B \in Ob(\mathcal{C})$ and the monodromy (isomorphism) $\mathcal{M}_{A,B} := \mathcal{R}_{B,A} \circ \mathcal{R}_{A,B}$, which commutes with the endomorphisms on $A \boxtimes B$ of the form $f \boxtimes g$. In particular, the monodoromy $\mathcal{M}_{S,M}$ in the right-hand side of (4.1) defines an endmorphism m_S in the left-hand side. If S is of finite order, that is, $S^{\boxtimes n} \simeq 1_{\mathcal{C}}$ for some $n \geq 1$, then $m_S^n = \mathrm{id}$, which implies an eigenspace decomposition of the Hom (super)space $\mathrm{Hom}_{\mathcal{C}}(M, M)$ and therefore,

$$M = \bigoplus_{a\in\mathbb{Z}_n} M_a, \quad M_a = \mathrm{Ker}_M(m_S - \zeta^a \,\mathrm{id}_M),$$

where $\zeta = \exp(2\pi\sqrt{-1}/n)$. It follows that we have a decomposition of $\mathcal{C}$ into full subcategories

$$\mathcal{C} = \bigoplus_{a\in\mathbb{Z}_n} \mathcal{C}_a, \quad \mathcal{C}_a = \{M \mid m_S = \zeta^a \,\mathrm{id}_M\}.$$

If we have a set of simple currents $\{S_g\}_{g\in G}$ parametrized by a finite abelian group G, then the above decomposition for each S_g gives rise to the simultaneous decomposition

$$\mathcal{C} = \bigoplus_{\xi\in G^\vee} \mathcal{C}_\xi, \quad \mathcal{C}_\xi = \{M \mid m_{S_g} = \xi(g) \,\mathrm{id}_M\},$$

where $G^\vee$ denotes the dual group $\mathrm{Hom}(G, \mathbb{C}^\times)$. Hence, if the tensor product $\boxtimes$ is bi-exact, then the fusion ring $\mathcal{K}(\mathcal{C})$ has a structure of $G^\vee$-grading by monodromy and $\mathbb{Z}[G]$-algebra structure by fusion product (with $\{S_g\}_{g\in G}$).

Now, suppose that the direct sum of the above simple currents $\mathcal{E} = \bigoplus_{g\in G} S_g$ has a structure of commutative algebra object, that is, we have an even morphism

$$\mu_{\mathcal{E}} \colon \mathcal{E} \boxtimes \mathcal{E} \to \mathcal{E},$$

which makes the following three diagrams commutative:

$$
\begin{array}{ccc}
1_{\mathcal{C}} \boxtimes \mathcal{E} & \longrightarrow & \mathcal{E} \boxtimes \mathcal{E} \\
& \searrow^{\simeq} & \downarrow \mu_{\mathcal{E}} \\
& & \mathcal{E},
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{E} \boxtimes (\mathcal{E} \boxtimes \mathcal{E}) & \xrightarrow{\mathrm{id} \boxtimes \mu_{\mathcal{E}}} & \mathcal{E} \boxtimes \mathcal{E} & \xrightarrow{\mu_{\mathcal{E}}} & \mathcal{E} \\
\simeq \downarrow & & & \nearrow_{\mu_{\mathcal{E}}} & \\
(\mathcal{E} \boxtimes \mathcal{E}) \boxtimes \mathcal{E} & \xrightarrow{\mu_{\mathcal{E}} \boxtimes \mathrm{id}} & \mathcal{E} \boxtimes \mathcal{E}, & &
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{E} \boxtimes \mathcal{E} & \xrightarrow{\mathcal{R}_{\mathcal{E},\mathcal{E}}} & \mathcal{E} \boxtimes \mathcal{E} \\
& \searrow^{\mu_{\mathcal{E}}} & \downarrow \mu_{\mathcal{E}} \\
& & \mathcal{E}.
\end{array}
$$

The meanings of the three diagrams are, the unit axiom, the associativity axiom, and the commutativity axioms respectively as in the case of the usual commutative algebras. We note that since we are working on the *supercategory*, the commutativity is actually means the *super-commutativity* of the algebra in the usual sense. An $\mathcal{E}$-module object in $\mathcal{C}$ is defined as a pair (M, μ_M) of $M \in Ob(\mathcal{C})$ and $\mu_M \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{E} \boxtimes M, M)$ which makes the following two diagrams commutative:

$$
\begin{array}{ccc}
1_{\mathcal{C}} \boxtimes M & \longrightarrow & \mathcal{E} \boxtimes M \\
& \searrow^{\simeq} & \downarrow \mu_M \\
& & M,
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{E} \boxtimes (\mathcal{E} \boxtimes M) & \xrightarrow{\mathrm{id} \boxtimes \mu_M} & \mathcal{E} \boxtimes M & \xrightarrow{\mu_M} & M \\
\simeq \downarrow & & & \nearrow_{\mu_M} & \\
(\mathcal{E} \boxtimes \mathcal{E}) \boxtimes M & \xrightarrow{\mu_{\mathcal{E}} \boxtimes \mathrm{id}} & \mathcal{E} \boxtimes M. & &
\end{array}
$$

An $\mathcal{E}$-module (M, μ_M) is called *local* if it further makes the following diagram commutative:

$$
\begin{array}{ccc}
\mathcal{E} \boxtimes M & \xrightarrow{\mathcal{M}_{\mathcal{E},M}} & \mathcal{E} \boxtimes M \\
& \searrow^{\mu_M} & \downarrow \mu_M \\
& & M,
\end{array}
$$

Let $\mathrm{Rep}(\mathcal{E})$ denote the (super)category of $\mathcal{E}$-module objects in $\mathcal{C}$ and $\mathrm{Rep}^0(\mathcal{E}) \subset \mathrm{Rep}(\mathcal{E})$ the full subcategory consisting of local $\mathcal{E}$-module objects. Then by [18, 16], $\mathrm{Rep}(\mathcal{E})$ has a natural structure of tensor supercategory induced by the that of $\mathcal{C}$ and $\mathrm{Rep}^0(\mathcal{E})$ a natural structure of braided tensor supercategory.

Example 3. *A typical example of a braided tensor supercategory is the module category A-mod of a unital, (super)commutative, (associative) $\mathbb{C}$-algebra A. Here we do not assume the simplicity of the unit object $1_{A\text{-mod}} = A$. The tensor product of the modules M and N is just the tensor product over A*

$$M \otimes_A N = M \otimes_{\mathbb{C}} N / \mathrm{span}\{am \otimes n - (-1)^{\bar{a}\bar{m}} m \otimes an \mid a \in A,\ m \in M, n \in N\}$$

and the braiding is just the swapping of the factors

$$\mathcal{R}_{M,N} \colon M \otimes N \to N \otimes M, \quad m \otimes n \mapsto (-1)^{\bar{m}\bar{n}} n \otimes m.$$

Here $\bar{\bullet} \in \mathbb{Z}_2$ denotes the parity of $\bullet$. Hence the monodromy $\mathcal{R}_{M,N}$ is just the identity $\mathrm{id}_{M \otimes_A N}$. Then the commutative algebra object B is just the A-(super)algebra and $\mathrm{Rep}(B)$ is naturally isomorphic to the supercategory of B-modules. In this case, $\mathrm{Rep}(B)$ coincides with $\mathrm{Rep}^0(B)$.

In general, we have the so-called induction functors

$$\mathcal{E} \boxtimes \bullet \colon \mathcal{C} \to \mathrm{Rep}(\mathcal{E}), \quad \mathcal{E} \boxtimes \bullet \colon \mathcal{C}_0 \to \mathrm{Rep}^0(\mathcal{E}),$$

where $\mathcal{C}_0 \subset \mathcal{C}$ denotes the full subcategory consisting of objects M such that $\mathcal{M}_{\mathcal{E},M} = \mathrm{id}_{\mathcal{E} \boxtimes M}$. This functor is a tensor (super)functor and is a straightforward generalization of the extension of coefficients $B \otimes_A \bullet$ in the above example.

We consider the case when the direct sum $\mathcal{E} = \oplus_{g \in G} S_g$ of the simple currents has a structure of commutative algebra object. In this article, we always assume that the restriction of the structure morphism $\mu_{\mathcal{E}} \colon S_g \boxtimes S_h \to S_{gh}$ does not vanish. The following is fundamental:

Proposition 4. *Suppose that $\mathcal{C}$ is of finite length and that the action of G on the set $\mathrm{Irr}(\mathcal{C})$ of isomorphism classes of simple objects in $\mathcal{C}$ induced by the fusion product is fixed-point free. Then $\mathcal{C}$ is semisimple if and only if $\mathrm{Rep}(\mathcal{E})$ is semisimple and moreover, every simple object M in $\mathrm{Rep}(\mathcal{E})$ is isomorphic to $\mathcal{E} \boxtimes N$ for some $N \in \mathrm{Irr}(\mathcal{C})$.*

An immediate corollary of this proposition is as follows:

Corollary 5. *Suppose that $\mathcal{C}$ satisfies the assumptions in Proposition 4 and that the fusion products $\boxtimes$ on $\mathcal{C}$ and $\boxtimes_{\mathcal{E}}$ on $\mathrm{Rep}(\mathcal{E})$ are bi-exact. Then we have an isomorphism of associative algebras*

$$\mathcal{K}(\mathrm{Rep}(\mathcal{E})) \simeq \mathcal{K}(\mathcal{C}) \otimes_{\mathbb{Z}[G]} \mathbb{Z},$$

which restricts to

$$\mathcal{K}(\mathrm{Rep}^0(\mathcal{E})) \simeq \left(\mathcal{K}(\mathcal{C}) \otimes_{\mathbb{Z}[G]} \mathbb{Z}\right)^G,$$

where the right-hand side means the trivial $G^\vee$-grading part.

5. Fusion rules of lattice cosets

We apply the previous results to the theory of vertex superalgebras.

Let V a simple $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded C_2-cofinite vertex operator superalgebras of CFT type. Here we do not assume that the $\frac{1}{2}\mathbb{Z}_{\geq 0}$-grading does not necessarily induce the super structure. Indeed, we have the $\mathbb{Z}_{\geq 0}$-grading and the $\mathbb{Z}_2$-grading (due to the super-structure)

$$V = \bigoplus_{\Delta \in \frac{1}{2}\mathbb{Z}_{\geq 0}} V_\Delta, \quad V = V_{\bar{0}} \oplus V_{\bar{1}},$$

which are compatible in the sense that

$$V_\Delta = V_{\Delta,\bar{0}} \oplus V_{\Delta,\bar{1}}, \quad V_{\Delta,a} = V_\Delta \cap V_a,\ (a \in \mathbb{Z}_2).$$

Then $V_{\mathbb{Z},\bar{0}} := \bigoplus_{\Delta\in\mathbb{Z}} V_{\Delta,\bar{0}}$ is a vertex operator algebra of CFT type, which is simple and C_2-cofinite by [30] since it is the fixed-point subalgebra of two involutions $\Pi = \mathrm{id}_{V_{\bar{0}}} - \mathrm{id}_{V_{\bar{1}}}$ and $\theta = \exp(2\pi\sqrt{-1}L_0)$, where L_0 denotes the grading operator of the $\frac{1}{2}\mathbb{Z}_{\geq 0}$-grading. Then the category $V_{\mathbb{Z},\bar{0}}$-mod of the grading-restricted generalized $V_{\mathbb{Z},\bar{0}}$-(super)modules has a structure of braided tensor (super)category which is $\mathbb{C}$-linear finite as an abelian (super)category by a general result of Huang, Lepowsky and Zhang. Then the original vertex superalgebra V is a commutative algebra object in V-mod and the category V-mod of the grading-restricted generalized V-(super)modules is naturally isomorphic to the category $\mathrm{Rep}^0(\mathcal{E})$ of local V-module objects in $V_{\mathbb{Z},\bar{0}}$-mod as braideded tensor supercategory and thus $\mathbb{C}$-linear finite as an abelian supercategory.

Let L be a $\mathbb{Z}$-lattice of finite rank equipped with a positive-definite $\mathbb{Z}$-bilinear form $(\cdot|\cdot)$. Then one may associate it with a lattice vertex operator superalgebra V_L. It is well-known that V_L is simple, $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded, rational, and C_2-cofinite and that V_L-mod coincides with the category of ordinary V-modules whose simple objects are $\mathrm{Irr}(V_L) = \{V_{a+L} \mid a \in L'/L\}$ where $L' = \{\mathbb{Q}\otimes L \mid (a|L) \subset \mathbb{Z}\}$ is the dual lattice of L. Furthermore, the fusion ring $\mathcal{K}(V_L) := \mathcal{K}(V_L\text{-mod})$ is isomorphic to a group ring

$$\mathcal{K}(V_L) \simeq \mathbb{Z}[L'/L], \quad [V_{a+L}] \mapsto [a].$$

We consider a simple current extension of the tensor product $V \otimes V_L$ of the form

$$\mathcal{E} = \bigoplus_{a \in N/L} S_a \otimes V_{a+L},$$

where N is a $\mathbb{Z}$-lattice such that $L \subset N \subset L'$ and $\{S_a\}_{a\in N/L}$ is a set of simple currents in V-mod parametrized by the finite abelian group N/L. Then we have the following.

Proposition 6 ([18, 12])**.** *Suppose that the twist $\theta_a := \exp(2\pi\sqrt{-1}L_0)|_{\mathcal{E}_a}$, $(a \in N/L)$, satisfies $\theta_a^2 = \mathrm{id}$ and $\theta_a\theta_b = \theta_{a+b}$. Then the vertex operator superalgebra $\mathcal{E}$ is simple, $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded vertex operator superalgebra. Moreover, $\mathcal{E}$ is rational if V is rational.*

We note that the first statement follows from our setting, the second statement is a corollary of Proposition 4.

For simplicity, let us assume that V is rational, (equivalently V-mod is semisimple). Then the fusion product $\boxtimes_V$ in V-mod is obviously bi-exact and so is $\boxtimes_{\mathcal{E}}$ on $\mathrm{Rep}(E)$ and $\mathrm{Rep}^0(\mathcal{E}) \simeq \mathcal{E}$-mod by Proposition 4. Then Proposition 5 implies

$$\mathcal{K}(\mathrm{Rep}(\mathcal{E})) \simeq \mathcal{K}(V \otimes V_L) \bigotimes_{\mathbb{Z}[N/L]} \mathbb{Z} \simeq (\mathcal{K}(V) \otimes \mathbb{Z}[L'/L]) \bigotimes_{\mathbb{Z}[N/L]} \mathbb{Z} \simeq \mathcal{K}(V) \bigotimes_{\mathbb{Z}[N/L]} \mathbb{Z}[L'/L],$$

where the tensor product in the rightmost-hand side means $[M \boxtimes_V S_a] \otimes [b] = [M] \otimes [-a+b]$. Then it implies

$$\mathcal{K}(E) \simeq \mathcal{K}(\mathrm{Rep}^0(\mathcal{E})) \simeq \left(\mathcal{K}(V) \bigotimes_{\mathbb{Z}[N/L]} \mathbb{Z}[L'/L]\right)^{N/L}.$$

In order to derive an inverse formula which describes $\mathcal{K}(V)$ in terms of $\mathcal{K}(\mathcal{E})$, we use the induction functor

$$\mathcal{E} \boxtimes \bullet \colon V\text{-mod} \to \mathrm{Rep}(\mathcal{E}),$$

which is an embedding since the simple currents $S_a \otimes V_{a+L}$, $(a \in N/L)$, are fixed-point free. Therefore, $\mathcal{K}(V)$ is a subalgebra of $\mathcal{K}(\mathrm{Rep}(\mathcal{E}))$. By using the forgetful functor $\mathrm{Rep}(\mathcal{E}) \to (V \otimes V_L)$-mod, we have a monodromy action of $V \otimes V_{a+L}$ $(a \in L'/L)$. Then the objects of V-mod among those of $\mathrm{Rep}(\mathcal{E})$ are characterized as the monodromy-free objects with respect to V_{a+L} $(a \in N'/L)$, where $N' \subset L'$ is the dual lattice of N. Thus we obtain an isomorphism $\mathcal{K}(V) \simeq \mathcal{K}(\mathrm{Rep}(\mathcal{E}))^{N'/L}$. Next, we describe the ring $\mathcal{K}(\mathrm{Rep}(\mathcal{E}))$ in terms of $\mathrm{Rep}^0(\mathcal{E})$. Note that $\mathrm{Rep}(\mathcal{E})$ has a set of simple currents $\mathcal{E}_a := \mathcal{E} \boxtimes_{V \otimes V_L} (V \otimes V_{a+L})$, $(a \in L'/L)$. Introduce full subcategories $\mathrm{Rep}^a(\mathcal{E})$ $(a \in L'/L)$ by the image of the fusion product with $\mathcal{E}_a$,

$$\mathrm{Rep}^a(\mathcal{E}) := \mathrm{Im}(\mathcal{E}_a \boxtimes \bullet \colon \mathrm{Rep}^0(\mathcal{E}) \to \mathrm{Rep}(\mathcal{E})).$$

Since $\mathcal{E}_a$ is an invertible object, the functor $\mathcal{E}_a \boxtimes \bullet$ is exact and thus gives an equivalence of abelian supercategories

$$\mathcal{E}_a \boxtimes \bullet \colon \mathrm{Rep}^0(\mathcal{E}) \xrightarrow{\simeq} \mathrm{Rep}^a(\mathcal{E}).$$

Note that $\mathcal{E}_a \boxtimes \bullet$ stabilizes $\mathrm{Rep}^0(\mathcal{E})$ if and only if $\mathcal{E}_a \in Ob(\mathrm{Rep}^0(\mathcal{E}))$, that is, $a \in N'/L$. Thanks to Proposition 4, we obtain a decomposition

$$\mathrm{Rep}(\mathcal{E}) = \bigoplus_{a \in L'/N'} \mathrm{Rep}^a(\mathcal{E}),$$

which satisfies $\mathcal{E}_a \boxtimes \bullet \colon \mathrm{Rep}^b(\mathcal{E}) \simeq \mathrm{Rep}^{a+b}(\mathcal{E})$. It follows that

$$\mathcal{K}(\mathrm{Rep}(\mathcal{E})) \simeq \mathcal{K}(\mathrm{Rep}^0(\mathcal{E})) \bigotimes_{\mathbb{Z}[N'/L]} \mathbb{Z}[L'/L]$$

and thus

$$\mathcal{K}(V) \simeq \mathcal{K}(\mathrm{Rep}(\mathcal{E}))^{N'/L} \simeq \left(\mathcal{K}(\mathrm{Rep}^0(\mathcal{E})) \bigotimes_{\mathbb{Z}[N'/L]} \mathbb{Z}[L'/L] \right)^{N'/L}.$$

To summarize, we have obtained the following formulas:

Theorem 7 (cf.[18, 33])**.** [2] *Suppose that V is rational. Then we have isomorphisms of associative algebras*

$$\mathcal{K}(\mathcal{E}) \simeq \left(\mathcal{K}(V) \bigotimes_{\mathbb{Z}[N/L]} \mathbb{Z}[L'/L] \right)^{N/L}, \quad \mathcal{K}(V) \simeq \left(\mathcal{K}(\mathcal{E}) \bigotimes_{\mathbb{Z}[N'/L]} \mathbb{Z}[L'/L] \right)^{N'/L}.$$

6. Applications

In this final section, we apply the formulas in Theorem 7 to several simple current extensions.

6.1. **Subregular $\mathcal{W}$-algebras.** We apply Theorem 7 to the simple current extension (3.3):

$$\mathcal{K}\left(\mathcal{W}_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{\mathrm{sub}})\right) \simeq \left(\mathcal{K}(\mathcal{W}_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r)) \bigotimes_{\mathbb{Z}[\mathbb{Z}_r]} \mathbb{Z}[\mathbb{Z}_{nr}] \right)^{\mathbb{Z}_n} \simeq \left(\mathcal{K}(L_n(\mathfrak{sl}_r)) \bigotimes_{\mathbb{Z}[\mathbb{Z}_r]} \mathbb{Z}[\mathbb{Z}_{nr}] \right)^{\mathbb{Z}_n}.$$

By using the Level-Rank duality of the fusion ring of affine vertex algebras [16, 31], we obtain the following:

Theorem 8 ([16])**.** *For $r \in \mathbb{Z}_{\geq 3}$ such that $\gcd(r+r, n-1) = 1$, we have an isomorphism of fusion rings*

$$\mathcal{K}\left(\mathcal{W}_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{\mathrm{sub}})\right) \simeq \mathcal{K}(L_r(\mathfrak{sl}_n)).$$

The above theorem generalizes a result in [11] for the cases when n is even and $r \in \mathbb{Z}_{\geq 0}$.

[2] This theorem holds beyond the rational cases. Indeed, the rigidity of V-mod and the self-duality of V is sufficient.

6.2. **Principal $\mathcal{W}$-superalgebras.** We first apply Proposition 4 to the simple current extension (3.4) and obtain the following:

Theorem 9 ([15, 16]). *For $r \in \mathbb{Z}_{\geq 3}$ such that $\gcd(r+r, n-1) = 1$, then the principal $\mathcal{W}$-superalgebra $\mathcal{W}_{-(n-1)+\frac{n-1}{n+r}}(\mathfrak{sl}_{1|n})$ is rational and C_2-cofinite.*

Next, we apply Theorem 7 to the simple current extension (3.4). It follows that the Kazama–Suzuki type coset construction Theorem 2 (3) implies a simple current extension

$$\mathcal{W}_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}} \supset \mathcal{W}_{-(n-1)+\frac{n-1}{n+r}}(\mathfrak{sl}_{1|n}) \otimes V_{\sqrt{(n+r)n}\mathbb{Z}},$$

such that the simple currents appearing in the decomposition is parametrized by the finite abelian group $\frac{n}{\sqrt{(n+r)n}}\mathbb{Z}/\sqrt{(n+r)n}\mathbb{Z} \simeq \mathbb{Z}_{n+r}$. Hence Theorem 7 implies the following:

Theorem 10 ([16]). *For $r \in \mathbb{Z}_{\geq 3}$ such that $\gcd(r+r, n-1) = 1$, we have an isomorphism of associative algebras*

$$\mathcal{K}\left(\mathcal{W}_{-(n-1)+\frac{n-1}{n+r}}(\mathfrak{sl}_{1|n})\right) \simeq \left(\mathcal{K}(L_r(\mathfrak{sl}_n)) \bigotimes_{\mathbb{Z}[\mathbb{Z}_n]} \mathbb{Z}[\mathbb{Z}_{(n+r)n}]\right)^{\mathbb{Z}_n}.$$

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