On (α, q)-Poisson Operators and Their Distributions

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Abstract

We propose an (α, q)-analogue of the Poisson operator on the Fock space of Bożejko-Ejsmont-Hasebe [6] and discuss a probability law of the operator. We show that the probability law is expressed by the q-Meixner distribution. Our results contain not only symmetric distributions as in [6], but also non-symmetric ones such as free Poisson, q and q^2-deformations of Poisson, Pascal, Gamma, and Meixner distributions. This is the summary of the paper [4].

1 Introduction

A deformation of the full Fock space with two parameters $\alpha, q \in (-1, 1)$, namely, the (α, q)-Fock space (or the Fock space of type B) $\mathcal{F}_{\alpha,q}(\mathscr{H})$ over a complex Hilbert space $\mathscr{H}$ is considered by Bożejko-Ejsmont-Hasebe [6]. The $(0, q)$-Fock space is the q-Fock space in the sense of [7][8]. Their starting point is to replace the Coxeter group of type A, that is, symmetric group $\mathfrak{S}_n$ for the q-Fock space by the Coxeter group of type B, $B(n) := \mathbb{Z}_2^n \rtimes \mathfrak{S}_n$, to construct $\mathcal{F}_{\alpha,q}(\mathscr{H})$ equipped with the $(\alpha.q)$-inner product $\langle \cdot, \cdot \rangle_{\alpha,q}$. This replacement allows us to define, for $f \in \mathscr{H}$, the (α, q)-creation $B^{\dagger}_{\alpha,q}(f)$ and annihilation $B_{\alpha,q}(f)$ operators acting on $\mathcal{F}_{\alpha,q}(\mathscr{H})$ and leads us to the problem finding a probability distribution $\nu_{\alpha,q}$ on $\mathbb{R}$ of the (α, q)-Gaussian operator (the Gaussian operator of type B) with respect to the vacuum state.

We propose an (α, q)-analogue of the Poisson operator on $\mathcal{F}_{\alpha,q}(\mathscr{H})$ and discuss a probability law of this operator. We remark that the Poisson operator of type B by Ejsmont [11] is essentially different from ours if $\alpha \neq 0$. In Section 3, we introduce a weighted $(-q, q^2)$-Poisson operator $\mathbf{Y}_{-q,q^2}$ after relationships between q-Meixner operator $\mathbf{X}_q$ of [17] and our (α, q^2)-Poisson operator are explained. We show in our main Theorem 3.1 that the probability law of $\mathbf{Y}_{-q,q^2}$ is equal to that of the scaled Meixner operator $\mathbf{Y}_q = \frac{\mathbf{X}_q}{1+q}$ with respect to appropriate vacuum states. As a result, one can treat not only symmetric distributions as in [6], but also non-symmetric ones such as free Poisson, q and q^2-deformations of Poisson, Pascal, Gamma, and Meixner distribution. Consult our paper [4] in detail.

2 (α, q)-operators and their distributions

2.1 q-analogue of the Meixner class of orthogonal polynomials

Let $[n]_q$ denote the q-number given by $[n]_q := 1 + q + \cdots q^{n-1}$ for $n \geq 1$. For given constants $q, \kappa_1, \kappa_2, \gamma, \delta$ with $0 \leq q < 1$, $\kappa_2 > 0$, $\delta \geq 0$, let $\mathbf{m}_q$ denote the probability measure

$\mu(q;\kappa_1,\kappa_2,\gamma,\delta)$ on $\mathbb{R}$ such that the sequence of monic polynomials $\{Q_n^{(q)}(x)\}$ given by the recurrence relation,

$$\begin{cases} Q_0^{(q)}(x) = 1, \quad Q_1^{(q)}(x) = x - \kappa_1, \\ xQ_n^{(q)}(x) = Q_{n+1}^{(q)}(x) + \big(\kappa_2 + \delta[n-1]_q\big)[n]_q Q_{n-1}^{(q)}(x) + (\kappa_1 + \gamma[n]_q)\, Q_n^{(q)}(x), \quad n \geq 1. \end{cases} \tag{2.1}$$

is orthogonal with respect to the $L^2(\mathbf{m}_q)$-inner product. We shall refer the measure $\mathbf{m}_q$ as the q-Meixner distribution. The existence of probability measure $\mathbf{m}_q$ is guaranteed by Favard's theorem, for example, in [10][12]. It is known [10] that the classical Meixner class of orthogonal polynomials and distributions ($q = 1$) can be classified into five types by parameters,

$$\begin{cases} \theta := \dfrac{\gamma}{\sqrt{\kappa_2}}, \quad \tau := \dfrac{\delta}{\kappa_2}, \\ D := \theta^2 - 4\tau. \end{cases}$$

A q-analogue of the classical case is discussed in [2] and characterized as well as $q = 1$ case by the same parameters [9][16]. See also [1][5][15] for the free case $q = 0$. More precisely, the q-Meixner distribution is classified into five types as follows:

(i) q-Gaussian: $\tau = 0,\ \theta = 0$.

(ii) q-Poisson: $\tau = 0,\ \theta \neq 0$.

(iii) q-Pascal: $\tau > 0,\ D > 0$.

(iv) q-Gamma: $\tau > 0,\ D = 0$.

(v) q-Meixner : $D < 0$.

Remark 2.1. For $\alpha, q \in (-1,1)$, let $\nu_{\alpha,q}$ be the orthogonalizing probability measure of the sequence of monic polynomials $\{P_n^{\alpha,q}(x)\}$ defined by the recurrence relation,

$$\begin{cases} P_0^{\alpha,q}(x) = 1,\ P_1^{\alpha,q}(x) = x, \\ xP_n^{\alpha,q}(x) = P_{n+1}^{\alpha,q}(x) + (1 + \alpha q^{n-1})[n]_q P_{n-1}^{\alpha,q}(x), \quad n \geq 1. \end{cases} \tag{2.2}$$

The measure $\nu_{\alpha,q}$ is symmetric and its explicit expression can be found in [3][6][13]. Since the equality $1+\alpha q^{n-1} = 1+\alpha-\alpha(1-q)[n-1]_q$ holds, $\{P_n^{\alpha,q}(x)\}$ for $\alpha \in (-1,0]$ can be considered as a special case of $\{Q_n^{(q)}(x)\}$ in the sense of Definition 2.1. Hence the measure $\nu_{\alpha,q}$ for $\{P_n^{\alpha,q}(x)\}$ coincides with $\mu\,(q; 0, 1+\alpha, 0, -\alpha(1-q))$ for $\alpha \in (-1,0]$.

2.2 (α, q)-Poisson operator on $\mathcal{F}_{\alpha,q}(\mathscr{H})$

To make our point clearer, we put basic facts on the (α,q)-creation $B_{\alpha,q}^\dagger$ and annihilation $B_{\alpha,q}$ operators acting on $\mathcal{F}_{\alpha,q}(\mathscr{H})$. One can see $B_{\alpha,q} = (B_{\alpha,q}^\dagger)^*$ with respect to the (α,q)-inner product $\langle\cdot,\cdot\rangle_{\alpha,q}$ For those who are not familiar with the (α,q)-deformation, refer Appendix A and also [6] written in detail.

Let us consider a self-adjoint operator $\mathbf{P}_{\alpha,q}(f)$ for $f \in \mathscr{H}$ defined by the form,

$$\mathbf{P}_{\alpha,q}(f) := B_{\alpha,q}^\dagger(f) + B_{\alpha,q}(f) + c_1 N_q(f) + c_2\mathbf{1},\ c_1 \geq 0, c_2 \in \mathbb{R}$$

where N_q is the q-number operator having $\mathscr{H}^{\otimes n}$ as the eigenspace with eigenvalue $[n]_q$. Moreover, we find the probability distribution of this operator with respect to the vacuum state $\langle\Omega, \cdot\Omega\rangle_{\alpha,q}$. We call $\mathbf{P}_{\alpha,q}(f)$ the (α,q)-Poisson operator (the Poisson operator of type B). In the case $c_1 = 0$, $\mathbf{P}_{\alpha,q}(f)$ is denoted simply by $\mathbf{G}_{\alpha,q}(f)$, called the (α,q)-Gaussian operator (the Gaussian operator of type B).

Remark 2.2. (1) It is easy to see that the operators $B_{0,q}^{\dagger}$ and $B_{0,q}$ are the same as the q-creation operator $b_q^{\dagger}$ and q-annihilation operator b_q on the q-Fock space $\mathcal{F}_q(\mathscr{H}) := \mathcal{F}_{0,q}(\mathscr{H})$, respectively. That is, $b_q = (b_q^{\dagger})^*$ with respect to the inner product $\langle\cdot,\cdot\rangle_q := \langle\cdot,\cdot\rangle_{0,q}$. (see [8]). In [14], the q-Poisson operator (the Poisson operator of type A) is examined as the sum of $b_q^{\dagger}, b_q$ and $b_q^{\dagger}b_q$ and its distribution is identified with the q-Poisson distribution with $\delta = 0$ $(\tau = 0)$ of the Meixner's classification.
(2) In [11], Ejsmont introduced the Poisson operator of type B, but his operator is essentially different from ours if $\alpha \neq 0$.

Definition 2.3. For $s \in \mathbb{R}$, we define the translation T_s of a probability measure μ by $T_s\mu(\cdot) = \mu(\cdot - s)$. For $\lambda \in \mathbb{R}, \lambda \neq 0$, we define the dilation D_λ of μ by $D_\lambda\mu(\cdot) = \mu(\cdot/\lambda)$.

Theorem 2.4. *Suppose $\alpha, q \in (-1,1)$ and $f \in \mathscr{H}$ with $\|f\| = 1$. Let $\rho_{\alpha,q,f}$ be the probability distribution of $\mathbf{P}_{\alpha,q}(f)$ with respect to the vacuum state $\langle\Omega,\cdot\Omega\rangle_{\alpha,q}$.*

(1) If $q \in (-1,1)$ and $-1 < \alpha\langle f,\overline{f}\rangle \leq 0$, then $\rho_{\alpha,q,f}$ is

$$\mu(q; c_2, 1+\alpha\langle f,\overline{f}\rangle, c_1, -\alpha(1-q)\langle f,\overline{f}\rangle).$$

(2) If $c_1 = 0$, $q \in (-1,1)$ and $-1 < \alpha\langle f,\overline{f}\rangle < 1$, then $\rho_{\alpha,q,f}$ is equal to $T_{c_2}\nu_{\alpha\langle f,\overline{f}\rangle,q}$, where it is the probability distribution of $\mathbf{G}_{\alpha,q}(f) - c_2\mathbf{1}$.

2.3 q-Meixner operator on $\mathcal{F}_q(\mathscr{H})$

Let us recall the double q-creation and annihilation operators, $(b_q^{\dagger})^2$ and $(b_q)^2$ respectively, acting on the q-Fock space $\mathcal{F}_q(\mathscr{H}) := \mathcal{F}_{0,q}(\mathscr{H})$ for $f \in \mathscr{H}$, defined by

$$\begin{cases} (b_q^{\dagger})^2\, f^{\otimes 2n} = f^{\otimes 2(n+1)}, & n \geq 0, \\ (b_q)^2\, f^{\otimes 2n} = [2n]_q\,[2n-1]_q\, f^{\otimes 2(n-1)}, & n \geq 1, \\ b_q^{\dagger}b_q\, f^{\otimes 2n} = [2n]_q\, f^{\otimes 2n}, & n \geq 1. \end{cases} \tag{2.3}$$

Yoshida [16] considered a self-adjoint operator $\mathbf{X}_q(c_3, c_4)$ on $\mathcal{F}_q(\mathscr{H})$ given by

$$\mathbf{X}_q(c_3,c_4) = (b_q^{\dagger})^2 + (b_q)^2 + c_3 b_q^{\dagger}b_q + c_4\mathbf{1}, \quad c_3 \geq 0, c_4 \in \mathbb{R},$$

and the probability distribution of this operator denoted by $\mu_{\mathbf{X}_q}$ with respect to the vacuum state $\langle\Omega,\cdot\Omega\rangle_q$. In this paper, $\mathbf{X}_q$, is called the q-Meixner operator acting on $\mathcal{F}_q(\mathscr{H})$.

It is our main concern in this section to clarify the relationship between probability distributions of $\mathbf{X}_q$ and $\mathbf{P}_{\alpha,q}$ in a sense. Let us first recall the following Theorem.

Theorem 2.5. *The probability distribution $\mu_{\mathbf{X}_q}$ of the operator $\mathbf{X}_q(c_3,c_4)$ with respect to the vacuum state $\langle\Omega,\cdot\Omega\rangle_q$ is given as follows:*
(1) If $c_3 > 0$, then

$$\mu_{\mathbf{X}_q} = \mu(q^2; c_4, 1+q, c_3(1+q), q(1+q)^2),$$

for $q \in [0,1)$.
(2) If $c_3 = 0$, then $\mu_{\mathbf{X}_q} = T_{c_4}\nu_{-q,q^2}$ for $q \in (-1,1)$.

Proof. Due to the equality,

$$(1+\alpha q^{2(n-1)})[n]_{q^2} = \frac{1}{q(1+q)}\,(\alpha + q - \alpha(1-q)[2n-1]_q)\,[2n]_q,$$

$\alpha = -q$ implies

$$[2n-1]_q[2n]_q = (1+q+q(1+q)^2[n-1]_{q^2})[n]_{q^2}. \tag{2.4}$$

Due to this identity, we get our assertion. ☐

3 Relationship between (α, q^2)-Poisson and q-Meixner operators

3.1 The case $\alpha = -q$

To see relationships between the operator $\mathbf{X}_q$ and with $(-q, q^2)$-operator, we shall define a scaled operator $\mathbf{Y}_q$ of $\mathbf{X}_q$ for $0 \le q < 1$,

$$\mathbf{Y}_q := \frac{1}{1+q}\mathbf{X}_q, \tag{3.1}$$

and a weighted Poisson type operator $\mathbf{Y}_{-q,q^2}$ defined by

$$\mathbf{Y}_{-q,q^2} := \frac{1}{1+q}\left\{B^\dagger_{-q,q^2} + \frac{1+q}{1-q}B_{-q,q^2} + c_1(1+q)N_{q^2} + c_2\mathbf{1}\right\}. \tag{3.2}$$

We remark here that if $c_1 = c_3 = 0$, the condition on q can be relaxed to $q \in (-1, 1)$.

Since $\mathbf{Y}_{-q,q^2}$ is not self-adjoint with respect to $\langle\cdot,\cdot\rangle_{-q,q^2}$ due to the second term in RHS of (3.2), which is a counterpart of $(b_q)^2$ in (3.1), we need to modify (α, q)-creation and annihilation operators by adding a weight $\beta > 0$ as follows:

Let $B^\dagger_{\beta,\alpha,q}(f)$ be the β-weighted (α, q)-creation defined as the (α, q)-creation operator $B^\dagger_{\alpha,q}(f)$ and $B_{\beta,\alpha,q}(f)$ be the β-weighted (α, q)-annihilation operator given by

$$B_{\beta,\alpha,q}(f) := \beta B_{\alpha,q}(f), \quad \beta > 0.$$

The above two β-weighted operators are adjoint each other with respect to the β-weighted (α, q)-inner product,

$$\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n\rangle_{\beta,\alpha,q} := \delta_{m,n}\beta^n\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n\rangle_{\alpha,q}, \ f_k, g_k \in \mathscr{H}.$$

By setting $\beta = \frac{1+q}{1-q}$, the operator $\mathbf{Y}_{-q,q^2}$ can be expressed as

$$\mathbf{Y}_{-q,q^2} = \frac{1}{1+q}\left\{B^\dagger_{\beta,-q,q^2} + B_{\beta,-q,q^2} + c_1(1+q)N_{q^2} + c_2\mathbf{1}\right\}$$

and hence $\mathbf{Y}_{-q,q^2}$ is the self-adjoint operator with respect to the inner product $\langle\cdot,\cdot\rangle_{\beta,-q,q^2}$.

Then we can clarify the relationship between probability distributions of $\mathbf{Y}_q$ and $\mathbf{Y}_{-q,q^2}$ with respect to the vacuum state.

Theorem 3.1. *Suppose $c_1 = c_3$ and $c_2 = c_4$. Then the probability law of $\mathbf{Y}_q$ with respect to $\langle\Omega, \cdot\Omega\rangle_q$ is equal to that of $\mathbf{Y}_{-q,q^2}$ with respect to $\langle\Omega, \cdot\Omega\rangle_{\beta,-q,q^2}$ with $\beta = \frac{1+q}{1-q}$. In fact, the probability distribution $\rho_{\mathbf{Y}}$ of these operators is given as follows:*
(1) If $c_1 > 0$, then $\rho_{\mathbf{Y}}$ is

$$D_a\mu_{\mathbf{X}_q} = \mu\left(q^2; \frac{c_2}{1+q}, \frac{1}{1+q}, c_1, q\right), \quad a = \frac{1}{1+q}$$

for $q \in [0, 1)$.
(2) If $c_1 = 0$, then $\rho_{\mathbf{Y}}$ is $D_a\mu_{\mathbf{X}_q} = D_aT_{c_2}\nu_{-q,q^2}$ for $q \in (-1, 1)$.

Moreover, the classification parameters θ and τ under $c_1 = c_3$ for the Meixner class are given by

$$\begin{cases} \theta = c_1\sqrt{1+q}, \\ \tau = q(1+q) \ge 0, \\ D = (1+q)(c_1^2 - 4q). \end{cases} \tag{3.3}$$

Therefore, we get

Proposition 3.2. *(1) If $q = 0$ ($\tau = 0$) and $c_1 = c_3 = 0$ ($\theta = 0$), then $\mu_{\mathbf{X}_0} = T_{c_2}\nu_{0,0}$ and $D_a\mu_{\mathbf{X}_0} = D_aT_{c_2}\nu_{0,0}$ are the free Gaussian.*
(2) If $q = 0$ ($\tau = 0$) and $c_1 = c_3 \neq 0$ ($\theta \neq 0$), then $\mu_{\mathbf{X}_0}$ and $D_a\mu_{\mathbf{X}_0}$ are the free Poisson.
(3) If $0 < q < 1$ ($\tau > 0$) and $c_1 = c_3 > 2\sqrt{q}$ ($D > 0$), then $\mu_{\mathbf{X}_q}$ and $D_a\mu_{\mathbf{X}_q}$ are the q^2-Pascal.
(4) If $0 < q < 1$ ($\tau > 0$) and $c_1 = c_3 = 2\sqrt{q}$ ($D = 0$), then $\mu_{\mathbf{X}_q}$ and $D_a\mu_{\mathbf{X}_q}$ are the q^2-Gamma.
(5) If $0 < q < 1$ ($\tau > 0$) and $0 \neq c_1 = c_3 < 2\sqrt{q}$ ($D < 0$), then $\mu_{\mathbf{X}_q}$ and $D_a\mu_{\mathbf{X}_q}$ are the q^2-Meixner.

We have shown by introducing the (α, q^2)-Poisson and the q-Meixner operators that non-symmetric probability distributions such as (2)(3)(4)(5) in Proposition 3.2 can be treated within the framework of the Fock space of type B. In [6], non-symmetric cases are not treated.

3.2 The case $\alpha = q$

Due to Proposition 3.2, $\mathbf{X}_q$ and $\mathbf{Y}_{-q,q^2}$ do not provide the q^2-Gaussian and q^2-Poisson laws. In particular, the $\mathbf{Y}_{q,q}$-operator given by

$$\mathbf{Y}_{q,q} := a\mathbf{P}_{q,q}, \ a = \frac{1}{1+q}$$

has these probability laws for $q \in (-1, 1)$. It is the self-adjoint operator with respect to the inner product $\langle \cdot, \cdot \rangle_{1+q,0,q^2}$. It is easy to see that if $c_1 = 0$, then $\mathbf{Y}_{q,q} = a(\mathbf{G}_{q,q} + c_2\mathbf{1})$. Hence we have

Proposition 3.3. *For $q \in (-1, 1)$, the probability law of $\mathbf{Y}_{q,q}$ with respect to $\langle \Omega, \cdot\Omega \rangle_{1+q,0,q^2}$ is as follows;*
(1) $c_1 = 0 \Longrightarrow$ the q^2-Gaussian, $D_aT_{c_2}\nu_{0,q^2}$.
(2) $c_1 > 0 \Longrightarrow$ the q^2-Poisson, $\mu\left(q^2; \frac{c_2}{1+q}, \frac{1}{1+q}, c_1, 0\right)$.

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A Appendix

This appendix is referred from [4]. Let $B(n)$ be the set of bijections σ of the $2n$ points $\{\pm 1, \pm 2, \cdots, \pm n\}$ with $\sigma(-k) = -\sigma(k)$. Equipped with the composition operation as a product, $B(n)$ becomes what is called a Coxeter group of type B. It is generated by $\pi_0 := (1, -1)$ and $\pi_i := (i, i+1)$, $1 \leq i \leq n-1$, which satisfy the generalized braid relations

$$\begin{cases} \pi_i^2 = e, & 0 \leq i \leq n-1, \\ (\pi_0\pi_{n-1})^4 = (\pi_i\pi_{i+1})^3 = e, & 1 \leq i \leq n-1, \\ (\pi_i\pi_j)^2 = e, & |i-j| \geq 2, \ 0 \leq i, j \leq n-1. \end{cases} \tag{A.1}$$

An element $\sigma \in B(n)$ expresses an irreducible form,

$$\sigma = \pi_{i_1} \cdots \pi_{i_k}, \quad 0 \leq i_1, \ldots, i_k \leq n-1,$$

and in this case

$$\ell_1(\sigma) := \text{the number of } \pi_0 \text{ in } \sigma,$$
$$\ell_2(\sigma) := \text{the number of } \pi_i,\ 1 \le i \le n-1, \text{ in } \sigma$$

are well defined.

Let $\mathscr{H}$ be a complex Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, where the inner product is linear on the right and conjugate linear on the left. For a given self-adjoint involution $f \mapsto \overline{f}$ for $f \in \mathscr{H}$, an action of $B(n)$ on $\mathscr{H}^{\otimes n}$ is defined by

$$\begin{cases} \pi_0(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes f_2 \otimes \cdots \otimes \overline{f}_n, & n \ge 1, \\ \pi_i(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes f_i \otimes f_{i+2} \otimes \cdots \otimes f_n, & n \ge 2,\ 1 \le i \le n-1. \end{cases}$$

Throughout this paper, we assume the involution $\overline{f}$ of $f \in \mathscr{H}$ is defined in such a way that $\langle f, \overline{f} \rangle \in \mathbb{R}$ holds and $\langle f, \overline{f} \rangle = 0$ is equivalent to $f = 0$.

Let $\mathcal{F}_{\text{fin}}(\mathscr{H})$ denote the algebraic full Fock space over $\mathscr{H}$,

$$\mathcal{F}_{\text{fin}}(\mathscr{H}) := \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathscr{H}^{\otimes n},$$

where Ω denotes the vacuum vector. We note that elements of $\mathcal{F}_{\text{fin}}(\mathscr{H})$ are expressed as finite linear combinations of the elementary vectors $f_1 \otimes \cdots \otimes f_n \in \mathscr{H}^{\otimes n}$. We equip $\mathcal{F}_{\text{fin}}(\mathscr{H})$ with the inner product

$$\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_{0,0} := \delta_{m,n} \prod_{k=1}^{n} \langle f_k, g_k \rangle, \quad f_k, g_k \in \mathscr{H}.$$

For $\alpha, q \in (-1, 1)$, define the symmetrization operator of type B on $\mathscr{H}^{\otimes n}$ as

$$P^{(n)}_{\alpha,q} = \sum_{\sigma \in B(n)} \alpha^{\ell_1(\sigma)} q^{\ell_2(\sigma)} \sigma, \quad n \ge 1,$$
$$P^{(n)}_{0,q} = \sum_{\sigma \in \mathfrak{S}_n} q^{\ell_2(\sigma)} \sigma, \quad n \ge 1,$$
$$P^{(0)}_{\alpha,q} = I_{\mathscr{H}^{\otimes 0}},\ P^{(n)}_{0,0} = I_{\mathscr{H}^{\otimes n}},$$

where we put $0^0 = 1$ and $\mathscr{H}^{\otimes 0} = \mathbb{C}\Omega$ by convention and

$$P_{\alpha,q} = \bigoplus_{n=0}^{\infty} P^{(n)}_{\alpha,q}$$

be the symmetrization operator of type B on $\mathcal{F}_{\text{fin}}(\mathscr{H})$. Since $P^{(n)}_{\alpha,q}$ is known to be strictly positive,

$$\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_{\alpha,q} := \langle f_1 \otimes \cdots \otimes f_m, P_{\alpha,q}(g_1 \otimes \cdots \otimes g_n) \rangle_{0,0}$$

becomes an inner product and $\langle \cdot, \cdot \rangle_{\alpha,q}$ is called the (α, q)-inner product with the convention $0^0 = 1$ and $g_{-k} = \overline{g_k},\ k = 1, 2, \ldots, n$.

Definition A.1. (1) For $\alpha, q \in (-1,1)$, the (algebraic) full Fock space $\mathcal{F}_{\rm fin}(\mathscr{H})$ with respect to $\langle\cdot,\cdot\rangle_{\alpha,q}$ is called the (α,q)-Fock space (the Fock space of type B) denoted by $\mathcal{F}_{\alpha,q}(\mathscr{H})$. In this paper, we do not take completion. In particular, $\mathcal{F}_{0,q}(\mathscr{H})$ is nothing but the q-Fock space (the Fock space of type A) $\mathcal{F}_q(\mathscr{H})$ equipped with the q-inner product $\langle\cdot,\cdot\rangle_q := \langle\cdot,\cdot\rangle_{0,q}$ of Bożejko-Speicher [8].
(2) Let $B^\dagger_{\alpha,q}(f)$ be defined as the usual left creation operator,

$$\begin{aligned} &B^\dagger_{\alpha,q}(f)\Omega = f, \\ &B^\dagger_{\alpha,q}(f)(f_1\otimes\cdots\otimes f_n) = f\otimes f_1\otimes\cdots\otimes f_n, \quad n\ge 1 \end{aligned}$$

and $B_{\alpha,q}(f)$ be its adjoint with respect to $\langle\cdot,\cdot\rangle_{\alpha,q}$, that is, $B_{\alpha,q} = (B^\dagger_{\alpha,q})^*$. $B^\dagger_{\alpha,q}$ and $B_{\alpha,q}$ are called the the (α,q)-creation and (α,q)-annihilation operators, respectively.

The following proposition is a direct consequence of the definition.

Proposition A.2. (1) *The (α,q)-annihilation operator $B_{\alpha,q}$ acts on the elementary vectors as follows:*

$$\begin{aligned} &B_{\alpha,q}(f)\Omega = 0, \quad B_{\alpha,q}(f)f_1 = \langle f, f_1\rangle\,\Omega, \\ &B_{\alpha,q}(f)(f_1\otimes\cdots\otimes f_n) = L + R \end{aligned}$$

where

$$\begin{aligned} L &= \sum_{k=1}^{n} q^{k-1}\langle f, f_k\rangle\, f_1\otimes\cdots\otimes \overset{\vee}{f}_k\otimes\cdots\otimes f_n, \\ R &= \alpha q^{n-1}\sum_{k=1}^{n} q^{k-1}\langle f, \overline{f}_{n-(k-1)}\rangle\, f_1\otimes\cdots\otimes \overset{\vee}{f}_{n-(k-1)}\otimes\cdots\otimes f_n, \end{aligned}$$

for $n\ge 2$ where $\overset{\vee}{f_k}$ means that f_k should be deleted from tensor product.
(2) *The (α,q)-creation and the (α,q)-annihilation operators satisfy the commutation relation,*

$$B_{\alpha,q}(f)B^\dagger_{\alpha,q}(g) - qB^\dagger_{\alpha,q}(g)B_{\alpha,q}(f) = \langle f, g\rangle I + \alpha\langle f, \overline{g}\rangle q^{2N}, \quad f, g\in\mathscr{H}.$$

The readers can refer to [6] for details.

Corollary A.3. *(1) The q-annihilation operator $b_q(x)$ acts on the elementary vectors as follows:*

$$\begin{aligned} &b_q(f)\Omega = 0, \quad b_q(f)f_1 = \langle f, f_1\rangle\,\Omega, \\ &b_q(f)(f_1\otimes\cdots\otimes f_n) = \sum_{k=1}^{n} q^{k-1}\langle f, f_k\rangle\, f_1\otimes\cdots\otimes \overset{\vee}{f_k}\otimes\cdots\otimes f_n \quad n\ge 2, \end{aligned}$$

where $\overset{\vee}{f_k}$ means that f_k should be deleted from tensor product.
(2) The q-creation and the q-annihilation operators satisfy the q-commutation relation (q-CCR)

$$b_q(f)b^\dagger_q(g) - q\, b^\dagger_q(g)b_q(f) = \langle f, g\rangle\mathbf{1} \qquad f, g\in\mathscr{H}.$$

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