Diffusion in Coulomb environments

Hirofumi Osada

Faculty of Mathematics, Kyushu University
osada@math.kyushu-u.ac.jp

Abstract

We announce results that are to be described in full in an article titled "Homogenization of diffusion processes in Coulomb environments and a phase transition" and in a follow-up article. The results are summarized as follows:
• Homogenization in periodic Coulomb environments $d = 2$.
• A phase transition of the effective constant in two dimensions.
• Homogenization in periodic Coulomb environments $d \geq 3$.
• Homogenization in the Ginibre environment.

1 Introduction

The results of a paper "Homogenization of diffusion processes in Coulomb environments and a phase transition" [9] and another forthcoming paper are announced. In these papers, we shall present full details of the context of this paper .

We consider a diffusion process X in Coulomb environments for which the drift terms of the diffusion are given as a sum of Coulomb forces. We study the diffusive scaling limit of X.

$$\lim_{\epsilon \to 0} \epsilon X_{t/\epsilon^2}. \tag{1}$$

If X describes Brownian motion B of a particle initially at the origin, then it obeys the scaling law $\epsilon B_{t/\epsilon^2} = B_t$ in distribution. Therefore, the limit (1) exists. In general, the coefficient of X is *stationary* in space and *mean free* with respect to an invariant measure, then we expect the limit (1) exists and converges to a constant multiple of the Brownian motion. The limit constant is called the effective conductivity and this scaling limit is called the homogenization of the diffusion processes. We shall study instances when the random environments are derived from a Coulomb potential with inverse temperature β. We investigate whether the effective constant vanishes depending on β.

Let $d \geq 2$ and $\sigma(d)$ be the surface volume of the unit ball.

$$\sigma(d) = 2\pi^{d/2}/\Gamma(d/2).$$

Let Ψ_d be $\sigma(d)/2$ times the fundamental solution of the Laplacian $-\frac{1}{2}\Delta$ in $\mathbb{R}^d$.

$$\Psi_d(x) = \begin{cases} \frac{1}{d-2}|x|^{2-d} & (d \geq 3) \\ -\log|x| & (d = 2) \end{cases} \tag{2}$$

We denote the d-dimensional Coulomb potential by Ψ. The Coulomb force derived by Ψ is then

$$\nabla\Psi_d(x) = -\frac{x}{|x|^d}. \tag{3}$$

The normalizing constant in (2) is chosen in such a way that (3) has a simple expression [8].

For $\mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{R}^d$ we define the d-dimensional lattice $\mathbb{L}$ and torus $\mathbb{T}$ by

$$\mathbb{L} = \{\sum_{i=1}^{d} n_i\mathbf{v}_i\,;\, n_i \in \mathbb{Z}\ (i = 1, \ldots, d)\}, \tag{4}$$

$$\mathbb{T} = \{\sum_{i=1}^{d} t_i\mathbf{v}_i\,;\, t_i \in [0, 1)\ (i = 1, \ldots, d)\}.$$

We choose $\mathbf{v}_i$ such that $|\mathbb{T}| = 1$.

At each site on the lattice $\mathbb{L}$, we place a single particle of unit charge. The total Coulomb force acting at $x \in \mathbb{R}^d$ (without inverse temperature) is

$$\mathrm{b}(x) = \lim_{q\to\infty} \sum_{\substack{|x-s_i|<q,\\ s_i\in\mathbb{L}}} -\nabla\Psi_d(x - s_i) = \lim_{q\to\infty} \sum_{\substack{|x-s_i|<q,\\ s_i\in\mathbb{L}}} \frac{x - s_i}{|x - s_i|^d}.$$

Then $\mathrm{b}(x)$ is a periodic function with singularity at each site $s_i \in \mathbb{L}$. Mathematically,

$$\mathrm{b}(x) = \lim_{q\to\infty} \sum_{s_i\in\mathbb{L}} \varphi_q(x - s_i)\frac{x - s_i}{|x - s_i|^d}. \tag{5}$$

Here $\varphi_q(x) = \varphi(x/q)$, $0 \leq \varphi \leq 1$, $\varphi(x) = \varphi(|x|)$, and $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$\varphi(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| \geq 2. \end{cases}$$

We place a particle $X^x = \{X_t^x\}$ in $\mathbb{R}^d$ with the same charge as each site s_i. We suppose $X = \{X_t\}$ is a diffusion process given by an stochastic differential equation. For x $\in \mathbb{R}^d$, let

$X_t^{\mathrm{x}} \in \mathbb{R}^d$ be the solution of

$$dX_t^{\mathrm{x}} = dB_t + \frac{\beta}{2} \lim_{q\to\infty} \sum_{\substack{|X_t^{\mathrm{x}}-s_i|<q \\ s_i\in\mathbb{L}}} \frac{X_t^{\mathrm{x}} - s_i}{|X_t^{\mathrm{x}} - s_i|^d} dt, \tag{6}$$

$$X_0^{\mathrm{x}} = \mathrm{x}. \tag{7}$$

Here B denotes d-dimensional Brownian motion and β the inverse temperature.

Lemma 1. *There exists a symmetric matrix $\gamma_{\mathrm{eff}}^{\beta}$ such that for all* x

$$\lim_{\varepsilon\to\infty} \varepsilon X_{t/\varepsilon^2}^{\mathrm{x}} = \sqrt{\gamma_{\mathrm{eff}}^{\beta}} B_t \quad \textit{weakly in } C([0,\infty);\mathbb{R}^d) \tag{8}$$

$$0 < \gamma_{\mathrm{eff}}^{\beta} < E \textit{ for all } \beta \geq 0.$$

Remark 1. (1) The constant matrix $\gamma_{\mathrm{eff}}^{\beta}$ corresponds to an effective conductivity.
(2) $\gamma_{\mathrm{eff}}^{\beta}$ is given by a solution of *Poisson*'s equation and the variational formula (cf. [6]).
(3) The positivity of the effective conductivity $0 < \gamma_{\mathrm{eff}}^{\beta}$ follows from a comparison with the periodic homogenization of reflecting barrier Brownian motion. We refer to [4, 5] for details regarding the homogenization of reflecting barrier Brownian motion. We have proved homogenization on random stationary open sets in $\mathbb{R}^d$. Its special cases are the periodic homogenization of reflecting barrier Brownian motion.

We present a quick review of the homogenization of reflecting barrier Brownian motion. This problem was initiated by Hiroshi Tanaka [13], who proved the periodic homogenization on the half domain of $\mathbb{R}^d$ having a reflecting boundary condition imposed on the boundary of the domain. Bhattacharaya [1] and Ochi [3] proved the periodic homogenization with periodic balls as obstacles and a particle that is reflected at the boundary of such obstacles. Zhikov [15] and the author [4] proved homogenization for a domain with randomly placed stationary obstacles. In [4, 5], the author proved the non-degeneracy of the limit using an isoperimetric inequality. Tanemura [14] proved the non-degeneracy of the limit for the Poisson blob model (percolation domains). Rhodes [10] and Chen, Crydon, and Kumagai [2] studied the homogenization of reflecting barrier Brownian motion in conical domains.

There is a difficulty in treating homogenization of reflecting barrier Brownian motion, as spatial shifts do not act on the whole domain. Because of this difficulty, we can not construct the so-called corrector using spatial shifts as for the standard homogenization problem. We adopt instead Kipnis-Varadhan theory.

2 Defect lattice homogenization and a phase transition

For simplicity, we assume throughout this paper that in Lemma 1 $\gamma_{\mathrm{eff}}^{\beta}$ is a matrix corresponding to an effective conductivity and satisfies

(A1) $\gamma_{\mathrm{eff}}^{\beta}$ is a scalar matrix.

Remark 2. Assumption (A1) is not restrictive. If $\mathbb{L}$ is a d-dimensional cubic lattice or a triangular lattice in $d=2$, then (A1) holds.

We remove m-sites $\{t_1,\dots,t_m\}$ from $\mathbb{L}$. Let $\mathbb{L}_\diamond$ be a defect lattice such that

$$\mathbb{L}_\diamond = \mathbb{L}\backslash\{t_1,\dots,t_m\}.$$

For $\mathrm{x}\in\mathbb{T}$, let $Y_t^{\mathrm{x}}\in\mathbb{R}^d$ be the solution of

$$dY_t^{\mathrm{x}} = dB_t + \frac{\beta}{2}\lim_{q\to\infty}\sum_{\substack{|Y_t^{\mathrm{x}}-s_i|<q\\ s_i\in\mathbb{L}_\diamond}}\frac{Y_t^{\mathrm{x}}-s_i}{|Y_t^{\mathrm{x}}-s_i|^2}dt, \tag{9}$$

$$Y_0^{\mathrm{x}} = \mathrm{x}. \tag{10}$$

Theorem 1 (a phase transition in β in two dimension)**.** *Assume $d=2$ and* (A1)*. Let*

$$\gamma_0(\beta) = \mathrm{trace}(\gamma_{\mathrm{eff}}^{\beta}).$$

Then $\gamma_0(\beta)/m$ is a critical point in the following sense:

$$\lim_{\varepsilon\to\infty}\varepsilon Y_{t/\varepsilon^2}^{\mathrm{x}} = \begin{cases} \textit{not } 0 & \textit{if } \beta<\gamma_0(\beta)/m \\ 0 & \textit{if } \beta\geq\gamma_0(\beta)/m. \end{cases} \tag{11}$$

Moreover, $0<\gamma_0(\beta)<2$.

Remark 3. (1) Clearly, $\gamma_0(0)=2$ because Brownian motion is invariant under diffusive scaling. Therefore, if $\gamma_0(\beta)$ is strictly decreasing in β, then there exists a unique $\gamma(\mathbb{L})$ such that $\gamma(\mathbb{L})=\gamma_0(\beta)$. The critical value $\gamma(\mathbb{L})$ depends only on the lattice.
(2) Theorem 1 indicates that the strength of the long-range interaction of the Coulomb potential. If we replace the Coulomb by a potential of Ruelle's class, the limit is always non-degenerate. Here, we recall that by definition a potential of Ruelle's class has an integrable tail [11, 12].

In the previous theorem, the limit dynamics starts at the origin. We consider a different type of initial condition. Specifically,

$$\mathrm{x} \neq 0.$$

We also specify the limit stochastic differential equation, for which it convenient to introduce $X^{\varepsilon,\mathrm{x}}$ and $Y^{\varepsilon,\mathrm{x}}$,

$$X_t^{\varepsilon,\mathrm{x}} = \varepsilon X_{t/\varepsilon^2}^{\mathrm{x}/\varepsilon}, \quad Y_t^{\varepsilon,\mathrm{x}} = \varepsilon Y_{t/\varepsilon^2}^{\mathrm{x}/\varepsilon} \tag{12}$$

Then by a straightforward calculation we have

$$dX_t^{\varepsilon,\mathrm{x}} = dB_t + \frac{\beta}{2} \lim_{q\to\infty} \sum_{\substack{|X_t^{\varepsilon,\mathrm{x}}-\varepsilon s_i|<q \\ s_i\in\mathbb{L}}} \frac{X_t^{\varepsilon,\mathrm{x}} - \varepsilon s_i}{|X_t^{\varepsilon,\mathrm{x}} - \varepsilon s_i|^2} dt, \quad X_0^{\varepsilon,\mathrm{x}} = \mathrm{x} \tag{13}$$

$$dY_t^{\varepsilon,\mathrm{y}_\varepsilon} = dB_t + \frac{\beta}{2} \lim_{q\to\infty} \sum_{\substack{|Y_t^{\varepsilon,\mathrm{y}_\varepsilon}-\varepsilon s_i|<q \\ s_i\in\mathbb{L}_\diamond}} \frac{Y_t^{\varepsilon,\mathrm{y}_\varepsilon} - \varepsilon s_i}{|Y_t^{\varepsilon,\mathrm{y}_\varepsilon} - \varepsilon s_i|^2} dt, \quad Y_0^{\varepsilon,\mathrm{y}_\varepsilon} = \mathrm{y}_\varepsilon \tag{14}$$

Remark 4. Note that the stochastic differential equations (13) and (14) are similar to (6) and (9). This similarity stems from the following invariance of the logarithmic function

$$\nabla \log(x/\epsilon) = \nabla \log(x) \quad \text{for all } \epsilon, x > 0. \tag{15}$$

This property plays a critical role in the present analysis. Another function that has the same invariance as (15) is the Heaviside step function H, specifically,

$$\nabla H(x/\epsilon) = \nabla H(x) \quad \text{for all } \epsilon, x > 0, \tag{16}$$

and play a critical role in the homogenization of reflecting barrier Brownian motion on conical domains.

Consider a subsidiary stochastic differential equation in $\mathbb{R}^2$.

$$dU_t^{\mathrm{y}} = \sqrt{\gamma_{\mathrm{eff}}^{\beta}} dB_t - \frac{m\beta}{2} \frac{U_t^{\mathrm{y}}}{|U_t^{\mathrm{y}}|^2} dt, \quad U_0^{\mathrm{y}} = \mathrm{y}. \tag{17}$$

Theorem 2. *Assume $d = 2$ and* (A1). *Assume that the initial starting points y_ε are set o*

$$\lim_{\varepsilon\to 0} \mathrm{y}_\varepsilon = \mathrm{y} \in \mathbb{R}^2.$$

Then

$$\lim_{\epsilon\to 0} Y^{\varepsilon,\mathrm{y}_\varepsilon} = U^{\mathrm{y}}. \tag{18}$$

Let $\sigma = \inf\{t > 0; U_t^{\mathrm{y}} = 0\}$. *Then*

$$P(\sigma < \infty) = 1 \quad \textit{for all } \beta > 0.$$

Furthermore,

$$P(U_t^{\mathrm{y}} = 0 \textit{ for all } t \geq \sigma) = 1 \quad \textit{for } \beta \geq \gamma_0(\beta)/m.$$

We next consider the higher dimensional case.

Theorem 3. *Let* $d \geq 3$ *and assume the initial starting points* y_ε *satisfy*

$$\lim_{\varepsilon\to 0} \mathrm{y}_\varepsilon = \mathrm{y} \neq 0.$$

Then for each $0 \leq \beta < \infty$

$$\lim_{\epsilon\to 0} Y_t^{\varepsilon,\mathrm{y}_\varepsilon} = \sqrt{\gamma_{\mathrm{eff}}^{\beta}} B_t + \mathrm{y}.$$

3 Diffusion in Ginibre environment

Let S be a configuration space over $\mathbb{R}^d$,

$$\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\, ;\, s_i \in \mathbb{R}^d,\ \mathsf{s}(|s| \leq r) < \infty\ (\forall r \in \mathbb{N})\}.$$

We equip S with vague topology under which S becomes a Polish space. A probability measure μ on S is called a point process (or a random point field) on $\mathbb{R}^d$. Point processes describe unlabeled particle systems with no accumulation in any compact set. There are a plurality of ways to formulate their configurations. Indeed, we may regard a configuration as a set consisting of countable points. The advantage of the former formulation is its natural topology.

A lattice $\mathbb{L}$ can be regarded as a periodic point process, with each site s_i regarded as a particle.

We recall the notion of a correlation function. Let m be a Radon measure on $\mathbb{R}^d$. Function $\rho^n(\mathbf{x}_n)$ is called an n-correlation function of μ with respect to m if

$$\int_{A_1^{k_1}\times\cdots\times A_m^{k_m}} \rho^n(\mathbf{x}_n) \prod_{i=1}^{n} m(dx_i) = \int_{\mathsf{S}} \prod_{i=1}^{m} \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i)-k_i)!} d\mu$$

for any disjoint $A_i \in \mathcal{B}(\mathbb{R}^d)$, $k_i \in \mathbb{N}$ such that $k_1 + \ldots + k_m = n$.

A point process μ is called a determinantal point process generated by (K, m) if its n-correlation function ρ^n with respect to m is given by

$$\rho^n(\mathbf{x}_n) = \det[K(x_i, x_j)]_{1\le i,j\le n}. \tag{19}$$

We now introduce the Ginibre point process, which is a determinantal point process on $\mathbb{R}^2$. For convenience, we regard $\mathbb{R}^2$ as $\mathbb{C}$ by the obvious correspondence: $\mathbb{R}^2 \ni (x, y) \mapsto x + \sqrt{-1}y \in \mathbb{C}$. We set

$$K(x, y) = e^{x\bar{y}} \qquad g(dx) = \pi^{-1}e^{-|x|^2}dx.$$

Then μ_{gin} is a determinantal point process generated by (K, g).

It is well known that μ_{gin} is a translation- and rotation-invariant point process on $\mathbb{R}^2$. Furthermore, μ_{gin} can be regarded as a Gibbs measure with interaction potential

$$-2\log|x|.$$

Very loosely, μ_{gin} is a translation-invariant *measure* on $(\mathbb{R}^2)^{\mathbb{N}}$ such that

$$\mu_{\mathrm{gin}}(\prod_{i\in\mathbb{N}} ds_i) = \frac{1}{\mathcal{Z}}\prod_{i<j}^{\infty}|s_i - s_j|^2 \prod_{k\in\mathbb{N}} ds_k \tag{20}$$

Naturally, the representation (20) is not rigorous because it contains an infinite product of Lebesgue measures. To justify (20), we introduce a notion of logarithmic derivative of μ [7].

Let μ_x be the (reduced) Palm measure of μ conditioned at x

$$\mu_x(\cdot) = \mu(\cdot - \delta_x|\mathsf{s}(x) \ge 1)$$

Let $\mu^{[1]}$ be the (reduced) 1-Campbell measure on $\mathbb{R}^d\times\mathsf{S}$.

$$\mu^{[1]}(A\times B) = \int_A \rho^1(x)\mu_x(B)dx. \tag{21}$$

We say a function f defined on S is local if f is $\sigma[\pi_R]$-measurable. Here $\pi_R : \mathsf{S}\to\mathsf{S}$ is such that $\pi_R(\mathsf{s}) = \mathsf{s}(\cdot \cap S_R)$ and $S_R = \{|x| < R\}$. For a local function f which is $\sigma[\pi_{R_0}]$-measurable and R such that $R_0 \le R$ there exists a symmetric function $\check{f}_R$ defined on $\sum_{m=0}^{\infty}(S_R)^m$ such that $f(\mathsf{s}) = \check{f}_R(s_1, \ldots, s_m)$ for $\mathsf{s} = \sum_{i=1}^{m}\delta_{s_i} \in \mathsf{S}_R^m$, where $\mathsf{S}_R^m = \{\mathsf{s} \in \mathsf{S}\,;\, \mathsf{s}(S_R) = m\}$. We say f is smooth if $\check{f}_R$ is smooth for all $R \ge R_0$. We note that $\check{f}_R$ is unique because of symmetry. We easily see that f is smooth if and only if $\check{f}_R$ is smooth for some $R > R_0$.

Definition 1. We call $\mathsf{d}_\mu \in L^1(\mathbb{R}^d \times \mathsf{S}, \mu^{[1]})$ the logarithmic derivative of μ if

$$\int_{\mathbb{R}^d \times \mathsf{S}} \nabla_x f d\mu^{[1]} = -\int_{\mathbb{R}^d \times \mathsf{S}} f \mathsf{d}_\mu d\mu^{[1]} \tag{22}$$

for all $f \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_\circ$. Here, ∇_x is the derivative operator on $\mathbb{R}^d$, and $\mathcal{D}_\circ$ is the space of all bounded, local smooth functions on S.

Very informally, the logarithmic derivative is given by

$$\mathsf{d}_\mu = \nabla_x \log \mu^{[1]} \tag{23}$$

The conventional method to define infinite volume Gibbs measures for given interaction potentials is to use the Dobrushin-Lanford-Ruelle equation. This formulation is, however, not available for Coulomb potentials. The significance of the notion of the logarithmic derivative is that it also makes sense for these long-ranged potentials.

In [7], we calculated the logarithmic derivative $\mathsf{d}_{\mu_{\mathrm{gin}}}$ of the Ginibre point process is

$$\mathsf{d}_{\mu_{\mathrm{gin}}}(x, \mathsf{s}) = 2 \lim_{q \to \infty} \sum_{|x - s_i| < q} \frac{x - s_i}{|x - s_i|^2}, \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^2 \times \mathsf{S}, \mu_{\mathrm{gin}}^{[1]}), \tag{24}$$

where $\mathsf{s} = \sum_i \delta_{s_i}$, (see [7]). Note that the sum in (24) is a conditional convergence because of the long-range nature of the logarithmic interaction potential. Therefore, to prove convergence is a sensitive problem. We use the rigidity of the Ginibre point process such that, for the variance of the number of particles in the disks $\{|s| \le R\}$, its order of growth is suppressed to order R. We remark that the order of growth of the same quantity for translation-invariant Poisson point processes on $\mathbb{R}^2$ is R^2.

The equality (24) is rigorous and justifies (20) because, taking (23) into account, we have

$$\nabla_x \log\{\frac{1}{\mathcal{Z}} \prod_j^\infty |x - s_j|^2\} = 2 \lim_{q \to \infty} \sum_{|x - s_i| < q} \frac{x - s_i}{|x - s_i|^2}.$$

For a configuration $\mathsf{s} = \sum_i \delta_{s_i}$, we write $\mathbb{L}[\mathsf{s}] = \{s_i\}$. For $\mu_{\mathrm{gin},0}$-a.s. s we consider

$$dX_t^{\mathrm{x}} = dB_t + \lim_{q \to \infty} \sum_{\substack{|X_t^{\mathrm{x}} - s_i| < q \\ s_i \in \mathbb{L}[\mathsf{s}]}} \frac{X_t^{\mathrm{x}} - s_i}{|X_t^{\mathrm{x}} - s_i|^2} dt, \quad X_0^{\mathrm{x}} = \mathrm{x}. \tag{25}$$

Here $\mu_{\mathrm{gin},0}$ is the reduced Palm measure of μ_{gin} conditioned at the origin. Using the logarithmic derivative $\mathsf{d}_{\mu_{\mathrm{gin}}}(x, \mathsf{s})$, we can rewirte (25) as

$$dX_t^{\mathrm{x}} = dB_t + \frac{1}{2} \mathsf{d}_{\mu_{\mathrm{gin}}}(X_t^x, \mathsf{s}) dt. \tag{26}$$

We remark that the solution $X^{\mathsf{x}} = \{X_t^{\mathsf{x}}\}$ depends on s although we suppress it from the notation.

Theorem 4. *In $\mu_{\text{gin},0}$-probability in $\mathsf{s} \in \mathsf{S}$,*

$$\lim_{\varepsilon\to\infty} \varepsilon X_{t/\varepsilon^2}^{\mathsf{x}} = 0 \quad \textit{weakly in } C([0,\infty);\mathbb{R}^2). \tag{27}$$

That is, for any bounded continuous function F on $C([0,\infty);\mathbb{R}^2)$ and $c>0$,

$$\lim_{\varepsilon\to\infty} \mu_{\text{gin},0}(\{\mathsf{s};\, |E[F(\varepsilon X_{\cdot/\varepsilon^2}^{\mathsf{x}})] - F(\mathbf{0})| \geq c\}) = 0. \tag{28}$$

Here $\mathbf{0}$ denotes the $0 = (0,0)$-valued, constant path.

• The proof is completely different from the periodic case.
• We expect that the same phase transition holds for this case as for the periodic case.
• I have not yet prove the positivity of the effective constant for the original Ginibre point process. That is, "for μ_{gin}-a.s. s".
• If this is done, then the rest of the proof is the same as for the periodic case.

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