Time Operators and Time Crystals in Ring Systems

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I. INTRODUCTION

A. Background

In quantum mechanics and quantum field theory, position operators, momentum operators and the position-momentum uncertainty relation are well-established. For a one dimensional system, the position-momentum uncertainty relation $\Delta x \Delta p \geq \hbar/2$ can be derived mathematically from the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$. The energy-time uncertainty relation $\Delta E \Delta T \sim \hbar$ is also known *experimentally*. In analogy with position and momentum, one exepts that the energy-time uncertainty relation can be obtained from a commutation relation as shown in figure 1. However, how to define a corresponding time operator is still an open problem. So, time is conventionally regarded as a parameter, not a physical observable [1]. How can we promote time from a parameter to a physical observable? We consider this problem for ring (S^1) systems with a periodic boundary condition.

In this résumé we discuss time operators and time crystals in ring systems [2, 3]. Time crystals are recently proposed phases of matter which seem to promote time to a physical quantity. So, we consider time operators in terms of time crystals. First, we discuss quantization of ring systems. Next, we define a time crystal and explain our model. Then, we define time operators from time crystals in ring systems. Finally, we discuss the connection between time operators and time crystals.

II. QUANTIZATION OF RING SYSTEMS

There is a "hierarchy" of operators which satisfy stronger and weaker versions of the canonical commutation relation [4–11]. Here, we consider the *generalized weak Weyl relation* (GWWR) introduced by A. Arai [7]. As we exemplify below, the GWWR is necessary for a consistent quantization of ring systems. Let $\hat{A}$ be a symmetric operator on a Hilbert space $\mathcal{H}$, let $\hat{B}$ be a self-adjoint operator on $\mathcal{H}$ and $\hat{K}(s)$ ($s \in \mathbb{R}$, s may

FIG. 1. A motivation to define time operators.

denote time, position, or some other variable) be a bounded self-adjoint operator on $\mathcal{H}$ with $D(\hat{K}(s)) = \mathcal{H}$ ($D(\cdot)$ denotes operator domain), $\forall s \in \mathbb{R}$. We say that the triplet of operators $(\hat{A}, \hat{B}, \hat{K})$ obeys the GWWR in $\mathcal{H}$ if for all $\psi \in D(\hat{A})$ and for all $s \in \mathbb{R}$ we have $e^{-is\hat{B}/\hbar}\psi \in D(\hat{A})$ and

$$\hat{A}e^{-is\hat{B}/\hbar}\psi = e^{-is\hat{B}/\hbar}(\hat{A} + \hat{K}(s))\psi. \tag{1}$$

$\hat{K}(s)$ is called the commutation factor of the GWWR. $\hat{A}$ and $\hat{B}$ may represent position, momentum, angular momentum, Hamiltonian, time operator, etc. Moreover, if $\hat{K}(s)$ is differentiable with respect to s, then we can differentiate both sides of Eq. (1) and set $s = 0$ to obtain the *generalized canonical commutation relation* (GCCR) for all $\psi \in D(\hat{A}\hat{B}) \cap D(\hat{B}\hat{A})$

$$[\hat{A}, \hat{B}]\psi = i\hbar\hat{K}'(0)\psi, \tag{2}$$

where $'$ denotes derivative with respect to s. For $\hat{K}(s) = \pm s$, Eq. (1) reduces to the *weak Weyl relation* [5, 6] and Eq. (2) reduces to the canonical commutation relation $[\hat{A}, \hat{B}]\psi = \pm i\hbar\psi$. We conjecture that the commutation factor $\hat{K}(s)$, in general, depends on the real-space topology of a quantum system. Several examples are given below.

A. One-dimensional system

The Hilbert space of a one-dimensional system is $\mathcal{H} = L^2(\mathbb{R})$. The position operator $\hat{x}$ and the momentum operator $\hat{p}$ are both self-adjoint. For $(\hat{A}, \hat{B}, \hat{K}(s)) = (\hat{x}, \hat{p}, s)$ we have $[\hat{x}, \hat{p}] = i\hbar$. For $(\hat{A}, \hat{B}, \hat{K}(t)) = (\hat{T}_{\mathbb{R}}, \hat{H}, -t)$ we have $[\hat{H}, \hat{T}_{\mathbb{R}}] = i\hbar$, where $\hat{H} = \hat{p}^2/2m$ and $\hat{T}_{\mathbb{R}} = -\frac{m}{2}(\hat{x}\hat{p}^{-1} + \hat{p}^{-1}\hat{x})$ is the Aharonov-Bohm time operator. $\hat{T}_{\mathbb{R}}$ is a symmetric operator but it is not self-adjoint. So, real eigenvalues and orthogonal eigenstates are not ensured, hence it is not suitable as a physical observable.

B. Ring systems

Let $\mathcal{H} = L^2([-\pi, \pi])$ be the Hilbert space of ring systems with the periodic boundary condition $\psi(\theta) = \psi(\theta + 2\pi)$, where $\theta = x/R$ is the angular coordinate.

Position operators are defined as a set of operators whose eigenvalues have one-to-one correspondence with points on a manifold. But, in the case of periodic boundary condition, $\hat{x}$ is a multivalued operator, hence it is not suitable as a position operator [9–11]. For instance, any continuous wave function ψ of a ring system must satisfy the periodic boundary condition but $\hat{x}\psi$ is not periodic. Moreover, many contradictions arise if $\hat{x}$ is not used with care. For example, the expectation value of the canonical commutation relation with momentum eigenstates ψ_p (including the ground state ψ_0) leads to two equivalent equations $\langle\psi_p, [\hat{x}, \hat{p}]\psi_p\rangle = p(\langle\psi_p, \hat{x}\psi_p\rangle - \langle\psi_p, \hat{x}\psi_p\rangle) = 0$ and $\langle\psi_p, [\hat{x}, \hat{p}]\psi_p\rangle = \langle\psi_p, i\hbar\psi_p\rangle = i\hbar$. These results lead to the contradiction $0 = i\hbar$. Similarly, one can verify that $\langle\psi_p, [\hat{H}, \hat{T}]\psi_p\rangle = 0$ for $p \neq 0$. These facts suggest that the pairs $(\hat{x}, \hat{p})$ and $(\hat{T}, \hat{H})$ are very sensitive to the domain on which their commutation relation is applied. In other words, the canonical commutation relation imposes a restriction on possible wave functions of a ring system [12, 13].

So, how can we obtain position operators, time operators and commutation relations for general periodic functions? Previous studies to solve this problem include refs. [9–11]. In all cases, the solution was to use periodic angle variables such as $f(\theta) = \theta \bmod 2\pi$, $f(\theta) = \cos\theta$, $f(\theta) = \sin\theta$, and $f(\theta) = e^{i\theta}$. In fact, these different formulations have the same mathematical structure, namely the GWWR. Instead of the multivalued position operator $\hat{x}$ or the angle operator $\hat{\theta} = \hat{x}/R$ we use a periodic position operator $\hat{f}$ such that $[\hat{f}\psi](\theta) = f(\theta)\psi(\theta)$ with $f(\theta+2\pi) = f(\theta)$. We also define $\hat{f}_s = e^{i\hat{\pi}_\theta s/\hbar}\hat{f}e^{-i\hat{\pi}_\theta s/\hbar}$ with $[\hat{f}_s\psi](\theta) = f(\theta+s)\psi(\theta)$, $\hat{\pi}_\theta = R\hat{p}$ is the canonical angular momentum operator. Then, from the GWWR with $\hat{A} = \hat{f}$ and $\hat{B} = \hat{\pi}_\theta$ we obtain $\hat{K}(s) = \hat{f}_s - \hat{f}$, $\hat{K}'(s) = \hat{f}'_s = d\hat{f}_s/ds$, and these operators satisfy the GCCR

$$[\hat{\pi}_\theta, \hat{f}] = -i\hbar\hat{f}'_s\big|_{s=0}. \tag{3}$$

Periodic operators can be constructed using Fourier series

$$\hat{f} = \sum_{n=-\infty}^{\infty} c_n \hat{W}^n. \tag{4}$$

Here, we have defined the unitary position operator $\hat{W} = \widehat{e^{i\theta}}$, $\hat{W}^\dagger = \hat{W}^{-1}$ [9, 11]. $\hat{\pi}_\theta$ and $\hat{W}$ satisfy the following eigenvalue equations and commutation relation in $\mathcal{H}$

$$\hat{\pi}_\theta \psi_l = l\hbar\psi_l, \tag{5}$$

$$\hat{W}^n \psi_l = \psi_{l+n} \tag{6}$$

$$[\hat{\pi}_\theta, \hat{W}^n] = n\hbar\hat{W}^n. \tag{7}$$

l and n are integers and one can verify that $\hat{W}^{-n} = \hat{W}^{\dagger n}$. From this definition with Eq. (3) and Eq. (7) we obtain

$$\hat{f}'_s\big|_{s=0} = \sum_{n=-\infty}^{\infty} c_n \frac{i}{\hbar}[\hat{\pi}_\theta, \hat{W}^n] = \sum_{n=-\infty}^{\infty} c_n in\hat{W}^n.$$

Besides, it is a known fact from spectral analysis that the real and imaginary parts of a bounded operator (i.e. an operator whose spectrum is bounded from above and below) are self-adjoint operators. $\hat{W}^n$s are bounded operators, so $\hat{f}$ is a bounded self-adjoint operator if $c_n = c^*_{-n}$. As specific examples of $\hat{f}$ we may take the self-adjoint sine operator $\hat{S}$ and the self-adjoint cosine operator $\hat{C}$ to specify a point on a ring [10, 11].

$$\hat{C} = \frac{\hat{W} + \hat{W}^\dagger}{2} \quad (c_n = c^*_{-n} = \frac{1}{2}\delta_{n,1}), \tag{8}$$

$$\hat{S} = \frac{\hat{W} - \hat{W}^\dagger}{2i} \quad (c_n = c^*_{-n} = \frac{i}{2}\delta_{n,1}), \tag{9}$$

Another important operator is the single-valued periodic angle operator [10]

$$\hat{\Theta} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{in}(\hat{W}^n - \hat{W}^{-n}) = -i\mathrm{Log}(\hat{W}) \tag{10}$$

which is the operator version of the "sawtooth function" $\Theta(\theta) = 2\sum_{n=1}^{\infty}(-1)^{n+1}n^{-1}\sin(n\theta) = -i\mathrm{Log}(e^{i\theta})$, i.e. the angle θ restricted in the region $[-\pi, \pi]$. These operators satisfy the commutation relations

$$[\hat{\pi}_\theta, \hat{C}] = i\hbar\hat{S}, \tag{11}$$

$$[\hat{\pi}_\theta, \hat{S}] = -i\hbar\hat{C}, \tag{12}$$

$$[\hat{\Theta}, \hat{\pi}_\theta] = i\hbar\{1 - \delta(\hat{\Theta} + \pi)\} \tag{13}$$

Next, we use quantum mechanics on S^1 to construct a model of time crystal.

III. QUANTUM TIME CRYSTAL BY DECOHERENCE

A. Background

In 2012, F. Wilczek defined a quantum time crystal as quantum mechanical ground state which spontaneously breaks time translation symmetry [14]. Here, spontaneous symmetry breaking means that a system or response from a system fails to preserve the symmetry of a Lagrangian or a Hamiltonian, as the system relaxes to an energetically stable phase. See Figure 2. In a QTC ground state there exists an operator

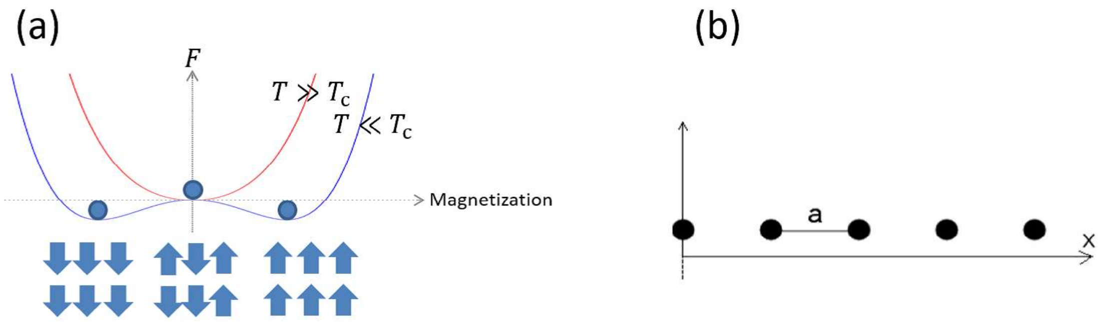

FIG. 2. Typical examples of spontaneous symmetry breaking. (a) The Ising Hamiltonian $\hat{H}_{\text{Ising}} = -J \sum_{\langle i,j \rangle} \hat{s}_i \hat{s}_j$ describes a system composed of spin 1/2 particles. At finite temperature, the phase of this system minimizes the free energy $F = \langle \hat{H} \rangle - TS$ (T is temperature and S is the entropy of the system). At high temperature, the free energy is minimizes if all spins $\hat{s}_i$ point up or down with equal probability. As temperature decreases, the free energy is minimized if all particles spins are oriented in the same direction. Although $\hat{H}_{\text{Ising}}$ is symmetric under the transformation $\hat{s}_i \to -\hat{s}_i$, the ground state breaks this symmetry. (b) Formation of a lattice structure can be explained likewise. Let us consider a many-body system described by the Hamiltonian $\hat{H} = \sum_i \frac{\hat{\mathbf{P}}_i}{2m} + \frac{1}{2} \sum_{i \neq j} U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)$. Here, $U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)$ is the potential between particle i and particle j. This system possesses continuous spatial translation symmetry $\mathbf{r}_i \to \mathbf{r}_i + \epsilon$ ($\forall i$) where ϵ is an infinitesimal displacement vector. However, as temperature decreases, the inter-particle distance can be fixed and form a lattice structure. In this case, continuous spatial translation symmetry spontaneous reduces to its discrete subgroup.

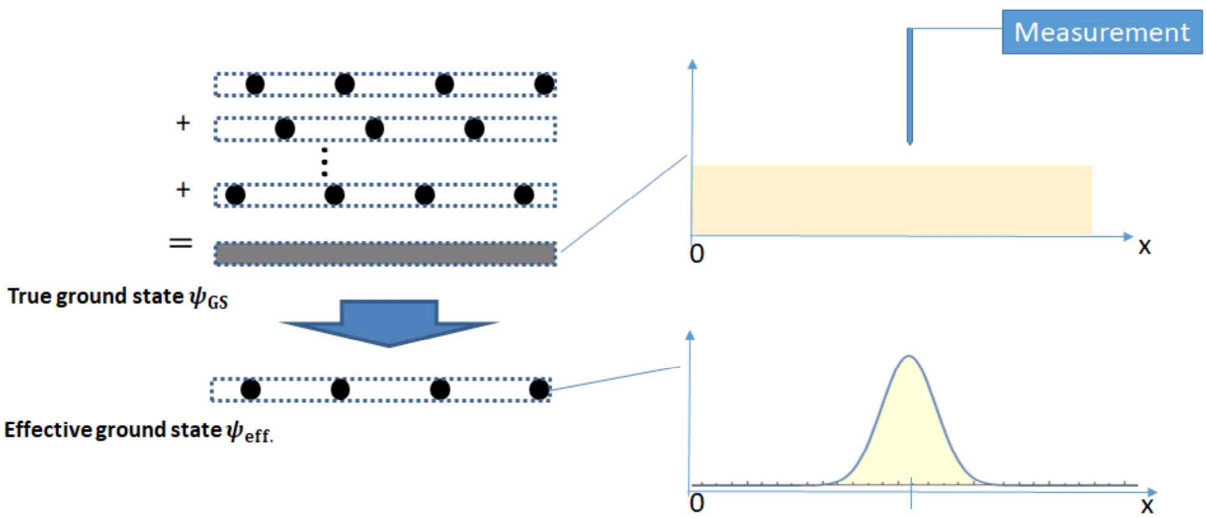

FIG. 3. Let us consider the above example of lattice formation in free space. At the ground state we can ignore degrees of freedom between individual particles and only consider the center of mass motion. In this case, this system can be regarded basically as a free particle. The ground state of this systems has zero center-of-mass momentum with zero momentum uncertainty $\Delta p = 0$. Then, from $\Delta x \Delta p \geq \hbar/2$, the position uncertainty diverges. This state can be interpreted as a superposition of lattices with different center of mass positions (different vertices) which preserves continuous spatial translation symmetry. A state with broken symmetry can be obtained by symmetry-breaking perturbations such as measurements. If the measured state and the symmetric ground state are energetically degenerate, then the measured state can be regarded as an effective ground state. Conventionally, spontaneous symmetry breaking from the true ground state to the effective ground state is expected to occurs in the infinite-volume limit $V \to \infty$.

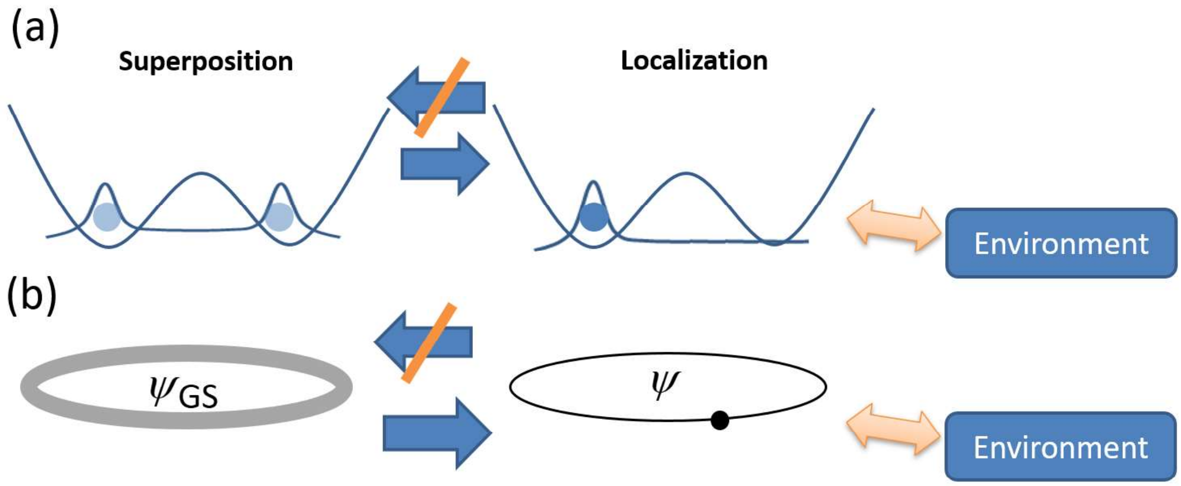

FIG. 4. Symmetry breaking by decoherence. (a) Example with two-state system. (b) Example with a particle on a ring. This "particle" corresponds to the phase of an incommensurate charge density wave in our model.

$\hat{Q}$ whose expectation value oscillates permanently with a well-defined "lattice constant" P, that is, with a well-defined period.

Several realization of Wilczek's idea have been proposed. However, Watanabe-Oshikawa [15] argued that spontaneous breaking of time translation symmetry cannot occur at the ground state. Their main argument is that, while spontaneous symmetry breaking requires the infinite-volume limit (Figure 3), oscillation of a time crystal vanish in this limit.

Recently, it was shown that there is a notion of spontaneous breaking of time translation symmetry in periodically driven (Floquet) states [16–18] and this idea was proved experimentally[19].

B. Symmetry Breaking by Decoherence

Here, we take a different approach. In section II, we argued that quantization of S^1 is different from quantization of $\mathbb{R}$. We take advantage of this difference. Since the infinite-volume limit is not applicable, we consider the possibility of a QTC state without spontaneous symmetry breaking: We consider symmetry breaking by *decoherence*. Decoherence is defined as the loss of quantum coherence of a system coupled to its environment.

The concept of translation symmetry breaking by decoherence is illustrated in Fig. 4. First, consider a simple two-state system (Fig. 4(a)). A particle can tunnel through a potential barrier and exist at two states simultaneously. However, if this system starts to interact with its surrounding environment, then the particle will localize at one of the states [20]. Similarly, the ground state of a free particle confined on a ring is a plane wave state (Fig. 4(b)). Coupling to environment will localize the particle and break rotational symmetry. This "particle" corresponds to the phase of an incommensurate charge density wave ring (ICDWring) in our model.

C. Our model

Our model consists of a ring-shaped incommensurate charge density wave (ICDW ring) threaded by a fluctuating magnetic flux. A charge density wave (CDW) is a periodic modulation of electric charge density which can occurs in quasi-one-dimensional crystals [21–23] (Fig. 5 (a)). Ring-shaped crystals and ring-shaped

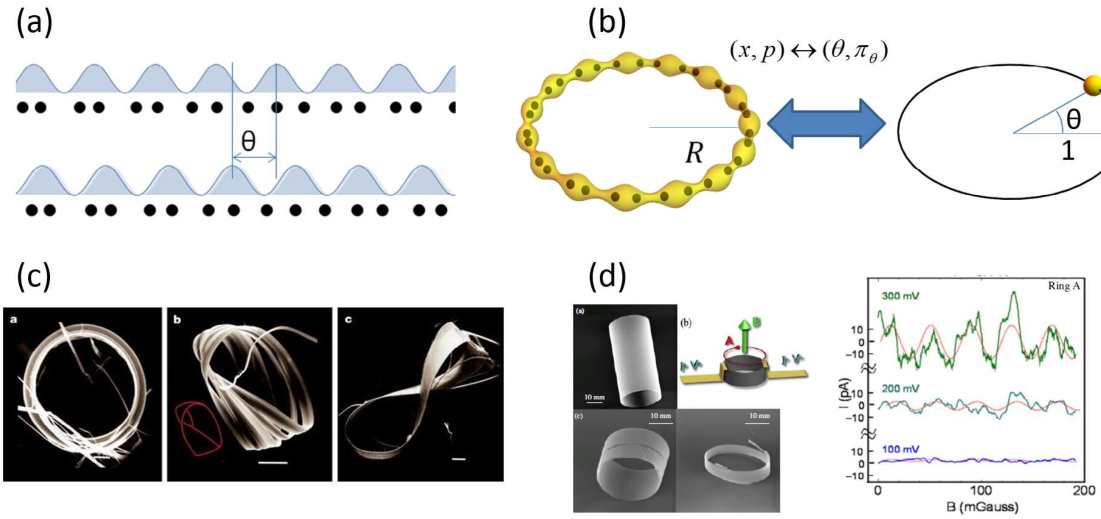

FIG. 5. (a) A description of charge density wave. The black dots represent ions of the uncerlying lattice and the wave describe the electron charge density. (b) An incommensurate charge density wave can be mapped to a free particle on a ring with unit radius. (c) Topological crystals with charge density waves. (d) Aharonov-Bohm oscillation of a charge density wave ring has been observed experimentally [26].

(I)CDWs have been produced experimentally [24] (FIG. 5 (c)). The presence of circulating CDW current [25] and Aharonov-Bohm oscillation (evidence of macroscopic wave function) [26] are verified experimentally (Fig. 5 (d)). The energy of an incommensurate CDW (ICDW) is independent of its phase (i.e. position), so an ICDW ring can be mapped to a free particle (boson) on a ring with unit radius (Fig. 5 (b)). The quantized Hamiltonian of an isolated ICDW ring is $\hat{H}_0 = \hat{\pi}_\theta^2/2I$ where $\hat{\pi}_\theta$ is the angular-momentum operator and I is the moment of inertia. Now, suppose that the charge density can be defined as the expectation value of the charge density operator

$$\hat{n}(x) = n_0 + \frac{n_1}{2}(\hat{W}e^{ix/\lambda} + \hat{W}^\dagger e^{-ix/\lambda}) \tag{14}$$

where n_0 is the uniform charge density and n_1 is the amplitude of a classical ICDW. Note that this operator is equivalent to the cosine operator $\hat{C}$ defined by Eq. (8), hence $\hat{n}$ is suitable to describe the phase of a quantum ICDW ring.

The ground state of an isolated ICDW ring is the plane-wave state with zero angular momentum $\psi_{\rm GS}(x) = 1/\sqrt{2\pi}$, hence the charge density expectation value at the ground state is uniform

$$\langle \psi_{\rm GS}, e^{i\hat{H}_0 t/\hbar}\hat{n}e^{-i\hat{H}_0 t/\hbar}\psi_{\rm GS}\rangle = n_0 + \frac{n_1}{2\pi}\int_{-\pi}^{\pi}\cos(x/\lambda+\theta)d\theta = n_0. \tag{15}$$

This state can be interpreted as a superposition of ICDWs with different phases, which preserves the continuous spatial translation symmetry and continuous time translation symmetry of the Hamiltonian.

Now, suppose that the ICDW ring starts to interact with its surrounding enviromnent at $t = 0$. Then, we expect decoherence of the phase θ. This interaction is modeled using the Caldeira-Leggett model [27] which is a model quantum Brownian motion. The environment is described as a set of harmonic oscillators. The Hamiltonian of the entire system is

$$\hat{H} = \frac{1}{2I}\left(\hat{\pi}_\theta - \sum_j c_j q_j\right)^2 + \sum_j \left(\frac{\hat{p}_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2\hat{q}_j^2\right). \tag{16}$$

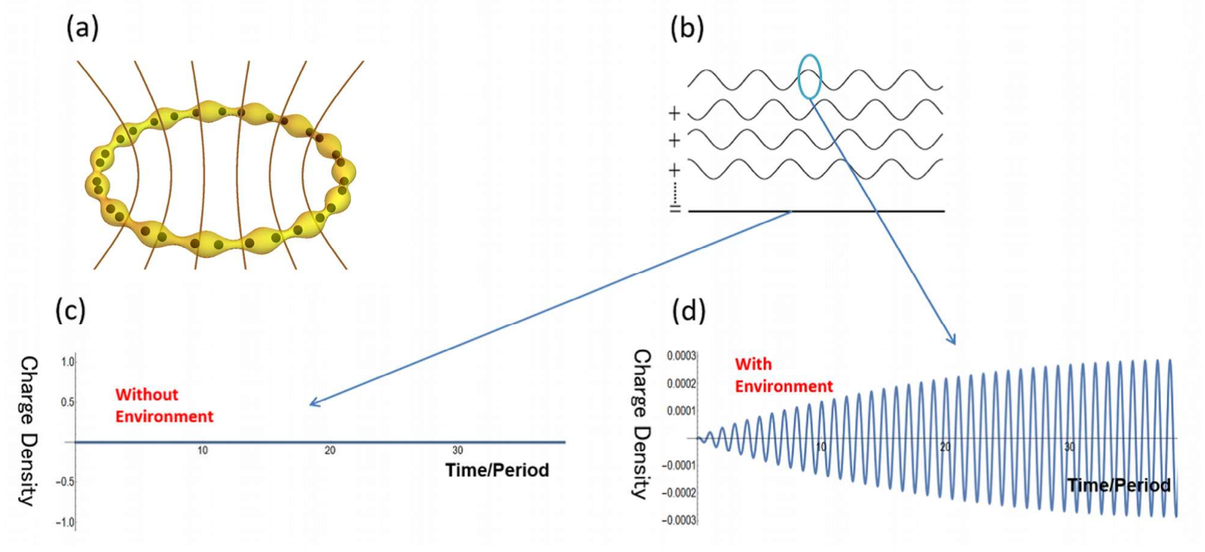

FIG. 6. (a) Incommensurate charge density wave (ICDW) ring threaded by a fluctuating magnetic flux (environment). (b) suppose that we measure the charge density at a specific position on the ring. The ground state of an ICDW ring is a uniform superposition of periodically oscillating standing waves with different phases, so the charge density expectation value is constant in time (c). (d) Coupling with environment will localize the phase and the charge density expectation value oscillates with a constant period $P = 4\pi I/\hbar$ where I is the moment of inertia of the ICDW ring.

The environment parameters are used to define a characteristic damping rate γ such that the classical solution has the form $\theta(t) \sim e^{-2\gamma t}$. For $\gamma t \ll 1$, the charge density expectation value is given by

$$\langle e^{i\hat{H}t/\hbar}\hat{n}(x)e^{-i\hat{H}t/\hbar}\rangle \approx n_0 + n_1 \mathrm{sinc}[\pi e^{-2\gamma t}]\cos(\hbar t/2I)\cos(x/\lambda). \tag{17}$$

The charge density oscillation is shown in FIG. 6. This kind of time-periodic oscillation is a consequence of using periodic angle operators.

Coupling to environment will inevitably introduce friction to the system such that the oscillation will eventually decay at $t = \tau_{\mathrm{damp}} = \gamma^{-1}$. However, for $t \ll \tau_{\mathrm{damp}}$ the oscillation period P is well defined. If friction is sufficiently weak such that $P \ll \tau_{\mathrm{damp}}$, then we have a model of effective QTC with life time $\sim \tau_{\mathrm{damp}}$ [28].

IV. TIME OPERATORS AND TIME CRYSTALS

One of the requirements to define time operators is that they satisfy a commutation relation similar to that of position and momentum: This requirement is necessary to derive time-energy uncertainty relations as well as for the unification of space and time in relativistic quantum mechanics. In this context, we propose that time operators in ring systems should be defined based on the GWWR. If periodic position operators $\hat{f}$ are used instead of $\hat{\theta}$, then time operators are expected to obey a commutation relation similar to Eq. (3).

A. Derivation of Generalized Self-Adjoint Time Operators

Using the quantization of ring systems based on the GWWR and the GCCR, we show that there exists a class of generalized self-adjoint time operators in ring systems with $\hat{K}'(0) = \hat{f}'_s\big|_{s=0}$. We take Eq. (3) as a starting point to derive these operators. Consider a free particle with moment of inertia I and Hamiltonian

$\hat{H} = \frac{\hat{\pi}_\theta^2}{2I}$. Consider the following commutation relation

$$[\hat{H}, \hat{W}^n]\psi = [\hat{H}, \hat{W}^n] \sum_{l=-\infty}^{\infty} \psi_l \langle \psi_l, \psi \rangle$$
$$= \sum_{l=-\infty}^{\infty} (E_{l+n} - E_l)\psi_{l+n} \langle \psi_l | \psi \rangle ,$$

where we used Eq. (5), $\hat{H}\psi_l = E_l \psi_l$ and the identity operator $\psi = \sum_{l=-\infty}^{\infty} \langle \psi_l, \psi \rangle \psi_l$. The desired time operator is obtained if we can get rid of $E_{l+n} - E_l = \frac{(2l+n)n\hbar^2}{2I}$. If n is an odd integer then $l = -\frac{n}{2}$ is not an eigenvalue of $\hat{\pi}_\theta$, so $1/(2l+n)$ is bounded. On the other hand, if n is an even integer, we can use the projection operator $\mathscr{P}_{-n/2}$ to remove the state $\psi - \frac{n}{2}$. In both cases we can define the time operator $\hat{T}$ as

$$\hat{T}\psi = -\sum_{n,l=-\infty}^{\infty} c_n n\hbar \frac{(1 - \delta_{l,-\frac{n}{2}})}{E_{l+n} - E_l} \psi_{l+n} \langle \psi_l, \psi \rangle .$$

This kind of time operator has been studied in refs. [13, 29, 30]. The commutation relation between this time operator and the Hamiltonian gives

$$[\hat{H}, \hat{T}]\psi = -\sum_{n,l=-\infty}^{\infty} c_n n\hbar (1 - \delta_{l,-\frac{n}{2}})\psi_{l+n} \langle \psi_l, \psi \rangle$$
$$= -\sum_{n=-\infty}^{\infty} c_n n\hbar \hat{W}^n \psi + \sum_{n=-\infty}^{\infty} c_{2n} 2n\hbar \psi_n \langle \psi_{-n}, \psi \rangle$$
$$= i\hbar \hat{f}'_s\big|_{s=0} \psi_\psi + \sum_{n=-\infty}^{\infty} c_{2n} 2n\hbar \psi_n \langle \psi_{-n}, \psi \rangle ,$$
$$\psi \in D(\hat{H}\hat{T}) \cap D(\hat{T}\hat{H}).$$

Here, ψ_n are linearly independent, hence the last term with c_{2n} vanishes only if $c_{2n} \langle \psi_{-n}, \psi \rangle = 0$ for all n. Finally, we can define the bounded self-adjoint operator which has the dimension of time

$$\hat{\mu}_n = \begin{cases} \frac{2I}{2\hat{\pi}_\theta + n\hbar} & , \ n = \text{odd} \\ \mathscr{P}_{-n/2} \frac{2I}{2\hat{\pi}_\theta + n\hbar} \mathscr{P}_{-n/2} & , \ n = \text{even} \end{cases}$$

and write our time operator as

$$\hat{T} = -\sum_n c_n \hat{W}^n \hat{\mu}_n$$
$$= -\sum_n \frac{(c_n \hat{W}^n \hat{\mu}_n + c_n^* \hat{\mu}_n \hat{W}^{-n})}{2}, \tag{18}$$
$$[\hat{H}, \hat{T}]\psi = i\hbar \hat{f}'_s\big|_{s=0} \psi, \tag{19}$$
$$\psi \in \Big\{ \psi \in D(\hat{T}) : c_{2n} \langle \psi_{-n}, \psi \rangle = 0, \forall n \Big\}.$$

The symmetrized form in the second line is obtained from the fact that $\hat{W}^n \hat{\mu}_n = \hat{\mu}_{-n} \hat{W}^n$ and $c_n^* = c_{-n}$. This time operator has the following properties:
1. $\hat{T}$ is a bounded symmetric operator, hence it is self-adjoint. Another reason that this operator is self-adjoint is the discreteness of energy spectrum [12, 13, 29, 30]. Discreteness of the energy spectrum is very important but, as we have seen, it is not the only factor that determines the time operators.

2. $\hat{T}$ satisfies the GWWR with $(\hat{A}, \hat{B}, \hat{K}(t)) = (\hat{T}, \hat{H}, \hat{T}(t) - \hat{T})$, $\hat{T}(t) = e^{i\hat{H}t/\hbar}\hat{T}e^{-i\hat{H}t/\hbar}$. One can readily show that $d\hat{K}(t)/dt|_{t=0} = -\hat{f}'_s|_{s=0}$ and obtain the correct generalized commutation relation.
3. We note that $\hat{T}$ is not unique because, for any self-adjoint operator $\hat{F}$ which commutes with the Hamiltonian, $\hat{T}$ and the new operator $\hat{T} + \hat{F}$ satisfy the same commutation relation with $\hat{H}$. However, if $\hat{f}$ is prescribed then $\hat{T}$ is unique up to addition of $\hat{F}$. Which $\hat{f}$ we choose depends on physical basis, i.e. on the phenomena (such as time of arrival, time crystal or other events) that we want to measure.
4. If we choose $\hat{f}$ such that $R\hat{f} \to \hat{x}$ in the infinite-volume limit $R \to \infty$, then the commutation relation Eq. (19) reduces to the canonical commutation relation. This limit must be taken for ψ in the domain $\mathcal{D} \subset D(\hat{H}\hat{T}) \cap D(\hat{T}\hat{H})$ such that $\psi \in \mathcal{D}$ is also square-integrable for $R \to \infty$. This domain includes the Gaussian-like minimum-uncertainty state considered by S. Tanimura [11].
5. $\hat{T}$ is defined as a linear combination of non-Hermitian operators $\hat{W}^n\mu_n$ with space-time inversion $(\mathcal{PT})$ symmetry. The significance of $\mathcal{PT}$-symmetric time operators is discussed in ref. [3].

Now, let us consider some special cases and discuss their physical significance. If $c_n = c^*_{-n} = \frac{i}{2}\delta_{n,1}$ we obtain

$$\hat{T}_{S^1} = \frac{1}{2i}[\hat{\mu}_1\hat{W}^\dagger - \hat{W}\hat{\mu}_1] = \mathrm{Im}[\hat{\mu}_1\hat{W}^\dagger], \tag{20}$$

which satisfies the commutation relation

$$[\hat{H}, \hat{T}_{S^1}] = i\hbar\hat{C}. \tag{21}$$

Note that $\hat{T}_{S^1}$ reduces to $\hat{T}_{\mathbb{R}}$ in the infinite-radius limit $R \to \infty$. Using $x = R\theta$, $k = l/R$, $p = \hbar k$, $\psi_l(\theta) = \psi_k(x)$, and $I = mR^2$, one can verify that

$$[\hat{T}_{S^1}\psi_l](\theta) = -\frac{m}{2}\left(\frac{i}{k^2\hbar} + \frac{2x}{k\hbar}\right)\psi_k(x) + O(R^{-2}).$$

Then, from the commutation relation $[\hat{x}, \hat{p}^{-1}] = -i\hbar/\hat{p}^2$ we obtain

$$[\hat{T}_{S^1}\psi_l](\theta) = [\hat{T}_{\mathbb{R}}\psi_k](x) + O(R^{-2}).$$

Similarly, one can show that $R\hat{S} \to \hat{x}$, $\hat{C} \to 1$ as $R \to \infty$, hence

$$\begin{aligned} &\hat{T}_{S^1} \to \hat{T}_{\mathbb{R}}, \\ &[\hat{S}, \hat{\pi}_\theta] = i\hbar\hat{C} \to [\hat{x}, \hat{p}] = i\hbar, \\ &[\hat{H}, \hat{T}_{S^1}] = i\hbar\hat{C} \to [\hat{H}, \hat{T}_{\mathbb{R}}] = i\hbar, \end{aligned}$$

as $R \to \infty$. Therefore, we conclude that $\hat{T}_{S^1}$ is a self-adjoint analogue of the Aharonov-Bohm time operator $\hat{T}_{\mathbb{R}}$ in S^1.
If $c_n = c^*_{-n} = \frac{1}{2}\delta_{n,1}$ we can define the time operator $\hat{T}^{\mathrm{Re}}_{S^1}$

$$\hat{T}^{\mathrm{Re}}_{S^1} = \hat{\mu}_1 - \frac{1}{2}[\hat{W}\hat{\mu}_1 + \hat{\mu}_1\hat{W}^\dagger] \tag{22}$$

which satisfies the commutation relation

$$[\hat{H}, \hat{T}^{\mathrm{Re}}_{S^1}] = -i\hbar\hat{S}. \tag{23}$$

Note that $\hat{\mu}_1$ in Eq. (22), which commutes with the Hamiltonian $\hat{H}$, was included so that the matrix elements of $\hat{T}^{\mathrm{Re}}_{S^1}$ do not diverge in the limit $R \to \infty$. The physical significance of $\hat{T}^{\mathrm{Re}}_{S^1}$ can be understood by taking the large radius limit. Using $I = mR^2$, $l = Rk = Rp/\hbar$ and $\theta = x/R$, we find that $\hat{T}^{\mathrm{Re}}_{S^1}$ has matrix elements

$$[\hat{T}^{\mathrm{Re}}_{S^1}\psi_l](\theta) = -\frac{1}{4\pi}\frac{\lambda_{\mathrm{dB}}}{v}\psi_l(\theta) + O(R^{-1}).$$

Here $\lambda_{\rm dB} = 2\pi\hbar/p$ is the de Broglie wavelength and $v = p/m$ is the group velocity of the particle. The term $\lambda_{\rm dB}/v$ describes a "matter wave clock", i.e. because of the periodicity of de Broglie wavelength, a particle moving with a fixed velocity v has an internal clock with period $\lambda_{\rm dB}/v$ [31, 32].
Note that $\hat{S}$ in Eq. (23) vanishes as $R \to \infty$. So, although $T^{\rm Re}_{S^1}$ is a time operator which satisfies the GWWR, it is not a time operator in the sense of the canonical commutation relation.
The third example is for the periodic angle operator $\hat{\Theta}$ defined in Eq. (10). The corresponding time operator is defined as

$$\hat{T}_\Theta = -\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{in} (\hat{W}^n \hat{\mu}_n - \hat{\mu}_n \hat{W}^{-n}) \tag{24}$$

which satisfies the commutation relation

$$[\hat{H}, \hat{T}_\Theta] = -i\hbar\{1 - \delta(\hat{\Theta} + \pi)\}. \tag{25}$$

The delta function $\delta(\hat{\Theta} + \pi)$ has a contribution only if $\theta + (2n + 1)\pi = 0$. Otherwise $\hat{T}_\Theta$ satisfies the canonical commutation relation and it is equivalent to Galapon's time operator for periodic system [12, 29, 33]. Therefore, $\hat{T}_\Theta$ is interpreted as a time-of-arrival operator.

B. Connection to Time Crystals

How are these time operators connected to QTC? In our previous model of QTC, the Heisenberg operator $\hat{C}(t) = e^{i\hat{H}t/\hbar}\hat{C}e^{-i\hat{H}t/\hbar}$ describes the local oscillation of an incommensurate charge density wave: this oscillation is intrinsic to a ring system [2]. Other periodic operators $\hat{f}$ can also be used to model QTC in ring systems: Note that $\hat{W}^n$, as a momentum raising operator, can be written as $\hat{W}^n = \sum_l |\psi_{l+n}\rangle\langle\psi_l|$. Then, because $(E_{l+n} - E_l)/(E_{0+n} - E_0) = 2l + n$ is an integer, the Heisenberg operators $\hat{W}(t) = e^{i\hat{H}t/\hbar}\hat{W}e^{-i\hat{H}t/\hbar}$, $\hat{f}(t) = e^{i\hat{H}t/\hbar}\hat{f}e^{-i\hat{H}t/\hbar}$ and $\hat{T}(t) = e^{i\hat{H}t/\hbar}\hat{T}e^{-i\hat{H}t/\hbar}$ have a periodic time evolution with period $P = 2\pi(E_1 - E_0)^{-1} = 4\pi I/\hbar$:

$$\begin{aligned}\hat{W} &= \sum_{l=-\infty}^{\infty} |\psi_{l+n}\rangle\langle\psi_l| e^{it(E_{l+n}-E_l)/\hbar} \\ &= \sum_{l=-\infty}^{\infty} |\psi_{l+n}\rangle\langle\psi_l| e^{i(t+P)(E_{l+n}-E_l)/\hbar} \\ &= \hat{W}(t + P),\end{aligned}$$

$$\Rightarrow \hat{T}(t) = \hat{T}(t + P) \text{ and } \hat{f}(t) = \hat{f}(t + P).$$

The period P diverges as $R \to \infty$, so this periodicity is intrinsic to ring systems. In fact, for a one-dimensional free particle, it is clear from $\hat{x}(t) = \hat{x} + \hat{p}t/m$ that

$$\hat{T}_\mathbb{R}(t) = \hat{T}_\mathbb{R} + t \tag{26}$$

is not periodic. The commutation factor $\hat{K}(t) = \hat{T}(t) - \hat{T}$ of a ring system is also periodic with a radius-dependent period. On the other hand, for a one-dimensional system ($\mathbb{R}$) we have $\hat{K}(t) = t$, hence time for a one-dimensional system is not necessarily periodic. Therefore, $\hat{K}(t)$ may be interpreted as a function which gives the "temporal structure" of a quantum system.

Because we are using a general mathematical structure (Eq. (1)) to construct time operators, our work is not limited to ring systems but applies to other QTC models as well. We surmise $\hat{K}(t)$ characterizes the time-periodic evolution of a QTC. For our previous QTC model, this quantity is the charge density operator

$\hat{C}(t)$. QTC states can also be realized in excited Floquet states [16–19, 34–36]. If a Floquet system were driven with a period P, a Floquet time crystal (FTC) would return to its initial state after period nP (n is an integer), hence time translation symmetry is spontaneously broken. So, we conjecture the following set of operators for a FTC

$$\begin{aligned} [\hat{H}(t), \hat{T}(t)] &= \dot{\hat{K}}(t), \\ \hat{H}(t+P) &= \hat{H}(t), \\ \hat{T}(t+nP) &= \hat{T}(t), \\ \hat{K}(t+nP) &= \hat{K}(t). \end{aligned} \tag{27}$$

For instance, the Hamiltonians $\hat{H}(t)$ of most of the FTC models proposed so far are composed of Pauli matrices σ_i. If we set $\hat{H}(t+P) = \hat{H}(t) = \begin{cases} \sigma_1, & 0 \leq t < \pi, \\ 0, & \pi \leq t < P \end{cases}$ as a prototype of a Floquet Hamiltonian then we can identify $\hat{T}(t) = U^\dagger(t)\sigma_2 U(t), \hat{K}(t) = \hat{T}(t) - T(0)$, and $\dot{\hat{K}}(t) = 2U^\dagger(t)\sigma_3 U(t)$ with the unitary time evolution $U(t) = e^{-\frac{i}{\hbar}\int_0^t \hat{H}(t')dt'}$. In this case, $\hat{T}$ and $\hat{K}(t)$ are periodic with period $2P$. As a GCCR we obtain the $SU(2)$ commutation relation $[\sigma_1, \sigma_2] = 2i\sigma_3$. Further investigation of this conjecture is left for future study.

Therefore, it seems like time operators and time crystals are interrelated. The periods of a QTC seems to promote time from a parameter to a physical quantity. So, QTC are promising systems to define time operators. In spite of recently proposed models of QTC in excited states [16–19, 34–36], the original idea of a QTC is to define a dynamical *ground state* which breaks time translation symmetry [14]. How to construct a QTC ground state is a very important open problem in quantum mechanics, quantum field theory, condensed matter physics, and related fields, because it concerns the question of what a ground state is. The existence of QTC ground states have been criticized by showing that spontaneous braking of time translation symmetry at ground state does not occur in the infinite-volume limit [15, 37]. However, if the periodic boundary condition is not negligible, our results suggest how a time crystal can be defined in ring systems. We will have a QTC ground state if $\hat{K}(t) = \hat{T}(t) - \hat{T}$ and $d\hat{K}(t)/dt$ have periodic expectation value at ground state. Instead of a ring system we can also consider other systems with non-trivial real-space topologies. Once we have the appropriate mathematical structure to define a time operator (such as Eq. (1)), then we can apply it to models of time crystals, possibly including ground states.

V. CONCLUSION

In this résumé we discuss time operators and time crystals in ring systems [2, 3]. First, we discuss quantization of ring systems based on the generalized Weak Weyl relation. The angle operator $\hat{\theta}$ is multivalued, so it is not suitable as a position operator. Instead, we used periodic angle operators to quantize ring systems. Next, we showed that time translation symmetry of a ring system with a macroscopic quantum ground state is broken by decoherence. In particular, we considered a ring-shaped incommensurate charge density wave (ICDW ring) threaded by a fluctuating magnetic flux: the Caldeira-Leggett model is used to model the fluctuating flux as a bath of harmonic oscillators. The charge density expectation value of a quantized ICDW ring coupled to its environment oscillates periodically. This model forms a metastable quantum time crystal with a finite length in space and in time. Then, we investigated time operators in the context of quantum time crystals in ring systems. The generalized weak Weyl relation is used to derive a class of self-adjoint time operators for ring systems with a periodic time evolution: The conventional Aharonov-Bohm time operator is obtained by taking the infinite-radius limit. Finally, we discuss the connection between time operators, time crystals and real-space topology.

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