# Weierstrass semigroups on double covers of plane curves of degree $7$ [1]

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**Abstract**

We study Weierstrass semigroups of ramification points on double covers of plane curves of degree $7$. We treat the cases where the Weierstrass semigroups are generated by at most $5$ elements and the ramification point is on a total flex

# 1 Introduction

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid $H$ of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0 \backslash H$ is finite. The cardinality of $\mathbb{N}_0 \backslash H$ is called the *genus* of $H$, denoted by $g(H)$. In this paper $H$ always stands for a numerical semigroup. A *curve* means a projective non-singular irreducible algebraic curve over an algebraically closed field $k$ of characteritic $0$. For a pointed curve $(C, P)$ we set

$$H(P) = \{\alpha \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = \alpha P\},$$

where $k(C)$ is the field of rational functions on $C$. $H(P)$ is a numerical semigroup of genus $g(C)$ where $g(C)$ is the genus of $C$. For positive integers $a_1, \ldots, a_s$ we denote by $\langle a_1, \ldots, a_s \rangle$ the monoid generated by $a_1, \ldots, a_s$. Let $C$ be a plane curve of degree $7$ and $P$ be a total inflection point of $C$, i.e.,$T_P.C = 7P$ where $T_P$ is the tangent line at $P$ on $C$. Then we have $H(P) = \langle 6, 7 \rangle$. We set

$$d_2(H) = \{h' \in \mathbb{N}_0 \mid 2h' \in H\},$$

which is a numerical semigroup. Let $\pi : \tilde{C} \longrightarrow C$ be a double covering of curves with a ramification point $\tilde{P}$. Then we have $d_2(H(\tilde{P})) = H(\pi(\tilde{P}))$. For example, If $C = \mathbb{P}^1$, then we have $H(\tilde{P}) = \langle 2, 2g+1 \rangle$ and $d_2(H(\tilde{P})) = \mathbb{N}_0$, where $g = g(\tilde{C})$.

We pose the following problem:

**DCPHurwitz' Problem.** *Let $C$ be a plane curve of degree $d$ and $\pi : \tilde{C} \longrightarrow C$ be a double covering with a ramification point $\tilde{P}$. Then determine $H(\tilde{P})$.*

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[1] This paper is an extended abstract and the details were published (see [10])
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If $d = 1, 2$, then $C$ is isomorphic to $\mathbb{P}^1$. We have $H(\tilde{P}) = \langle 2, 2g+1\rangle$ and $d_2(H(\tilde{P})) = \mathbb{N}_0$, where $g = g(\tilde{C})$. If $d = 3$. then $C$ is isomorphic to an elliptic curve. In this case, we have $H(\tilde{P}) = \langle 4, 6, 4g-3\rangle$ or $\langle 4, 6, 4g-1, 4g+1\rangle$ and $d_2(H(\tilde{P})) = \langle 2, 3\rangle$, where $g = g(\tilde{C})$ (for example, see [6]). For $d \geqq 4$ we introduce our previous results. If $d = 4$, then DCPHurwitz' Problem is solved in [5], [2], [3] and [4]. In the case $d = 5$, if $g(\tilde{C}) \geqq 15$ and $\mathrm{ord}_{\pi(\tilde{P})}(T_{\pi(\tilde{P})}.C) = 5$ or 4, then DCPHurwitz' Problem is solved in [7]. In the case $d = 6$, if $g(\tilde{C}) \geqq 30$ and $\mathrm{ord}_{\pi(\tilde{P})}(T_{\pi(\tilde{P})}.C) = 6$ or 5, then DCPHurwitz' Problem is solved in [8] and [9].

We will investigate the case $d = 7$ where $g(\tilde{C}) \geqq 45$ and $\mathrm{ord}_{\pi(\tilde{P})}(T_{\pi(\tilde{P})}.C) = 7$. We get the two results. The first theorem is the following:

**Theorem 1.1** *Let $H$ be a numerical semigroup of genus $\geqq 45$ and $d_2(H) = \langle 6, 7\rangle$. Assume that $H$ is generated by at most 4 elements. If $H \neq 2\langle 6, 7\rangle + \langle n, n+8\rangle$, then $H$ is attained by a ramification point on a double covers of a plane curve of degree 7. In this case we say that $H$ is DCP7.*

To state the second theorem we prepare some notation. Let $H$ be a numerical semigroup with $d_2(H) = \langle 6, 7\rangle$ and $g(H) \geqq 45$. We set

$$n = \min\{h \in H \mid h \text{ is odd}\}.$$

We denote $30 + \frac{n-1}{2} - g(H)$ by $r(H)$, which is a non-negative inetger less than 16 (see [11]).

**Theorem 1.2** *Let $H$ be a numerical semigroup with $d_2(H) = \langle 6, 7\rangle$ and $g(H) \geqq 45$. Assume that $H$ is generated by 5 elements with $r(H) \leqq 6$. Then $H$ is DCP7.*

## 2 The proof of Theorem 1.1

There are our previous results as follows:

**Proposition 2.1** ([11]) *Let $H = 2\langle 6, 7\rangle + n\mathbb{N}_0$ with an odd integer $n \geqq 35$. Then $H$ is DCP7.*

To state the next proposition we need some notation. Let $m$ be the minimum positive integer in $H$. We set

$$s_i = \min\{h \in H \mid h \equiv i \bmod m\}$$

for $i = 1, \ldots, m-1$. The set $\{m, s_1, \ldots, s_{m-1}\}$ is denoted by $S(H)$, which is called the *standard basis* for $H$.

**Proposition 2.2** ([7]) *Let $n$ be an odd number with $n \geqq 35$. We set $H_7 = \langle 6, 7\rangle$. Let $H$ be a numerical semigroup which is one of the following:*

*(1) $2_7 + \langle n, n+2t\rangle$ with $t = 35 - l(7-1)$ where $l$ is a positive integer with $l \leqq 5$ and $n \geqq 30 + 1 + 2l$.*

*(2)* $2H_7 + \langle n, n+2t\rangle$ *with* $t = s_{7-m} - (7-1)$ *where* $m$ *is an integer with* $3 \leqq m \leqq 6$ *and* $n \geqq 30 - 1 + 2m$.

*(3)* $2H_7 + \langle n, n+2t\rangle$ *with* $t = s_{7-m} - 2(7-1)$ *where* $m$ *is an integer with* $3 \leqq m \leqq 5$ *and* $n \geqq 30 - 3 + 4m$.

*Then* $H$ *is DCP7.*

**Remark 2.3** *By Propositions 2.1 and 2.2 the remaining numerical semigroups* $H$ *with* $d_2(H) = \langle 6, 7\rangle$ *generated by* 4 *elements, which we do not know whether* $H$ *is DCP7 or not, are the following:*
(1) $2\langle 6,7\rangle + \langle n, n+20\rangle$ (2) $2\langle 6,7\rangle + \langle n, n+8\rangle$ (3) $2\langle 6,7\rangle + \langle n, n+6\rangle$

First, we will prove that $2\langle 6,7\rangle + \langle n, n+20\rangle$ is DCP7.

**Lemma 2.4** *Let* $(C, P)$ *be a pointed non-singular plane curve of degree* 7 *and* $H$ *be a numerical semigroup with* $d_2(H) = H(P)$ *and* $g(H) \geqq 45$. *Set* $n = \min\{h \in H \mid h$ *is odd*$\}$. *We note that* $g(H) = 30 + \dfrac{n-1}{2} - r$ *with some non-negative integer* $r$. *Let* $Q_1, \ldots, Q_r$ *be points of* $C$ *different from* $P$ *with* $h^0(Q_1 + \cdots + Q_r) = 1$. *Moreover, assume that* $H$ *has an expression*

$$H = 2d_2(H) + \langle n, n+2l_1, \ldots, n+2l_s\rangle$$

*with positive integers* $l_1, \ldots, l_s$ *such that*

$$h^0(K - (l_i - 1)P - Q_1 - \cdots - Q_r) = h^0(K - l_iP - Q_1 - \cdots - Q_r)$$

*where* $K$ *is a canonical divisor on* $C$. *Then there is a double cover* $\pi : \tilde{C} \longrightarrow C$ *with a ramification point* $\tilde{P}$ *over* $P$ *satisfying* $H(\tilde{P}) = H$.

See [10] for the details of the proof of Lemma 2.4 .

**Lemma 2.5** *(Cayley-Bacharach) Let* $C$ *be a non-singular plane curve. Let* $X_1$ *and* $X_2$ *be two plane curves of degree* $d$ *and* $e$ *respectively, meeting in a collection* $\Gamma$ *of* $de$ *points of* $C$ *with multiplicity. Let* $Y$ *be a curve of degree* $d + e - 3$ *such that the intersection* $Y.C$ *contains all but one point of* $\Gamma$. *Then* $Y.C$ *contains that remaining point also.*

For example, see p. 671 in [1].

**Lemma 2.6** *The plane curve of degree* 7 *defined by the equation*

$$(yz^2 - x^3)\left(\frac{1}{2}z^4 + ax^4\right) + (yz^2 + x^3 - 2y^3)\left(\frac{1}{2}z^4 + by^4\right) = 0$$

*is nonsingular for general* $a$ *and* $b$.

For the proof of Lemma 2.6 see [10].

**Proposition 2.7** *The numerical semigroup* $H = 2\langle 6,7\rangle + \langle n, n+20\rangle$ *is DCP7.*

**Proof.** Let $C$ be the non-singular plane curve of degree 7 in Lemma 2.6. We set $P = (0:0:1)$. We take six points

$$Q_1 = (1:1:1), Q_2 = (1:1:\omega), Q_3 = (1:1:\omega^2),$$

$$Q_4 = (1:-1:-1), Q_5 = (1:-1:-\omega) \text{ and } Q_6 = (1:-1:-\omega^2)$$

where $\omega$ is a primitive cubic root of unity. Using Lemma 2.5 we can apply Lemma 2.4. Hence, $H$ is DCP7. □

**Proposition 2.8** *The numerical semigroup $H = 2\langle 6,7\rangle + \langle n, n+6\rangle$ is DCP7.*

In this proof we use the plane curve of degree 7 defined by the equation

$$(yz^2 - x^3)\left(\frac{1}{2}z^4 + ax^4\right) + (yz^3 + x^3z - 2y^4)\left(\frac{1}{2}z^3 + by^3\right) = 0$$

for general $a$ and $b$. See [10] for the details of the proof.

By Remark 2.3, Propositions 2.7 and 2.8 we get Theorem 1.1.

# 3 The proof of Theorem 1.2

**Remark 3.1** *( [7]) Any numerical semigroup $H$ with $d_2(H) = \langle 6,7\rangle$ and $r(H) \leqq 6$ which is generated by 5 elements is DCP7 except the following four semigroups:*

$$2\langle 6,7\rangle + \langle n, n+22, n+32\rangle, 2\langle 6,7\rangle + \langle n, n+22, n+30\rangle,$$

$$2\langle 6,7\rangle + \langle n, n+16, n+32\rangle \text{ and } 2\langle 6,7\rangle + \langle n, n+16, n+34\rangle.$$

**Proposition 3.2** *$H = 2\langle 6,7\rangle + \langle n, n+16, n+32\rangle$ is DCP7.*

**Proof.** In this case $r(H) = 6$. Let $(C, P)$ be a pointed plane curve of degree 7 with $H(P) = \langle 6,7\rangle$. Let $L_P$ and $L'_P$ be distinct lines through $P$ different from $T_P$. Let us take $Q_1, \ldots, Q_4$ such that the four points lie on the line $L_P$. Let us take $Q_5$ and $Q_6$ such that the two points lie on the line $L'_P$ . Let $C_4$ be a curve of degree 4 with $C_4.C \geqq 7P + E_6$ where we set $E_6 = Q_1 + \cdots + Q_6$. Then we have $C_4 = T_P L_P C_2$ where $C_2$ is a conic containing $Q_5$ and $Q_6$. Hence we get

$$h^0(K - 7P - E_6) = h^0(K - 8P - E_6).$$

Moreover, let $C'_4$ be a curve of degree 4 with $C'_4.C \geqq 15P + E_6$. Then we should have $C'_4 = T_P^2 L_P L'_P$, which implies that

$$h^0(K - 15P - E_6) = h^0(K - 16P - E_6) = 1.$$

It follows from Proposition 2.4 that $H$ is DCP7. □

**Proposition 3.3** *Let $H$ be one of the following numerical semigrouips:*

$2\langle 6,7\rangle + \langle n, n+22, n+32\rangle$, $2\langle 6,7\rangle + \langle n, n+22, n+30\rangle$ *and* $2\langle 6,7\rangle + \langle n, n+16, n+34\rangle$.

*Then it is DCP7.*

See [10] for the proof of Proposition 3.3. By Remark 3.1, Propositions 3.2 and 3.3 we obtain Theorem 1.2.

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