Structural Abstraction and Study of Symmetry

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1. Introduction

This paper is a continuation of the search for the tools to achieve clarification of the ideas present in mathematics and in some other disciplines which are expressed with the wild card word "structure." [1] Our interest is not only in the answer to the question "What is a structure?" asked without any reference to specific context, but also in related to this question problems of the meaning of frequently used, but not explicitly defined terms such as cryptomorphism (equivalence of structures which cannot be described in terms of the standard form of isomorphism due to differences in the conceptual framework of their definitions). Finally, the tools which could help in defining the general concept of a structure and in formalizing the equivalence relation between such structures may guide us in their generalization or increasing their level of abstraction. Such generalizations at present, for instance of the concept of a topological space or of a general algebras, follow arbitrary preferences of their authors rather than clearly specified process. Moreover, essentially the same structures are often considered as belonging to different sub-disciplines of mathematics (e.g. topologies defined on finite sets and quasiorders defined on these sets).

More detailed study of the concept of symmetry (invariance with respect to a group of transformations) was reported in earlier publications, therefore only an annotated brief report is included here for the purpose of explanation of relevant aspects. [2-3] Our main objectives in this paper are to report some observations related to the subject, sometimes rather in the form of questions or open problems than answers or solutions.

The paper starts from the explanation of the reasons for obstacles in getting answers to the questions stated above followed by the attempt to systematize further study.

2. Logical and Historical Processes of Structural Abstraction: Topology

One of the early attempts to develop a general theory of structures is associated with lattice theory. In the case of algebraic structures there was a hope that the lattice of subalgebras of a given algebra could provide sufficient information identifying the algebra independently from its specific signature (n-arities of operations listed in the definition). There are some obvious problems with this subalgebra lattice classification when algebra is defined with nullary operations, but this problem could be eliminated when we make an ad hoc decision to exclude the empty set from being a subalgebra. This hope was one of the reasons for the early interest in lattice theory. [4-5] However, by the time the second edition of Birkhoff's classic book *Lattice Theory* was published in 1948 it was already known that non-isomorphic groups may have isomorphic lattices of subgroups. [6] So, the question was not so much whether we can identify algebras by lattices of their subalgebras, but rather in what degree these lattices characterize structures of algebras.

Thus, it is a well known and frequently addressed fact that lattices of subalgebras of an algebra is only one of the important, but not unique characteristics of algebra. Subalgebras form not only a lattice, but complete lattice and they can be considered closed subsets with respect to the transitive closure operator extending any subset of a group to the least subalgebra including the set. This closure may be defined with respect to a various selections of operations. In the case of groups, if we assume that the closure of an empty set is the set which consists of the unity, we have that no matter which of all four different signatures we decide to choose for the group, the closure is the same. But this does not help us with the issue that non-isomorphic groups with the same signature may have the same subgroup closure.

Introductory group theory tells us about the importance of normal groups in the study of the structure of a group. We can consider the fact that the normal subgroups of a group can be considered subalgebras of the algebra with operations consisting of the group operation(s) together with all inner automorphisms as unary operations. However, all simple groups have only trivial normal subgroups and therefore trivial subalgebra closure with closed sets consisting of the singleton subset with the unity and the entire group. Thus the closure of this type is the same for all simple groups and therefore it does not characterize the structure of the group well except for decomposability into products.

If we consider all group automorphisms as unary operations, the subalgebra closure has as closed subsets all characteristic subgroups, but every characteristic subgroup are normal, so this closure tells us about the group even less. Klein four group has five normal subgroups (all subgroups are normal as it is commutative), but only trivial two characteristic subgroups (the singleton with unity and the entire group). The conclusion is that we cannot think about any specific subalgebra closure as a tool for the unique structural classification, unless we can increase the set of closed subsets (not decrease to normal, characteristic, fully characteristic, etc. subgroups). However, there is no natural way to achieve this increase.

It is important to be aware of the inconsistency between the logical interrelations between structures of increasing generality and the historical development of their study. The logical order follows the rules of genus-species definition where we define new concept by distinguishing its genus (more general concept) already defined or accepted as primitive, characterized by axioms, from which we derive our concept as its species made distinct by its differentia, i.e. by properties which make it different from other species of the genus. This means that the logical order is in the downward direction of decreasing generality. First we define a general structure (genus) and then we proceed to distinguishing more specific types by additional conditions. However, in actual, historical perspective the order is up, from more specific instances (prototypes) to more general concepts. Since there is no general theory of structures, this historical process of generalization was haphazard with individual preferences and interests as guides.

Topology started from the most restricted type of what we call now a topological space in the form of a metric space. With the accumulation of the examples where the assumption of the existence of the metric was too restrictive, increasingly general topological spaces were considered. The variety of original examples generated immense diversity of the definitions formulated in terms of very different concepts. For instance, every textbook for topology presents the series of Axioms of Separation with increasingly restrictive conditions which can be formulated in terms of the closure operator as follows.

Traditionally, topologies were and still are most frequently introduced in terms of open subsets or closed subsets and therefore were very early associated with Kuratowski's axioms based on the concept of a closure operation. In this case the complete lattice of closed subsets determines topological spaces in a unique way, although there are approaches to topology where the instead of closure operation a more general non-transitive pre-closure operation is considered (Eduard Čech School). What makes all approaches to topology uniform is the property of topological spaces which can be described as the additivity of the closure (or pre-closure) operation.

Definition 1: *A closure operator on set S is defined as f: $2^S \to 2^S$ such that:*

(i) $\forall A \subseteq S: A \subseteq f(A)$,

(ii) $\forall A,B \subseteq S$: *If* $A \subseteq B$, *then* , $f(A) \subseteq f(B)$.

(iii) $\forall A \subseteq S: f(f(A)) = f(A)$.

Pre-closure does not require the third condition called *transitivity*.

We write *f-Cl* and *f-Op* for the families of *f*-closed or *f*-open subsets of S.

Virtually all studies of topology assume the *finite additivity* condition:

(fA) $\forall A,B \subseteq S: f(A \cup B) = f(A) \cup f(B)$.

The condition of so called *normalization* **(N)** $f(\varnothing) = \varnothing$ is frequent, but it is apparently considered a matter of convenience than necessity.

We use here a convention to symbolize the conditions imposed on a closure operator by a capital letter with modifications indicated by small preceding letters or indices. The set of closure operators on a set S is represented by **I(S)** (I stands for all three conditions (i)-(iii) for the closure operator) with possible additional conditions indicated by the additional letters corresponding to appropriate conditions. Thus, the set of all topological spaces on a set S is represented by **NfAI(S).**

The series of so called Axioms of Separation can be written as:

(T_0) $\forall a,b \in S: f(\{a\}) = f(\{b\}) \Rightarrow a = b$.

(T_1) $\forall a \in S: f(\{a\}) = a$.

Of course $T_1(S) \subseteq T_0(S)$ as T_1 is stronger than T_0.

(T_2) $\forall x,y \in S\ \exists U,V \in$ f-Op: $U \cap V = \varnothing$ & $x \in U$ & $y \in V$.

(T_3) $\forall x \in S\ \forall A \in$ f-Cl $\exists U,V \in$ f-Op: $U \cap V = \varnothing$ & $x \in U$ & $A \subseteq V$.

(T_4) $\forall A,B \in$ f-Cl, $A \cap B = \varnothing\ \exists U,V \in$ f-Op: $U \cap V = \varnothing$ & $A \subseteq U$ & $B \subseteq V$.

Then we have: T_2**NfAI(S)** $\subseteq$ T_1**NfAI(S),**

T_1T_3**NfAI(S)** $\subseteq$ T_2**NfAI(S) &** T_1T_4**NfAI(S)** $\subseteq$ T_3**NfAI(S).**

What is the reason for the distinction of these classes of topological closure operators? The properties defining them were the byproducts of the search for the condition(s) of metrizability, i.e. conditions for the general topological closure operator to be realized by topology defined by some metric. The conditions were identified in several steps and metrizable spaces on a set S are defined by a closure operator belonging to the subset of $T_1T_2T_3T_4$**NfAI(S).** This class of closure operators is slightly smaller than the class of metrizable spaces, so it defines sufficient, but not necessary conditions for metrizability. The series of Separation Axioms was a result of the search for the return to the prototype of the topology of a metric space. This was a legitimate, objective task to find the way down from higher level of abstraction. It is much more difficult to find rationale in the way up.

For instance, what does it make the finite additivity a *sine qua non* condition for a topological space, but not the transitivity condition or Separation Axioms? After all, the transitivity condition is of fundamental importance and its absence ramifies the closure space theory into a theory of a myriad of different incomparable pre-closure structures. For instance, in the absence of transitivity very different spaces may have the same family of open subsets. Yet, it was considered dispensable.

Here too we could hope that the lattice theory provides the answer. This hope could be generated by the fact that that the lattice of closed subsets for the standard topological space is distributive. So, the condition of distributivity follows from finite additivity, but is it equivalent to finite additivity? The answer is no, it is too weak for the finite additivity, as the following example shows.

Proposition 1: *There are closure operators with the distributive lattice of closed subsets which are not finitely additive.*

Proof: We can provide examples of such closure operators.

Example 1: Let $T, U \subseteq S$ and $T \cup U \neq S$ and $T \cap U = \varnothing$. Define a closure operator f by: $\forall \mathbf{A} \subseteq \mathbf{S}$: $f(\varnothing) = \varnothing$, $f(A) = T$ if $A \subseteq \mathbf{T}$, $f(A) = U$ if $A \subseteq U$, and $f(A) = S$ otherwise. Then f-Cl = $\{\varnothing, T, U, S\}$ is a distributive lattice (actually Boolean lattice) with respect to inclusion, but for $A = B \cup C$ where $B \subseteq T$ and $C \subseteq U$, we have $f(A) = f(B \cup C) = S$, but $f(B) \cup f(C) = T \cup U \neq S$. Thus $f \notin$ fAI(S).

There is a natural question whether this is an extreme pathological case with a very simple distributive lattice which can be excluded from our consideration. However, the counterexample can be modified to construct multiple similar counterexamples with distributive lattices of arbitrary dimension. The pattern of the construction can be recognized from the next counterexample.

Example 2: Let $T, U, W, S_{TU}, S_{TW}, S_{UW} \subseteq S$; $T,U \subseteq S_{TU}$ and $T,W \subseteq S_{TW}$ and $U,W \subseteq S_{UW}$; $T \cup U \cup W \neq S$ and $T \cup U \neq S_{TU}$ and $T \cup W \neq S_{TW}$ and $U \cup W \neq S_{UW}$; $T \cap U = \varnothing$, $T \cap W = \varnothing$, $W \cap U = \varnothing$ and $S_{TU} \cap S_{TW} = \varnothing$ and $S_{TW} \cap S_{UW} = \varnothing$ and $S_{TU} \cap S_{UW} = \varnothing$. Define $f(\varnothing) = \varnothing$ and $\forall \mathbf{A} \subseteq S$ and $A \neq \varnothing$: $f(A) = T$ if $A \subseteq \mathbf{T}$, $f(A) = U$ if $A \subseteq U$, $f(A) = W$ if $A \subseteq W$; $f(A) = S_{TU}$ if $A \subseteq S_{TU}$, but $A \not\subset T$ and $A \not\subset U$; $f(A) = S_{TW}$ if $A \subseteq S_{TW}$, but $A \not\subset T$ and $A \not\subset W$; $f(A) = S_{UW}$ if $A \subseteq S_{UW}$, but $A \not\subset U$ and $A \not\subset W$; $f(A) = S$ otherwise. Then f-Cl = $\{\varnothing, T, U, W, S_{TU}, S_{TW}, S_{UW}, S\}$ is a distributive lattice (Boolean lattice) with respect to inclusion. However, for $A = B \cup C$ where $B \subseteq T$ and $C \subseteq U$, we have $f(A) = f(B \cup C) = S_{TU}$, but $f(B) \cup f(C) = T \cup U \neq S_{TU}$. Thus $f \notin$ fAI(S).

Distributivity is a very strong property of lattices and the examples show that even stronger condition for the lattice of closed subsets *f-Cl* of being a Boolean lattice is not sufficient for f to be finitely additive closure operator**.** How do we know that we need so strong property of finite additivity to study topology? Isn't it sufficient to require distributivity of the lattice of closed subsets?

3. Logical and Historical Processes of Structural Abstraction: Algebra

Similar question regarding the axioms for geometry was studied by the author before in the context of generalizations of geometry carried out through selections of axioms in terms of closure spaces. [7] In the literature of the subject the key characteristic of geometric structures was and still is the finite character property of closure operators. This property is usually defined in one of the three equivalent ways.

Definition 2: *A closure operator f: $2^S \to 2^S$ on a set S is of finite character, i.e. $f \in$* ***fC(S)*** *if it satisfies one and therefore all of the following equivalent conditions:*

i) $\forall A \subseteq S$: $A = f(A)$ iff $\forall B \in Fin(A)$: $f(B) \subseteq A$.

ii) $\forall A \subseteq S\, \forall x \in S$: $x \in f(A)$ iff $\exists B \in Fin(A)$: $x \in f(B)$,

iii) $\forall A \subseteq S$: $f(A) = \cup\{f(B): B \in Fin(A)\}$.

The three conditions can be written in a simplified but equivalent form after the removal of the redundancy:

i) $\forall A \subseteq S$: $[\forall B \in Fin(A): f(B) \subseteq A] \Rightarrow A = f(A)$

ii) $\forall A \subseteq S \forall x \in S$: $x \in f(A) \Rightarrow [\exists B \in Fin(A): x \in f(B)]$

iii) $\forall A \subseteq S$: $f(A) \subseteq \cup\{f(B): B \in Fin(A)\}$

Proof: The second and third conditions are obviously equivalent. If the third condition is satisfied, then obviously the first one is, as always $[\forall B \in Fin(A): f(B) \subseteq A] \Rightarrow \cup\{f(B): B \in Fin(A)\} \subseteq A$, and by the third condition we have $f(A) \subseteq A$, i.e. $A = f(A)$. Now, let's assume that the first condition is satisfied. The proof that it implies the third condition is slightly less obvious. Let $C = \cup\{f(B): B \in Fin(A)\}$ and $D \in Fin(C)$. Since D is finite, it is a subset of a finite sub-union of the union C: $D \subseteq f(B_1) \cup f(B_2) \cup ... \cup f(B_k) \subseteq f(B_1 \cup B_2 \cup ... \cup B_k)$ with $B_1 \cup B_2 \cup ... \cup B_k \in Fin(C)$. We did not restrict the choice of D except that it is a finite subset of C, therefore this implies by the first condition that $C = f(C)$ and therefore from $A \subseteq C$, we get $f(A) \subseteq \cup\{f(B): B \in Fin(A)$.

Starting from the condition of finite character fC together with the conditions of normalization N and separation T_1 multiple additions of a variety of selections of axioms serve the closure operator formulation of different types of geometries. No matter which geometry is defined on a set S by a closure operator *f*, always $f \in$ **INT₁fC(S).** [7-8]

On the other hand Garrett Birkhoff and Orrin Frink [9] showed that whenever closure operator on a set S has the finite character, there exists on this set an algebra (i.e. algebraic structure), so that the closure is its subalgebra closure. For this reason closure operators of finite character are frequently called algebraic. This shows that the guidance by intuitive understanding of the meaning of structures is not very reliable in the process of generalization. The same structure is defined sometimes as geometric, sometimes as algebraic without any reference to the other discipline.

In the earlier paper of the author arguments were presented that the more justified choice of the condition for the geometric structures is the condition for the closure operator to be of character n (i.e. to belong to $\mathbf{INC_n(S)}$) with n equal to the dimension of the space. [7]

Definition 3: *A closure operator f: $2^S \rightarrow 2^S$ on a set S is of the character n where n is any natural number including 0, which we write in the symbolic way as* $f \in IC_n(S)$, *if it satisfies the following condition:*

(C_n) $\forall A \subseteq S$: $A = f(A)$ iff $\forall B \subseteq A$: $|B| \leq n \Rightarrow f(B) \subseteq A$, *where $|B|$ is the number of elements in set B.*

The implication from the left to the right is always true, so this condition can be formulated as an implication from the right to the left.

It has to be noticed that unlike the case of finite character property here the analogs of the other forms of the condition are not equivalent as the union of closures of sets with n or less elements is not necessarily closed even if $f \in C_n(S)$. It can be easily shown that the property $\mathbf{C_n}$ is equivalent to the narity of closure operators defined by Stanley Burris and H.P. Sankappanavar in a different way using the transitive modification of the function $A \to \cup\{f(B): B\in Fin(A)\}$.[10]

The property of being of character n is stronger than being of character n+1 and this is stronger than finite character.

Proposition 2: *For every closure operator* ***f*** *on S and every natural n:* $f\in C_n(S) \Rightarrow f\in C_{n+1}(S) \Rightarrow f\in fC(S)$, *or in other words* $C_n(S) \subseteq C_{n+1}(S) \subseteq fC(S)$, *but in general the first implication cannot be reversed and the first inclusion is strict.*

Proof: The proof of the implications is straightforward. If $f\in C_n(S)$ and $\forall B \subseteq A$: $|B| \leq n+1 \Rightarrow f(B) \subseteq A$, then $\forall B\subseteq A$: $|B| \leq n \Rightarrow f(B) \subseteq A$, therefore $A = f(A)$. Now, If $f\in C_{n+1}(S)$ and $\forall B\in Fin(A)$: $f(B) \subseteq A$, then $\forall B\subseteq A$: $|B| \leq n+1 \Rightarrow f(B) \subseteq A$, and therefore $A = f(A)$, i.e. $f\in fC(S)$.

The proof that the first inclusion is strict is based on the example of closure operator $f\in C_n(S)$ such that $f\notin C_{n-1}(S)$. Let $n > 0$ and $n < |S|$. Define f by: $\forall A\subseteq S$: $f(A) = A$, if $|A| < n$ and $f(A) = S$ otherwise. We have $f\in C_n(S)$, because for every subset A of S the condition $\forall B \subseteq A$: $|B| \leq n \Rightarrow f(B) \subseteq A$ implies $A = f(A)$. It is obvious when $|A| < n$, because then by the definition $f(A) = A$, so the consequent is true and therefore the implication is true. If $A \neq S$ and $|A| \geq n$, then the antecedent $\forall B \subseteq A$: $|B| \leq n \Rightarrow f(B) \subseteq A$ is obviously false, as $A \subseteq A$ and $f(A) = S \neq A$, and the implication is true by the *ex falso quod libet* rule. Finally for $A = S$ the implication is true, because again $f(A) = f(S) = S = A$.

Now, we will show that $f\notin C_{n-1}(S)$, i.e. that $\forall B \subseteq A$: $|B| \leq n-1 \Rightarrow f(B) \subseteq A$ does not imply $A = f(A)$. Let $|A| = n$. Then A is not a closed subset as $f(A) = S$, but all its subsets with at most n-1 elements are closed, so their closures are subsets of A. Since, for our proof we need only $n > 0$, we have the proof complete for the first inclusion to be strict.

The author's conjecture is that the second inclusion is strict too, but at the moment he is not able to provide the proof.

Corollary 3: *From Proposition 2 follows that for every closure operator f of finite character either there is no natural n such that* $f\in C_n(S)$, *or there is the least such n.*

Proof: It follows directly from the contraposition of the first implication in Proposition 2: $f\notin C_{n+1}(S) \Rightarrow f\notin C_n(S)$ which by the finite induction gives $f\notin C_n(S) \Rightarrow f\notin C_m(S)$, if $m<n$.

Definition 4: *We call the least natural number n such that* $f\in C_n(S)$ *the essential character n of the closure operator f. If* $f\in fC(S)$, *but for no natural n:* $f\in C_n(S)$, *the closure operator is called to be of essential finite character.*

The existence of the essential character for a closure operator follows from Corollary 3.

The theorem proved by Birkhoff and Frink stating that for every closure operator of finite character there exists an algebra for which the closed subsets are its subalgebras has a limited value in the search for the description and identification of structures. The construction used in the proof requires the consideration of algebras with operations of arbitrarily high signatures. Of course, this does not bring any objections to the proof, but in applications of algebra to other disciplines of mathematics we very rarely consider algebras with operations different from nullary, unary, and binary. The result which tells us about exotic algebras with operations involving arbitrary high number of operands seems exceedingly formal.

Thus, the theorem which tells us that for a closure operator of character n there exists an algebra with operations which have the number of operands not exceeding n explains why the property of character n is so important. [7,10] Because every closure operator $f \in C_n(S)$ is automatically characterized by being of any character m higher than n and the existence of the essential (i.e. minimal) character we can identify the direction of further study of equivalent algebraic structures in their representation at the level of the essential character of the subalgebra closure operator.

4. Structural Abstraction in Terms of Symmetry

There is a clear tendency in mathematics to use abstraction as a method to formulate theories within the limits of the predicate calculus of first order. This is rather a matter of clarity than of the logical concerns. Some expressions of this tendency can be identified in the development of category theory whose morphisms represent functions and objects represent sets, but which formulates theorems without any direct reference to the underlying lower level of set theoretical concepts. Of course, logic and set theory can be formulated in this conceptual framework too.

The programs of searching for a general mathematical theory of structures sometimes followed the same tendency. The attempts to find a general characterization and classification of structures, in particular of algebraic structures in terms of lattices of subalgebras or lattices of the congruences of algebras were made in the same spirit of lifting the study to the higher level of abstraction where the meaning of structures such as algebras and their subalgebras was hidden and they became elements of lattices.

The question is whether this can be the reason why in so long time the general theory of structures and their cryptomorphisms did not progress significantly. This was one of the motivations for the present author to explore an alternative approach. This alternative approach is the search of the methods of inquiry in symmetry, but symmetry in sufficiently general conceptual framework without the use of coordinatization. Of course, symmetry was studied in a myriad ways and contexts, but almost exclusively in terms of invariance with respect to the groups of transformations described in terms of coordinates belonging to some field.

In the approach initiated in recent years, I proposed an approach which does not require coordinatization. The key concept here is again a *closure space*, i.e. a set S with a closure operator f: $2^S \rightarrow 2^S$ from Definition 1 which satisfies:

(i) $\forall A \subseteq S$: $A \subseteq f(A)$,

(ii) $\forall A,B \subseteq S$: If $A \subseteq B$, then , $f(A) \subseteq f(B)$.

(iii) $\forall A \subseteq S$: $f(f(A)) = f(A)$.

It is symptomatic that every closure space can be defined in an equivalent (cryptomorphic) way by

a Moore family of subsets of S, i.e. family closed with respect to arbitrary intersections and including the set S.

Every Moore family $\mathcal{M}$ defines a transitive operator: $\boldsymbol{f}(\mathbf{A}) = \cap \{\mathbf{M} \in \mathcal{M} : \mathbf{A} \subseteq \mathbf{M}\}$ and in turn the family $\boldsymbol{f}$-**Cl** = $\{\mathbf{M} \subseteq \mathbf{S}: \boldsymbol{f}(\mathbf{M}) = \mathbf{M}\}$ is a Moore family.

The set theoretical inclusion defines a partial order on f-Cl with respect to which it is a complete lattice $\mathcal{L}_f$. To this structure we will refer as the *logic* $\mathcal{L}_f$ *of a closure space* <*S,f*>.

With these concepts in hand we can introduce the concept of symmetry in an arbitrary closure space $<S,f>$.

Symmetry in Closure Space $<S,f>$ (A brief report of the main idea) [3]

Let $G = Aut(\mathscr{L}_f)$ be the group of automorphisms of the logic $\mathscr{L}_f$ of a closure space $<S,f>$ & $H◀G$ be a subgroup of the group $G = Aut(\mathscr{L}_f)$

Let $\mathscr{B} \subseteq \mathscr{L}_f$ be a configuration of closed subsets (e.g. in geometry on a plane of points or lines). Let φ^* be an automorphism induced on $\mathscr{L}_f$ by a bijection φ on S.

We get a correspondence between subgroups H of the group of automorphisms of $<S,f>$ *a*nd invariant families of configurations $\mathscr{B}$:

Proposition 4: *Let $H ◀ G = Aut(\mathscr{L}_f)$. Define the family $\mathscr{J}_H$ of subsets of $\mathscr{L}_f$ by $\forall \mathscr{B} \subseteq \mathscr{L}_f$: $\mathscr{B} \in \mathscr{J}_H$ iff $\forall A \in \mathscr{B}\, \forall \varphi \in H$: $\varphi^*(A) \in \mathscr{B}$. Then $\mathscr{J}_H$ is a complete lattice with respect to the order of inclusion of sets.*

Proposition 5: *The following two functions form a Galois connection between:*

$\Phi(H) = \mathscr{J}_H$ defined by $\forall \mathscr{B} \subseteq \mathscr{L}_f$: $\mathscr{B} \in \mathscr{J}_H$ iff $\forall A \in \mathscr{B}\, \forall \varphi \in H$: $\varphi^*(A) \in \mathscr{B}$ and

$\Psi(\mathscr{J}) = H$ defined by $H = \nabla\{K ◀ G: \mathscr{J} \subseteq \mathscr{J}_K\} = \{\varphi \in G: \varphi(\mathscr{J}) \subseteq \mathscr{J}\}$, where the last equality is a consequence of the fact that $\{\varphi \in G: \varphi(\mathscr{J}) \subseteq \mathscr{J}\}$ is a subgroup of G.

We will consider configuration $\mathscr{B}$ of closed subsets and conditions for its invariance.

Let G = Aut($\mathscr{L}_f$) be the group of automorphisms of the logic $\mathscr{L}_f$ of a closure space $<S,f>$ & H◀G be a subgroup of the group G = Aut($\mathscr{L}_f$)

Let $\mathscr{B} \subseteq \mathscr{L}_f$ be a configuration of closed subsets. Then there is a mutual correspondence between subgroups H of the group of automorphisms of $<S,f>$ *a*nd invariant families of configurations $\mathscr{B}$ defining a Galois connection between the lattice of subgroups of $G = Aut(\mathscr{L}_f)$, *i.e.* $\mathscr{L}_G$ or $\mathscr{L}_{Aut(\mathscr{L}_f)}$ and the lattice of families of closed subsets of the closure space $<S,f>$. The Galois connection is defined by two mappings $\Phi: \mathscr{L}_G \rightarrow \mathscr{J}$ & $\Psi: \mathscr{J} \rightarrow \mathscr{L}_G$: $\Phi(H) = \mathscr{J}_H$ *and* $\Psi(\mathscr{J}) = H = \{\varphi \in G: \varphi(\mathscr{J}) \subseteq \mathscr{J}\}$

This Galois connection defines anti-isomorphism of the lattice of subgroups of G and the lattice of invariant families of closed subsets of $<S,f>$.

Thus, the invariant families $\mathscr{B}$ are symmetric with respect to corresponding subgroup of the group of automorphisms of the logic of closure space. The symmetric configurations are distinguished as those closed with respect to Galois closure (different from *f* of course).

We can observe that in this approach it is necessary to consider at least two, possibly more levels of sets, families of such sets and families of families of such sets. Let $\wp$(S) indicate the power set of the set S, $\wp(\wp(S))$ the power set of $\wp$(S), etc. Then we have here S, $\wp$(S) and $\wp(\wp(S))$. The reason is that the concept of symmetry involves two concepts: that of a fixed subset of a group of transformations (set of elements which are not changed by the transformations) and that of invariant subset (set which remains the same, although its elements are permuted by the transformations). The two concepts are very different, although they are related across the distinction between sets and their power sets. Invariant subsets at the lower level are fixed points at the higher level.

At the level of $\wp(\wp(S))$, we have the Galois connection defined by $\Phi: \mathscr{L}_G \rightarrow \mathscr{J}$ & $\Psi: \mathscr{J} \rightarrow \mathscr{L}_G$ which links symmetric configurations of closed subsets of $<S,f>$ with subgroups of Aut($\mathscr{L}_f$).

The level of $\wp(S)$: $\Phi(H) = \mathcal{J}_H$ & $\Psi(\mathcal{J}) = H$, where $\Phi(H) = \mathcal{J}_H$ defined by $\forall \mathcal{B} \subseteq \mathcal{L}_f: \mathcal{B} \in \mathcal{J}_H$ *iff* $\forall A \in \mathcal{B} \forall \varphi \in H: \varphi^*(A) \in \mathcal{B}$ and $\Psi(\mathcal{J}) = H$ def. by $H = \bigvee\{K \blacktriangleleft G: \mathcal{J} \subseteq \mathcal{J}_K\} = \{\varphi \in G: \varphi(\mathcal{J}) \subseteq \mathcal{J}\} \blacktriangleleft G$

The level of S: There is a more general bijective correspondence between group G acting on the set S which preserves closure *f* and the group $Aut(\mathcal{L}_f)$.

So, at the lowest level we have simply a group G acting on S. The group G does not have to be the symmetric group Sym(S). It is a subgroup of Sym(S) selected by the choice of closure *f*. The distinction of these three levels serves the distinction between subsets of fixed points and invariant subsets.

The approach is based on the group $Aut(\mathcal{L}_f)$. There could be a legitimate concern that for the closure operation *f* of subalgebras of a given algebra, $\mathcal{L}_f$ does not determine uniquely the algebra, e.g. non-isomorphic groups can have isomorphic lattices of subgroups. However, the symmetry is not described as a distinction of this lattice, but it is described by the Galois connection. In the same way, symmetry is not simply giving a privileged position to the lattice $\mathcal{L}_G$ of subgroups of a given group, but it is involving it in the Galois connection.

The relationship between a stabilizer group and symmetry group (symmetry group is a stabilizer group, but at the level of power set) suggests that we can consider higher level symmetry. At this higher level symmetry we can consider invariance of the families of configurations stabilized by symmetry at the lower level. This suggests the continuing process of increasing structural abstraction.

5. What Symmetry Can Tell Us About Groups?

Let's assume that we have not arbitrary closure space $<S,f>$, but that we have a group G with *f* standing for the closure defined by extension of a subset to the subgroup (subgroup closure) and we want to study its structure. The following will be exclusively in terms of group action of G on some set X, i.e. a homomorphism $\varphi: G \rightarrow \text{Sym}(X)$.

Since the values of φ are bijective functions (transformations of X) these values will bewritten φ_g and then we can write: $\varphi_g(x) = y$ for $x,y \in X$. Group action on a non-empty set which is transitive and faithful is a representation of G. If it is transitive and free (and therefore faithful), it is a regular representation.

Group theory started from Cayley Theorem: $\varphi: G \rightarrow \text{Sym}(G)$ defined by $\varphi_g(x) = gx$ is a regular representation. Introductory courses in group theory start from another group action on itself where action is defined by conjugation: $\varphi: G \rightarrow Aut(G)$ defined by $\varphi_g(x) = gxg^{-1}$. In this case we have that every φ_g is an automorphism of G and $\varphi(G)$ is a normal subgroup of inner automorphisms *Inn*(G). If we follow this way we get a characterization of the structure of the group from the point of view of reducibility into product group. But when the group is simple (i.e. the only normal subgroups are trivial one {e} and entire G), we get nothing, but information that the group is actually simple one.

With each type of group action on itself there is associated a closure space. For instance Cayley's action with the lattice $\mathcal{L}_G$ of all subgroups of G. If we choose invariance of subgroups with respect to *Inn*(G) as the criterion for action, we get the lattice $\mathcal{L}_N$ of normal subgroups of G.

What was the requirement of invariance in Cayley's action? In this case we have the trivial subgroup {e}. We want to have a method for the study of the structure of all groups including simple groups. We have to include outer automorphisms.

Reminder: Always e(x)=x $\Rightarrow$ $\{e\} \blacktriangleleft_N$ *Aut(G)*. The symbol ◄ indicates the subgroup relation and the symbol $\blacktriangleleft_N$ indicates the normal subgroup relation.

Proposition 6: *The following diagram shows a Galois connection between the lattice of subgroups of the group of all automorphisms* ***Aut(G)*** *and the lattice* $\mathscr{L}_G$ *of closure operators on the set G. This Galois connection describes the structural characteristics of the group G. The diagram shows only the distinctive elements of the lattices, but the connection links each element of one lattice with the corresponding element of the other.*

$\mathscr{L}_G$ - The lattice of all subgroups (of finite character)

$\mathscr{L}_N$ - The lattice of all normal subgroups (modular)

$\mathscr{L}_f$ – The distributive lattice with two closure operators *f, g* defined by $\forall A \subseteq G$: *f*(A)=A and *g*(A)=G.

$\{e\} \blacktriangleleft_N$ **Inn(G)** $\blacktriangleleft_N$ **Aut(G)**

↑ ↑ ↑

↓ ↓ ↓

$\mathscr{L}_G \geq \mathscr{L}_N \geq \mathscr{L}_f$

References

[1] Schroeder, M.J. Structures and Their Cryptomorphic Manifestations: Searching for Inquiry Tools. In Adachi, T. (ed.) *Algebraic Systems, Logic, Language and Related Areas in Computer Science. RIMS Kokyuroku, Kyoto: Research Institute for Mathematical Sciences, Kyoto University*, **2019**, No. 2130-08, pp. 10. (available at www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/2130.html)

[2] Schroeder, M.J. Search for an Algebraic Structure of Not-Necessarily Algebraic Structures. In Kunimochi, Y. (ed.) *Algebras, logics, languages and related areas, RIMS Kokyuroku, Kyoto: Research Institute for Mathematical Sciences, Kyoto University*, **2018**, No. 2096, pp. 59-68. (available at www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/2096.html)

[3] Schroeder, M. J. Concept of Symmetry in Closure Spaces as a Tool for Naturalization of Information . In Horiuchi, K. (Ed.) *Algebraic System, Logic, Language and Computer Science. RIMS Kokyuroku, Kyoto: Research Institute for Mathematical Sciences, Kyoto University*, **2016**, No. 2008, pp. 29-36. Available at http://www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/ 2008.html

[4] Ore, O. Structures and Group Theory I. *Duke Math. J.*, 3 (1937), 149-174.

[5] Ore, O. Structures and Group Theory II. *Duke Math J.,* 4 (1938), 247-269.

[6] Birkhoff, G. *Lattice Theory, 3rd. ed.* American Mathematical Society Colloquium Publications, Vol XXV, Providence, R. I.: American Mathematical Society, 1967.

[7] Schroeder, M.J. On Classification of Closure Spaces: Search for the Methods and Criteria. In: A. Yamamura (Ed.), *Algebraic Systems and Theoretical Computer Science, RIMS Kokyuroku* 1809*,* Research Institute for Mathematical Sciences, Kyoto University, Kyoto, **2012**, pp. 145-154. (available at http://www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/1809.html)

[8] Jónsson, B. Lattice-theoretic Approach to Projective and Affine Geometry. In L. Henkin, P. Suppes, A. Tarski (Eds.) *The Axiomatic Method. Vol. 27 Studies in Logic and the Foundations of Mathematics.* North-Holland, Amsterdam, 1959.

[9] Birkhoff, G., Frink, O. Representations of lattices by sets. Trans. Amer. Math. Soc. 64, (1948), *299-316.*

[10] Burris, S., Sankappanavar, H.P. A Course in Universal Algebra, Springer, Berlin, 2012.