Report of computer experiments on the Rauzy fractals

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Abstract

For a Pisot primitive unimodular substitution over the alphabet $\mathcal{A}$ with d letters, a substitution dynamical system consisting of a subset of the full $\mathcal{A}$ shift and a shift map is constructed. And we obtain $d-1$ dimensional domain, so called the Rauzy fractals as a geometrical realization of the substitution dynamical system. The authors conducted computer experiments to observe geometrical properties of the Rauzy fractals. In this report, examples of the Rauzy fractals are given.

1 Introduction

Let d be a positive integer and $\mathcal{A} = \{1, 2, \ldots, d\}$ be the alphabet. We denote by $\mathcal{A}^*$ (resp. by F_d) the free monoid (resp. the free group) with the empty word λ over $\mathcal{A}$. An endomorphism σ on $\mathcal{A}^*$ satisfying $\sigma(i) \neq \lambda$ for any $i \in \mathcal{A}$ is called a substitution of rank d. A substitution σ is naturally extended to an endomorphism on F_d by $\sigma(i^{-1}) := (\sigma(i))^{-1}$ for $i \in \mathcal{A}$. Let $\mathcal{A}^{\mathbb{N}}$ be the set of one-sided infinite sequences of letters in $\mathcal{A}$, that is, the full shift over $\mathcal{A}$. A substitution σ is also extended to an endomorphism on $\mathcal{A}^{\mathbb{N}}$ by $\sigma(s_0 s_1 s_2 \cdots) := \sigma(s_0)\sigma(s_1)\sigma(s_2)\cdots$ for $s_0 s_1 s_2 \cdots \in \mathcal{A}^{\mathbb{N}}$.

Let $\mathbf{f} : F_d \to \mathbb{Z}^d$ be a canonical homomorphism defined by $\mathbf{f}(\lambda) = o$ and $\mathbf{f}(i^{\pm 1}) = \pm e_i,\ i \in \mathcal{A}$, where e_i are fundamental vectors. The incidence matrix L_σ of σ defined by $(\mathbf{f}(\sigma(1)), \mathbf{f}(\sigma(2)), \cdots, \mathbf{f}(\sigma(d)))$ satisfies the following commutative diagram:

$$\begin{array}{ccc} F_d & \xrightarrow{\ \sigma\ } & F_d \\ {\scriptstyle \mathbf{f}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{f}} \\ \mathbb{Z}^d & \xrightarrow[L_\sigma]{} & \mathbb{Z}^d \end{array} .$$

For a substitution σ, there are the following three conditions:

- (Pisot condition) The maximum root of the characteristic polynomial of L_σ is Pisot number, that is, the dominant eigenvalue of L_σ is greater than one and the others have modulus less than one,
- (Primitive condition) L_σ is primitive, that is, $L_\sigma^N > 0$ for an enough large integer N,
- (Unimodular condition) $\mid \det L_\sigma \mid = 1$.

Let us consider the substitution σ_R of rank 3 defined by $\sigma_R(1) = 12$, $\sigma_R(2) = 13$, $\sigma_R(3) = 1$, which is called the Rauzy substitution or the tribonacci substitution. Then, the incidence matrix is

$$L_{\sigma_R} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and its characteristic polynomial is $x^3 - x^2 - x - 1$. Therefore, σ_R is a Pisot primitive unimodular substitution. It is also invertible, since $\sigma_R^{-1}(1) = 3$, $\sigma_R(2) = 3^{-1}1$, $\sigma_R(3) = 3^{-1}2$. Note that $\sigma_R^{n+1}(1) = \sigma_R^n(1)\sigma_R^n(2)$ for any positive integer n. By iterating σ_R for the letter 1 infinitely many times, we get the one sided infinite sequence ω:

$$\omega = 1213121121312 \cdots \in \mathcal{A}^{\mathbb{N}},$$

and ω is a fixed point of σ_R. Let Ω_{σ_R} be

$$\overline{\{S^n(\omega) \mid n \in \mathbb{Z}_{\geq 0}\}} \subset \mathcal{A}^{\mathbb{N}}, \tag{1}$$

where S is the shift map on $\mathcal{A}^{\mathbb{N}}$ defined by $S(s_0 s_1 s_2 \cdots) := s_1 s_2 \cdots$ for $s_0 s_1 s_2 \cdots \in \mathcal{A}^{\mathbb{N}}$ and $\overline{A}$ means the topological closure of A with the metric ρ defined by

$$\rho(s,t) = \begin{cases} 2^{-\min\{s_k \neq t_k\}} & \text{if } s \neq t \\ 0 & \text{if } s = t \end{cases}$$

for $s = (s_k)$, $t = (t_k) \in \mathcal{A}^{\mathbb{N}}$. The symbolic dynamical system (Ω_{σ_R}, S) is called the substitution dynamical system.

In 1982, G.Rauzy found the set X in $\mathbb{R}^2$ called the Rauzy fractal, the map T on X called the domain exchange transformation, and the dynamical system (X,T) as a geometrical realization of (Ω_{σ_R}, S) for the substitution σ_R (cf. [5]).

In the following, let us define the Rauzy fractals in general. Let σ be a Pisot primitive unimodular substitution of rank d with a fixed point $\omega = \omega_0\omega_1\omega_2\cdots$. Let u be an eigenvector of L_σ corresponding to the dominant eigenvalue which is a Pisot number, P be the $d-1$ dimensional contractive eigenspace of L_σ spanned by the eigenvectors except u, and $\pi : \mathbb{R}^d \to P$ be the projection along u. For each $i \in \mathcal{A}$, set

$$X_i := \overline{\{\pi(\mathbf{f}(\omega_0\omega_1\omega_2\cdots\omega_{k-1})) \mid \omega_k = i\}}$$

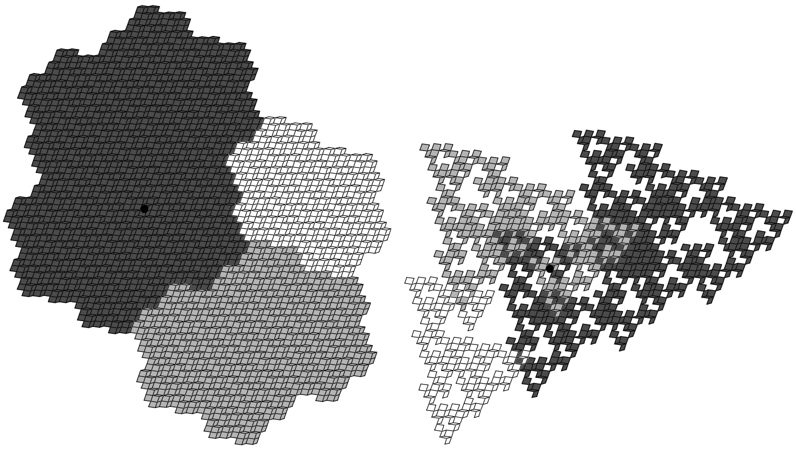
and

$$X(= X(\sigma)) := \bigcup_{i=1}^{d} X_i,$$

where $\overline{A}$ is the closure of A with the d dimensional Euclidean metric. The sets X and X_i $(i \in \mathcal{A})$ with fractal boundaries are called the Rauzy fractals. The domain exchange transformation $T : X \to X$ is defined by

$$T(x) := x + \pi(e_i) \text{ if } x \in X_i.$$

From these definitions, we see $T^n(o) \in X_{\omega_n}$ for any $n \in \mathbb{Z}_{\geq 0}$, that is, the fixed point ω gives the itinerary of the orbit $(T^n(o))$ of the origin point by T. So, the dynamical system (X,T) is considered as a geometrical realization of (Ω, S), where Ω is defined by (1) for the substitution σ. The left figure in Fig. 1. is the approximation of the Rauzy fractals for the Rauzy substitution σ_R: the darkest domain is X_1, the second darkest one is X_2, and the brightest one is X_3. All figures are drawn by using a geometrical version of substitution which was given by P. Arnoux and S. Ito in [1]. We see also a figure of the Rauzy fractals for the substitution defined by $\sigma'_R(1) = \mathbf{21}$, $\sigma'_R(2) = 13$, $\sigma'_R(3) = 1$ in [1] (see Fig. 1). The difference between σ_R and σ'_R is just the image of the letter 1, but big differences occur in these pictures from the point of view of geometrical property: the Rauzy fractals for σ_R are disk-like and the Rauzy fractals for σ'_R are not disk-like. Motivated by these examples, the authors did computer experiments. In the next section, results of computer experiments will be shown.



$X(\sigma_R)$ $X(\sigma_{R'})$

Figure 1: Approximation of the Rauzy fractals for σ_R and σ'_R

2 Results of computer experiments

Let L be

$$L := L_{\sigma_R^3} = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

A computer experiment gives a list of substitutions which have the incidence matrix L by permutating letters appearing in each word $\sigma_R^3(i)$, $i = 1, 2, 3$, and figures of the Rauzy fractals for these substitution.

Let us recall results in the case of $d = 2$ briefly.

Theorem 1. *([2])*
Let σ be a Pisot primitive unimodular substitution of rank 2. Then the Rauzy

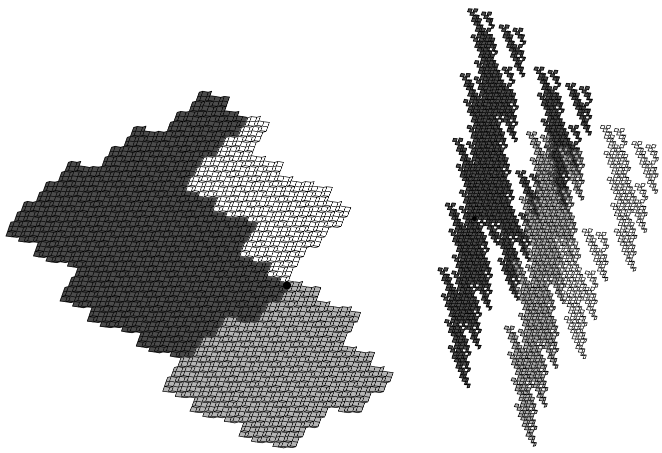
fractals X_1, X_2 and X are intervals, that is, connected if and only if σ is invertible.

We expect that in the case of rank 3, a substitution is invertible if and only if the Rauzy fractals for the substitution are connected. But the following example found by the computer experiment shows that it is not true. Let us consider the substitutions σ_1 and σ_2 with the incidence matrix L defined by

$$\sigma_1(1) = 1231121,\ \sigma_1(2) = 213121,\ \sigma_1(3) = 3121$$

$$\sigma_2(1) = 1212113,\ \sigma_2(2) = 131212,\ \sigma_2(3) = 1312.$$

When we see the Rauzy fractals in Fig. 2, we might think σ_1 is invertible and σ_2 is not invertible. Actually, these are the exact opposite.



$X(\sigma_1)$ $X(\sigma_2)$

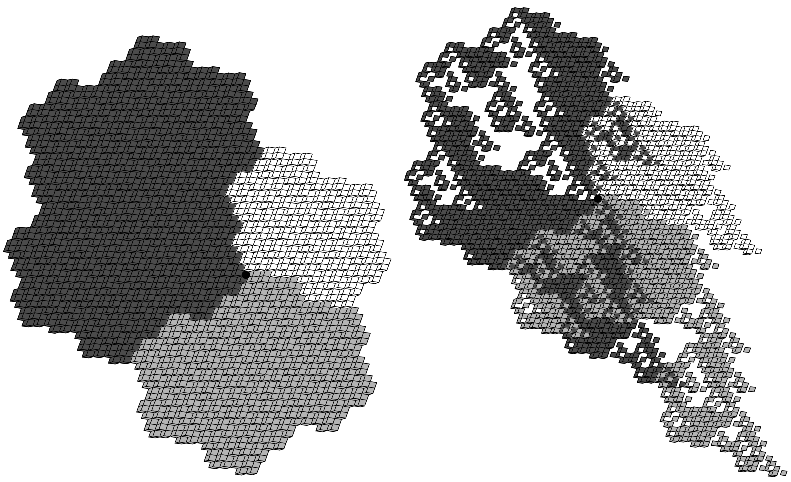
Figure 2: Approximation of the Rauzy fractals for σ_1 and σ_2

Recall that, as in Fig. 1 in Section 1, small differences between σ_R and σ'_R make big differences to the shape of the Rauzy fractals for them. Is that always the case? Take σ_3 and σ_4 defined by

$$\sigma_3(1) = 12131\mathbf{21},\ \sigma_3(2) = 213121,\ \sigma_3(3) = 3121$$

$$\sigma_4(1) = 12131\mathbf{12},\ \sigma_4(2) = 213121,\ \sigma_4(3) = 3121.$$

The difference between them are two consecutive letters appearing in the images of the letter 1. The Rauzy fractals for σ_4 keep a part of the Rauzy fractals for σ_3 (see in Fig. 3), and you may think that the two shapes are a bit similar. In addition, you may also notice that the Rauzy fractals for σ_3 in Fig. 3 are the same as the Rauzy fractals for σ_R in Fig. 1. From $\sigma_R^3(1) = 1213121$, $\sigma_R^3(2) = 121312$, $\sigma_R^3(3) = 1213$, we see $\sigma_3(i) = \overline{\sigma_R^3(i)}$ for $i = 1, 2, 3$, where $\overline{w_1 w_2 \cdots w_k} = w_k w_{k-1} \cdots w_1$ for $w_1 w_2 \cdots w_k \in \mathcal{A}^*$. In general, for a given substitution σ, the Rauzy fractals for the substitution σ' satisfying $\sigma'(i) = \overline{\sigma(i)}$ for $i = 1, 2, 3$ are the same as the Rauzy fractals for σ by 180-degree rotation and translation.



$X(\sigma_3)$ $X(\sigma_4)$

Figure 3: Approximation of the Rauzy fractals for σ_3 and σ_4

Let us recall some facts in the case of rank 2 and observe connectedness of the Rauzy fractals. A Sturmian sequence is an infinite sequence in $\{1, 2\}^{\mathbb{N}}$ such that the number $P(n)$ of subwords of length n occurring in the sequence, that is, the complexity for n is exactly $n + 1$ for any positive integer n. It is known that a sequence in $\{1, 2\}^{\mathbb{N}}$ is ultimately periodic if and only if there exists an positive integer n such that $P(n) < n + 1$ (see in [3]). It means that a Sturmian sequence has the smallest growth rate of words. Moreover, the following theorem is well-known:

Theorem 2. *(see for instance [4]) A sequence is a Sturmian sequence which is a fixed point of some non-trivial substitution of rank 2 if and only if it is a fixed point of some primitive and invertible substitution.*

From Theorem 1 and 2, we expect that the Rauzy fractals for a substitution of rank 3 whose fixed point has small growth rate of words are connected. In the case of the Rauzy substitution, it is known that $P(n) = 2n + 1$ for any positive integer n. Therefore, we choose the following substitutions σ_5, σ_6 and σ_7 such that the number of subwords of length 10 occurring in the sequence $\sigma^4(1)$ is exactly $21 = 2 \times 10 + 1$ and observe the Rauzy fractals for them:

$$\sigma_5(1) = 1321121,\ \sigma_5(2) = 121132,\ \sigma_5(3) = 1321$$

$$\sigma_6(1) = 1211321,\ \sigma_6(2) = 121132,\ \sigma_6(3) = 1321$$

$$\sigma_7(1) = 1112132,\ \sigma_7(2) = 112132,\ \sigma_7(3) = 1321.$$

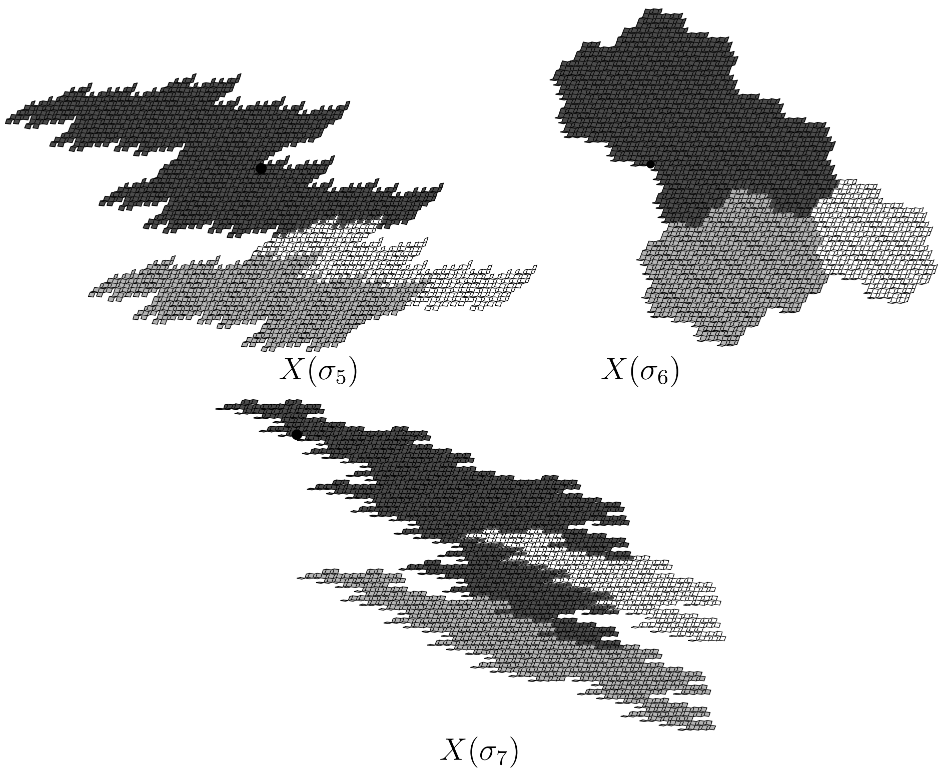


Figure 4: Approximation of the Rauzy fractals for σ_5, σ_6 and σ_7

The Rauzy fractals for these substitutions in Fig. 4 are all connected. In this computer experiment, we could not find a substitution such that the number of subwords of length 10 occurring in the sequence $\sigma^4(1)$ is exactly 21 and the Rauzy fractals for it are disconnected.

Next, we take σ_8, σ_9 and σ_{10} defined by

$$\sigma_8(1) = 1131212,\ \sigma_8(2) = 121312,\ \sigma_8(3) = 1213$$

$$\sigma_9(1) = 1311212,\ \sigma_9(2) = 121312,\ \sigma_9(3) = 1213$$

$$\sigma_{10}(1) = 1322111,\ \sigma_{10}(2) = 121312,\ \sigma_{10}(3) = 1213.$$

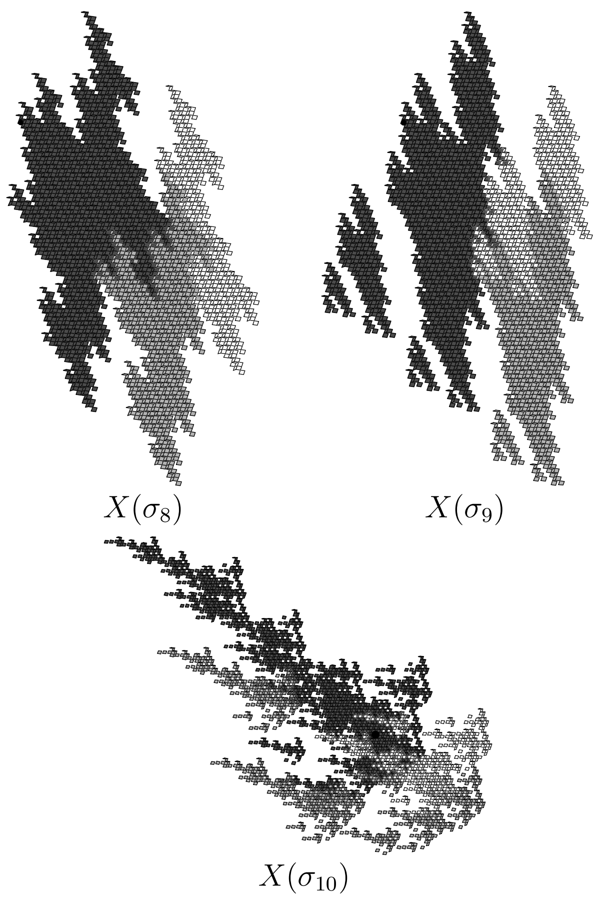


Figure 5: Approximation of the Rauzy fractals for σ_8, σ_9 and σ_{10}

The numbers of subwords of length 10 occurring in the sequence $\sigma_k^4(1)$ for $k =$

$8, 9, 10$ are 28, 27 and 41, respectively. By the observation, the shape of the Rauzy fractals for a substitution whose fixed point has the large complexity $P(n)$ are disconnected and intricate. Therefore, it seems that the complexity of a fixed point of a substitution tells us some information about geometrical properties of the Rauzy fractals. But it is unclear at the moment.

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