On principal affine W-superalgebras for $\mathfrak{sl}_{n|1}$

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1. Quantum Symmetry from Vertex Algebras
2. Duality in Principal $\mathcal{W}$-algebras
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Origin of Vertex Algebras

The notion of a **vertex algebra** encodes an algebraic structure of **"qunatum observables"** acting on a space of **"qunatum states"** with respect to the **operator product expansion**[1].

In the early days, such a structure appeared in the representation theory of **affine Lie algebras** [Lepowsky–Wilson '79, Frenkel–Kac '80, ...].

After that, it has turned out that vertex algebras are ubiquitous in

- *2d conformal field theory* [Belavin–Polyakov–Zamolodchikov '84, ...],
- *3d topological quantum field theory* [Witten '89, ...],
- *4d superconformal field theory* [Alday–Gaiotto–Tachikawa '10, ...],

and so on.

[1] The notion of OPE firstly appeared in the work of K. G. Wilson ('69).

Axioms of Vertex Algebras

Roughly speaking, a **vertex algebra** consists of

- a vector space V over $\mathbb{C}$,
- a bilinear mapping $(?) \underset{z}{\times} (?) \colon V \times V \to V(\!(z)\!)$,
- a non-zero element $\mathbf{1}$ in V

satisfying the following conditions: for $A, B, C \in V$,

1. $\mathbf{1} \underset{z}{\times} A = A$ and $A \underset{z}{\times} \mathbf{1} \equiv A \bmod V[\![z]\!]z$ (**unitality**);
2. $(A \underset{z_1 - z_2}{\times} B) \underset{z_2}{\times} C \approx A \underset{z_1}{\times} (B \underset{z_2}{\times} C)$ (**associativity**);
3. $A \underset{z_1}{\times} (B \underset{z_2}{\times} C) \approx B \underset{z_2}{\times} (A \underset{z_1}{\times} C)$ (**locality**).

Note that we refer to $A(z) := A \underset{z}{\times} (?)$ as a quantum observable.

Analogy to Commutative Algebras

More precisely, the locality axiom[2] is given by

$$(z_1 - z_2)^n \big[A(z_1), B(z_2)\big] = 0 \quad \text{for sufficiently large } n.$$

Standard categorical notions for vertex algebras (e.g., morphisms, subquotients, simplicity, modules, . . .) can be defined in a similar way to those for **unital associative commutative algebras**.

For example, we have the following lemma:

Lemma 1.1 (Tensor Products)

The tensor product of finitely many vertex algebras over $\mathbb{C}$ carries a natural vertex algebra structure.

[2]See, e.g., Kac's textbook ('98, AMS) for detail.

Well-studied Building Blocks

The following two examples are building blocks in our discussion:

- **affine vertex algebras** $V^\ell(\mathfrak{g})$ ($\leftrightsquigarrow$ affine Lie algebras $\widehat{\mathfrak{g}}$);
- **lattice vertex algebras** V_L ($\leftrightsquigarrow$ integral lattices L).

Loosely speaking, an appropriate representation category of $V^\ell(\mathfrak{g})$ (resp. V_L) has an explicit description in terms of the corresponding quantum enveloping algebra[3] $U_q(\mathfrak{g})$ (resp. the corresponding finite abelian group $\mathrm{Hom}(L,\mathbb{Z})/L$ with some $\mathbb{C}^\times$-valued 3-cocycle[4]).

[3] See, e.g., [Kazhdan–Lusztig '93,'94, Finkelberg '96].

[4] See, e.g., Etingof–Gelaki–Nikshych–Ostrik's textbook ('15, AMS).

Constructions of New Vertex Algebras

More examples are obtained by the following constructions:

Definition 1.2 (Extensions and Cosets)

Let $U \hookrightarrow V$ be an embedding of vertex algebras. Then

- *V is called a* **vertex algebra extension** *of U,*
- *the commutant vertex subalgebra*

$$\mathrm{Com}(U,V) := \left\{ A \in V \;\middle|\; \left[A(z_1), B(z_2)\right] = 0 \text{ for any } B \in U \right\}$$

is called the **coset vertex algebra** *of U in V.*

As a special case, we call $\mathcal{Z}(V) := \mathrm{Com}(V,V)$ the **center** of V.

2d Chiral Conformal Symmetry

The **Virasoro algebra** is the universal central extension of the Lie algebra of vector fields on $S^1 = \{z = e^{2\pi\sqrt{-1}\theta}\}$, which appears as the chiral symmetry of **2d conformal field theory** (**CFT**).

A vertex algebra V with a **conformal vector** ω, whose "modes"

$$L_n := \frac{1}{2\pi\sqrt{-1}} \oint \omega(z) z^{n+1} \mathrm{d}z \in \mathrm{End}(V)$$

generate the Virasoro algebra of some central charge, is referred to as a **vertex operator algebra** (**VOA**).

It is well-known that the **Sugawara construction** provides affine[5] and lattice vertex algebras with their standard conformal vectors.

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[5] We need to assume that the level ℓ is not equal to the **critical level** $-h^\vee$.

Axioms of Modules

A **module** of a VOA (V, ω) consists of

- a vector space M over $\mathbb{C}$,
- a bilinear mapping $(?) \underset{z}{\circ} (?) \colon V \times M \to M(\!(z)\!)$

satisfying the following conditions: for $A, B \in V$ and $m \in M$,

1. $\mathbf{1} \underset{z}{\circ} m = m$ (**unitality**);
2. $(A \underset{z_1 - z_2}{\times} B) \underset{z_2}{\circ} m \approx A \underset{z_1}{\circ} (B \underset{z_2}{\circ} m)$ (**associativity**);
3. $A \underset{z_1}{\circ} (B \underset{z_2}{\circ} m) \approx B \underset{z_2}{\circ} (A \underset{z_1}{\circ} m)$ (**locality**);
4. $(L_{-1}A) \underset{z}{\circ} m = \frac{\partial}{\partial z}(A \underset{z}{\circ} m)$ (**flatness condition**);
5. L_0 is locally finite with lower bounded eigenvalues on M.

Fundamental Problem

Let (V, ω) be a VOA and $V\text{-mod}$ the $\mathbb{C}$-linear abelian category of V-modules **of finite length**, i.e., having finite composition series.

When V is **C_2-cofinite**, the number of inequivalent simple objects in $V\text{-mod}$ turns out to be **finite** [Zhu '96, Gaberdiel–Neitzke '03].

Adding mild conditions[6], Y.-Z. Huang proved that $V\text{-mod}$ carries a **braided monoidal category** structure with respect to the **fusion product** $(?) \boxtimes (?) \colon V\text{-mod} \times V\text{-mod} \to V\text{-mod}$ [Huang '09, ...].

Problem 1.3 (Kazhdan–Lusztig Correspondence)

Confirm such (non-symmetric) braided monoidal categories to be rigid and various conjectural connections to quantum supergroups.

[6] We further assume that V is $\mathbb{N}$-graded by L_0 and $\ker(L_0 \colon V \to V) = \mathbb{C}\mathbf{1}$.

Origin of Non-Symmetric Braiding

For distinct n-points $\boldsymbol{p} = (p_1, \ldots, p_n)$ on the projective line $\mathbf{P}^1(\mathbb{C})$ and an n-tuple $\boldsymbol{M} = (M_1, \cdots, M_n)$ of V-modules, one can define the vector space of (genus-zero) **n-point conformal blocks**[7] by

$$\mathrm{CB}\big(\mathbf{P}^1(\mathbb{C}), \boldsymbol{p}, \boldsymbol{M}\big) := \Big(\bigotimes_{i=1}^{n} M_i \Big/ \text{(conformal constraints)}\Big)^{*}.$$

Then the following functor

$$V\text{-mod} \to \mathbb{C}\text{-mod};\ \ M \mapsto \mathrm{CB}\big(\mathbf{P}^1(\mathbb{C}), (0,1,\infty), (M_2, M_1, M^*)\big)$$

is represented by the fusion product $M_1 \boxtimes M_2$ if it exists, and the square σ^2 is the **monodromy** of four-point conformal blocks.

[7] They glue to form a $\mathscr{D}$-module on the n-point configuration space of $\mathbf{P}^1(\mathbb{C})$.

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$\mathcal{W}$-algebras as Extensions

The smallest example of **$\mathcal{W}$-algebra** is the Virasoro VOA $\mathcal{W}^\ell(\mathfrak{sl}_2)$.

The second smallest $\mathcal{W}$-algebra $\mathcal{W}^\ell(\mathfrak{sl}_3)$ is originally introduced by A. Zamolodchikov ('85) as a higher-spin extension of the Virasoro VOA, which is no longer generated by an "elementary" Lie algebra.

General $\mathcal{W}$-algebras are obtained as extensions of $\mathcal{W}^\ell(\mathfrak{sl}_2)$ and play a fundamental role in the (conjectural) **2d chiral CFT/4d $\mathcal{N}=2$ SCFT correspondence** [Beem et. al. '15,...].

We first review the most standard class, called the **pricipal** case.

Center of Enveloping Algebra

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be a triangular decomposition of a simple Lie algebra and κ the normalized symmetric invariant form on $\mathfrak{g}$.

The **center** $Z(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ is isomorphic to

1. the commutant subalgebra $\mathrm{Com}\big(\mathfrak{g}, U(\mathfrak{g})\big)$ by definition;
2. the Weyl group-invariant subalgebra $U(\mathfrak{h})^W$ of $U(\mathfrak{h})$ through the Harish-Chandra homomorphism [Harish-Chandra '51];
3. the opposite algebra of $\mathfrak{g}$-endomorphisms[8] on the Whittaker module $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{n}_+}(\chi)$, where $\chi(?) = \kappa(f, ?) \colon \mathfrak{n}_+ \to \mathbb{C}$ is defined by a **principal** nilpotent element $f = f_{\text{prin}} \in \mathfrak{n}_-$ [Kostant '78].

[8] By the Frobenius reciprocity, they correspond to Whittaker vectors.

Principal Affine $\mathcal{W}$-algebras

Roughly speaking, the **universal principal affine $\mathcal{W}$-algebra** is an "affinization" of the center $Z(\mathfrak{g})$ at level $\ell \in \mathbb{C}$, denoted by $\mathcal{W}^\ell(\mathfrak{g})$.

$\mathcal{W}$-algebras are **NOT** generated by affine Lie algebras in general!!

Modules of the principal $\mathcal{W}$-algebra $\mathcal{W}^\ell(\mathfrak{g})$ are obtained by

1. *coset construction* [Goddard–Kent–Olive '85, ...];
2. *free field realization* [Fateev–Lukyanov '88, Feigin–Frenkel '92, ...];
3. *semi-infinite cohomology*[9] [Feigin–Frenkel '90, ...].

[9] This case is also known as Becchi–Rouet–Stora–Tyutin (BRST) cohomology.

Free Field Realization

The **free field realization** of the principal affine $\mathcal{W}$-algebra $\mathcal{W}^\ell(\mathfrak{g})$ is a vertex algebraic analog of the Harish-Chandra Homomorphism

$$\overline{\Upsilon}\colon Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{h}),$$

which is known as the **Miura map**

$$\Upsilon\colon \mathcal{W}^\ell(\mathfrak{g}) \hookrightarrow V^{\tau_\ell}(\mathfrak{h}).$$

Here τ_ℓ stands for a certain symmetric invariant form on $\mathfrak{h}$.

The image of the Miura map coincides with the union of kernels of $\mathrm{rank}(\mathfrak{g})$ **screening operators**[10].

[10] These operators are a vertex algebraic analog of simple reflections.

Generators of Principal $\mathcal{W}$-algebra

Let $D = \{d_i \,|\, i = 1, \dots, \mathrm{rank}(\mathfrak{g})\}$ denote the multi-set of degrees of homogeneous polynomial generators for $U(\mathfrak{h})^W = \mathbb{C}[\mathfrak{h}^*]^W$.

It is known that the set D always contains 2 which corresponds to the **quadratic Casimir element** Ω in $Z(\mathfrak{g})$.

The counterpart to Ω gives a conformal vector ω in $\mathcal{W}^\ell(\mathfrak{g})$.

Theorem 2.1 (e.g., Feigin–Frenkel '90)

The Virasoro L_0-operator induced by ω defines an $\mathbb{N}$-gradation

$$\mathcal{W}^\ell(\mathfrak{g}) = \bigoplus_{d=0}^{\infty} \mathcal{W}^\ell(\mathfrak{g})_d$$

and there exists a finite set of generators $\{J^{d_i} \in \mathcal{W}^\ell(\mathfrak{g})_{d_i}\}$ which contains the conformal vector $\omega = J^2$ of $\mathcal{W}^\ell(\mathfrak{g})$.

Langlands Dual Groups

Recall that connected complex reductive groups are determined by their **root data** up to isomorphism[11].

Two connected complex reductive groups are said to be **Langlands dual** to each other when their root data are dual to each other.

Let G be the **simply-connected** simple group associated to $\mathfrak{g}$ and $\check{G}$ denote its Langlands dual group associated to $\check{\mathfrak{g}} = \mathrm{Lie}(\check{G})$.

We note that $\check{G}$ is the **adjoint** group of the simple Lie algebra $\check{\mathfrak{g}}$.

Example 2.2 (Duality Between Classical Groups)

We have $\check{\mathrm{SL}}_n = \mathrm{PSL}_n$, $\check{\mathrm{Spin}}_{2n} = \mathrm{SO}_{2n}/\mathbb{Z}_2$, $\check{\mathrm{Spin}}_{2n+1} = \mathrm{Sp}_{2n}/\mathbb{Z}_2$.

[11] See, e.g., Springer's textbook ('98, Birkhäuser) for detial.

Feigin–Frenkel Duality

The next theorem is known as the **Feigin–Frenkel duality**:

Theorem 2.3 (Feigin–Frenkel '92, Aganagic–Frenkel–Okounkov '18)

For arbitrary $(\ell, \check{\ell})$ *satisfying* $r^\vee(\ell + h^\vee)(\check{\ell} + \check{h}^\vee) = 1$, *where* $r^\vee$ *is the lacing number of* $\mathfrak{g}$, *there exists a vertex algebra isomorphism* $V^{\tau_\ell}(\mathfrak{h}) \simeq V^{\check{\tau}_{\check{\ell}}}(\check{\mathfrak{h}})$ *which restricts to* $\mathcal{W}^\ell(\mathfrak{g}) \simeq \mathcal{W}^{\check{\ell}}(\check{\mathfrak{g}})$.

Remark 2.4 (Local Geometric Langlands Correspondence)

By taking a suitable limit, we obtain natural isomorphism(s)

$$\Big(\mathcal{Z}(V^{-h^\vee}(\mathfrak{g})) \simeq\Big)\, \mathcal{W}^{-h^\vee}(\mathfrak{g}) \simeq \mathcal{W}^{\infty}(\check{\mathfrak{g}})$$

of Poisson vertex algebras and the enveloping algebra of the last is naturally dual to the moduli space of $\check{G}$*-opers on* $\mathrm{Spec}\big(\mathbb{C}(\!(z)\!)\big)$.

Beyond Principal Non-Super $\mathcal{W}$-algebras

Naïve Question (cf. Gaiotto–Rapčák '19)

Can we generalize the Feigin–Frenkel duality to outside of principal non-super $\mathcal{W}$-algebras? Are there any relationships among

- *principal $\mathcal{W}$-**super**algebras,*
- ***non-principal** $\mathcal{W}$-algebras,*

and relevant (super)geometric objects[a]*?*

[a] See, e.g., [Zeitlin '15] for the $\mathfrak{osp}_{1|2}$-Gaudin model and SPL_2-superopers.

Today's Main Topic: Feigin–Semikhatov Duality

*In 2004, B. Feigin and A. Semikhatov found a mysterious clue of a possible **super**/**non-principal** duality which is recently proved by Creutzig–Linshaw and Creutzig–Genra–Nakatsuka, independently.*

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Generalization to Non-Principal Case

Let f be a general nilpotent element in $\mathfrak{g}$ and $\chi(?) = \kappa(f, ?)$.

The **finite $\mathcal{W}$-algebra**[12] $U(\mathfrak{g}, f)$ is the deformation quantization of the **Slodowy slice**, which is a Poisson transversal at χ in $\mathfrak{g}^*$.

Informally speaking, the **universal affine $\mathcal{W}$-algebra** $\mathcal{W}^\ell(\mathfrak{g}, f)$ is an "affinization" of the finite $\mathcal{W}$-algebra $U(\mathfrak{g}, f)$ at level ℓ.

Now let's go into a bit more detail of its definition for later use.

[12] Originally introduced by A. Premet ('02) and generalized by Gan–Ginzburg ('02).

Good Gradings for Lie Superalgebras

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a complex simple Lie **super**algebra equipped with a suitably normalized **super**symmetric invariant form κ.

Definition 3.1 (Kac–Roan–Wakimoto '03)

A $\mathbb{Z}/2\mathbb{Z}$-homogeneous $\frac{1}{2}\mathbb{Z}$-gradation $\Gamma\colon \mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ is said to be a **good grading** *adapted to a nilpotent element $f \in \mathfrak{g}_{\bar{0}}$ if*

1. *the nilpotent element f lies in $\mathfrak{g}_{-1}$,*
2. *$\mathrm{ad}(f)$ is injective for $j \geq 1/2$; surjective for $j \leq 1/2$.*

A good grading is said to be **even** *if it is a $\mathbb{Z}$-gradation.*

Example 3.2 (Principal Non-Super Case)

The principal $\mathbb{Z}$-gradation Γ_{prin} of a simple Lie algebra gives an even good grading adapted to a principal nilpotent element f_{prin}.

Definition of Universal $\mathcal{W}$-superalgebras

Let $\Gamma\colon \mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ be an even good grading adapted to f and regard $X := \Pi\mathfrak{g}_{>0} \oplus \Pi\mathfrak{g}^*_{>0}$ as a symplectic vector superspace[13].

The **quantum BRST cohomology complex** (e.g., [de Boer–Tjin '93])

$$\left(U(\mathfrak{g}) \otimes \overline{\mathcal{C}\ell}(X) \overset{\mathsf{gr}}{\simeq} \mathbb{C}[\mathfrak{g}^*] \otimes \mathbb{C}[X],\ \bar{\mathrm{d}} = \bar{\mathrm{d}}_{\mathsf{CE}} + \bar{\mathrm{d}}_f\right)$$

admits a vertex superalgebra analog (e.g., [Kac–Roan–Wakimoto '03])

$$\left(\mathcal{C}^\ell(\mathfrak{g}, f; \Gamma) := V^\ell(\mathfrak{g}) \otimes \mathcal{C}\ell(X),\ \mathrm{d} = \mathrm{d}_{\mathsf{CE}} + \mathrm{d}_f\right).$$

Then the corresponding cohomology $H^*\left(\mathcal{C}^\ell(\mathfrak{g}, f; \Gamma), \mathrm{d}^{\mathsf{ch}}\right)$ turns out to be independent[14] of the choice of Γ and is denoted by $\mathcal{W}^\ell(\mathfrak{g}, f)$.

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[13] Here $\Pi(?)$ stands for the $\mathbb{Z}/2\mathbb{Z}$-parity reversing functor.

[14] Different choices of Γ may give different conformal vectors on $\mathcal{W}^\ell(\mathfrak{g}, f)$.

Trivial & Principal Non-Super Cases

1. Since $\Gamma_{\mathsf{triv}}\colon \mathfrak{g} = \mathfrak{g}_0$ is adapted to $f = 0$, we have

$$\left(\mathcal{C}^\ell(\mathfrak{g}, 0; \Gamma_{\mathsf{triv}}) = V^\ell(\mathfrak{g}),\ \mathrm{d} = 0\right)$$

   and the corresponding cohomology $\mathcal{W}^\ell(\mathfrak{g}, 0)$ coincides with the **universal affine vertex superalgebra** $V^\ell(\mathfrak{g})$.

2. When $\mathfrak{g}$ is a Lie algebra, we have

$$\left(\mathcal{C}^\ell(\mathfrak{g}, f_{\mathsf{prin}}; \Gamma_{\mathsf{prin}}) = V^\ell(\mathfrak{g}) \otimes V_{\mathbb{Z}}^{\otimes \dim(\mathfrak{n}_+)},\ \mathrm{d}\right).$$

   Then $\mathcal{W}^\ell(\mathfrak{g}, f_{\mathsf{prin}})$ provides a cohomological definition of the universal principal non-super $\mathcal{W}$-algebra $\mathcal{W}^\ell(\mathfrak{g})$.

Miura Map for $\mathcal{W}$-superalgebras

Let $\Gamma\colon \mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ be an even good grading adapted to f.

Theorem 3.3 (Arakawa '17, Genra '17, Nakatsuka '21)

For arbitrary ℓ, there exist a supersymmetric invariant form τ_ℓ on $\mathfrak{g}_0$ and an injective vertex superalgebra homomorphism

$$\Upsilon_\Gamma\colon \mathcal{W}^\ell(\mathfrak{g}, f) \hookrightarrow V^{\tau_\ell}(\mathfrak{g}_0),$$

whose image is the union of kernels of certain screening operators.

Note that De Sole–Kac–Valeri ('16) proved its Poisson analog.

Example 3.4 (Principal Non-Super Case)

When $(\mathfrak{g}, f, \Gamma) = (\mathfrak{g}_{\bar{0}}, f_{\mathsf{prin}}, \Gamma_{\mathsf{prin}})$, *we get* $(\mathfrak{g}_0, \tau_\ell) = (\mathfrak{h}, (\ell + h^\vee)\kappa)$.

Generators of $\mathcal{W}$-superalgebras

Let $\Gamma\colon \mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ be an even good grading adapted to f and set $\mathfrak{g}^f$ to be the centralizer of f in $\mathfrak{g}$.

Theorem 3.5 (Kac–Wakimoto '04)

For a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$-homogeneous basis $\{x_i \in \mathfrak{g}^f \cap \mathfrak{g}_{-j_i}\}$ of $\mathfrak{g}^f$, one can construct a set of generators

$$\left\{ J^{\{x_i\}} \in \mathcal{W}^\ell(\mathfrak{g}, f)_{j_i+1} \;\middle|\; i = 1, \dots, \dim \mathfrak{g}^f \right\}$$

containing the conformal vector $\omega_\Gamma = J^{\{f\}}$ for $\mathcal{W}^\ell(\mathfrak{g}, f)$.

Example 3.6 (Principal Non-Super Case)

When $(\mathfrak{g}, f, \Gamma) = (\mathfrak{g}_{\bar{0}}, f_{\mathsf{prin}}, \Gamma_{\mathsf{prin}})$, *we have* $\mathfrak{g}^f = \bigoplus_i (\mathfrak{g}^f \cap \mathfrak{g}_{-d_i+1})$.

Subregular $\mathcal{W}$-algebras of type A

Let $\mathfrak{g} = \mathfrak{sl}_n$ and $f = f_{\text{sub}}$, a **subregular**[15] nilpotent element of $\mathfrak{g}$.

Then there exists an even good grading Γ adapted to f such that we have $\mathfrak{g}_0 \simeq \mathfrak{sl}_2 \oplus \mathbb{C}^{n-1}$ and $\mathfrak{g}^f \cap \mathfrak{g}_0 = \mathbb{C}x_0$.

As a corollary, the element $H_{\text{sub}} := J^{\{x_0\}}$ generates a Heisenberg subalgebra π_{sub} of $\mathcal{W}^\ell(\mathfrak{g}, f)$ iff $\ell \neq -n + \frac{n}{n-1}$.

Lemma 3.7 (Creutzig–Genra–Nakatsuka '21)

The Heisenberg coset $\pi^\perp := \mathrm{Com}\big(\Upsilon_\Gamma(\pi_{\text{sub}}), V^{\tau_\ell}(\mathfrak{g}_0)\big)$ *is a rank* n *Heisenberg vertex algebra and we have a free field realization*

$$\Upsilon_\Gamma|\colon\ \mathrm{Com}\big(\pi_{\text{sub}}, \mathcal{W}^\ell(\mathfrak{g}, f)\big) \hookrightarrow \pi^\perp.$$

[15] The corresponding partition (the shape of Jordan cells) is $n = (n-1) + 1$.

Principal $\mathcal{W}$-superalgebras of type A

Let $\check{\mathfrak{g}} = \mathfrak{sl}_{1|n}$ $(= \mathfrak{sl}_{n|1})$ and $\check{f} = f_{\text{prin}}$ in the even part $\check{\mathfrak{g}}_{\bar{0}} = \mathfrak{gl}_n$.

Then there exists an even good grading $\check{\Gamma}$ adapted to $\check{f}$ such that we have $\check{\mathfrak{g}}_0 \simeq \mathfrak{gl}_{1|1} \oplus \mathbb{C}^{n-1}$ and $\check{\mathfrak{g}}^{\check{f}} \cap \check{\mathfrak{g}}_0 = \mathbb{C}\check{x}_0$.

As a corollary, the element $H_{\text{prin}} := J^{\{\check{x}_0\}}$ generates a Heisenberg subalgebra π_{prin} of $\mathcal{W}^\ell(\check{\mathfrak{g}}) := \mathcal{W}^\ell(\check{\mathfrak{g}}, \check{f})$ iff $\ell \neq -(n-1) + \frac{n-1}{n}$.

Lemma 3.8 (Creutzig–Genra–Nakatsuka '21)

The Heisenberg coset $\check{\pi}^\perp := \mathrm{Com}\big(\Upsilon_{\check{\Gamma}}(\pi_{\text{prin}}), V^{\check{\tau}_\ell}(\check{\mathfrak{g}}_0)\big)$ *is a rank* n *Heisenberg vertex algebra and we have a free field realization*

$$\Upsilon_{\check{\Gamma}}|\colon\ \mathrm{Com}\big(\pi_{\text{prin}}, \mathcal{W}^\ell(\check{\mathfrak{g}})\big) \hookrightarrow \check{\pi}^\perp.$$

Feigin–Semikhatov Duality

The next theorem was conjectured by Feigin–Semikhatov ('04).

Theorem 3.9 (Creutzig–Genra–Nakatsuka '21, cf. Creutzig–Linshaw '20$^+$)

Set $(\ell_0, h^\vee; \check{\ell}_0, \check{h}^\vee)$ *to be* $(-n + \frac{n}{n-1}, n; -(n-1) + \frac{n-1}{n}, n-1)$. *Then, for arbitrary* $(\ell, \check{\ell}) \neq (\ell_0, \check{\ell}_0)$ *satisfying* $(\ell + h^\vee)(\check{\ell} + \check{h}^\vee) = 1$, *there is a vertex algebra isomorphism* $\pi^\perp \simeq \check{\pi}^\perp$ *which restricts to*

$$\mathbf{FS}\colon\ \mathrm{Com}\big(\pi_{\mathsf{sub}}, \mathcal{W}^\ell(\mathfrak{sl}_n, f_{\mathsf{sub}})\big) \simeq \mathrm{Com}\big(\pi_{\mathsf{prin}}, \mathcal{W}^{\check{\ell}}(\mathfrak{sl}_{1|n})\big)$$

through their Miura maps.

Note that a similar duality between subregular $\mathcal{W}$-algebras of **type B** and principal $\mathcal{W}$-superalgebras of **type C** is obtained in loc. cit.

Kazama–Suzuki Duality

The following theorem is a generalization of the **Kazama–Suzuki** and **Feigin–Semikhatov–Tipunin** coset construction for $\mathfrak{g} = \mathfrak{sl}_2$.

Theorem 3.10 (Creutzig–Genra–Nakatsuka '21)

There exist two diagonal Heisenberg vertex subalgebras of rank one

$$\Delta(\pi_{\mathsf{sub}}) \subset \mathcal{W}^\ell(\mathfrak{g}, f) \otimes V_{\mathbb{Z}}, \quad \Delta(\pi_{\mathsf{prin}}) \subset \mathcal{W}^{\check{\ell}}(\check{\mathfrak{g}}) \otimes V_{\sqrt{-1}\mathbb{Z}}$$

such that we have natural isomorphisms

$$\mathbf{KS}\colon \mathcal{W}^{\check{\ell}}(\check{\mathfrak{g}}) \xrightarrow{\simeq} \mathrm{Com}(\Delta(\pi_{\mathsf{sub}}), \mathcal{W}^\ell(\mathfrak{g}, f) \otimes V_{\mathbb{Z}}),$$

$$\mathbf{FST}\colon \mathcal{W}^\ell(\mathfrak{g}, f) \xrightarrow{\simeq} \mathrm{Com}(\Delta(\pi_{\mathsf{prin}}), \mathcal{W}^{\check{\ell}}(\check{\mathfrak{g}}) \otimes V_{\sqrt{-1}\mathbb{Z}}),$$

which are compatible with their Miura maps.

How About Representations?

So far, we obtain the following three constructions

$$\mathbf{FS}\colon\ \mathrm{Com}\big(\pi_{\mathsf{sub}}, \mathcal{W}^{\ell}(\mathfrak{g}, f)\big) \simeq \mathrm{Com}\big(\pi_{\mathsf{prin}}, \mathcal{W}^{\check{\ell}}(\check{\mathfrak{g}})\big),$$

$$\mathbf{KS}\colon \mathcal{W}^{\check{\ell}}(\check{\mathfrak{g}}) \xrightarrow{\simeq} \mathrm{Com}(\Delta(\pi_{\mathsf{sub}}), \mathcal{W}^{\ell}(\mathfrak{g}, f) \otimes V_{\mathbb{Z}}),$$

$$\mathbf{FST}\colon \mathcal{W}^{\ell}(\mathfrak{g}, f) \xrightarrow{\simeq} \mathrm{Com}(\Delta(\pi_{\mathsf{prin}}), \mathcal{W}^{\check{\ell}}(\check{\mathfrak{g}}) \otimes V_{\sqrt{-1}\mathbb{Z}}).$$

The representation theory of a **$\mathcal{W}$-superalgebra** can be described in terms of that of the corresponding **affine vertex superalgebra**, **but** the latter has been well-studied **only in the non-super case**.

Our Problem: From Algebras to Representations

To describe the representation theory of $\mathcal{W}^{\check{\ell}}(\check{\mathfrak{g}}) = \mathcal{W}^{\check{\ell}}(\mathfrak{sl}_{1|n}, f_{\mathsf{prin}})$ *by using the dualities and* **relative semi-infinite cohomology**.

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Category of Weight Modules

Let (V, ω) be a conformal vertex superalgebra and π its Heisenberg vertex subalgebra generated by an abelian Lie algebra $\mathfrak{a}$.

A V-module M is **π-weight** if it decomposes into a direct sum

$$M = \bigoplus_{\lambda \in \mathfrak{a}^*} \Omega_\lambda(M) \otimes \pi_\lambda$$

of π-modules, where π_λ stands for the Heisenberg Fock π-module, such that the coefficient $\mathrm{Com}(\pi, V)$-module $\Omega_\lambda(M)$ decomposes into **finite-dimensional** generalized L_0-eigenspaces.

We write $\mathscr{C}_{\mathsf{sub}}$ for the category of π_{sub}-weight $\mathcal{W}^\ell(\mathfrak{g}, f)$-modules and $\mathscr{C}_{\mathsf{prin}}$ for that of π_{prin}-weight $\mathcal{W}^{\check{\ell}}(\check{\mathfrak{g}})$-modules.

Diagonal Coset Functor

Recall that we have

$$\mathbf{KS}\colon \mathcal{W}^{\check{\ell}}(\mathfrak{sl}_{1|n}) \xrightarrow{\simeq} \mathrm{Com}(\Delta(\pi_{\mathsf{sub}}), \mathcal{W}^{\ell}(\mathfrak{sl}_n, f_{\mathsf{sub}}) \otimes V_{\mathbb{Z}}),$$

$$\mathbf{FST}\colon \mathcal{W}^{\ell}(\mathfrak{sl}_n, f_{\mathsf{sub}}) \xrightarrow{\simeq} \mathrm{Com}(\Delta(\pi_{\mathsf{prin}}), \mathcal{W}^{\check{\ell}}(\mathfrak{sl}_{1|n}) \otimes V_{\sqrt{-1}\mathbb{Z}}).$$

Let $\mathfrak{a} = \mathbb{C}H_{\mathsf{sub}}$ and $\check{\mathfrak{a}} = \mathbb{C}H_{\mathsf{prin}}$ be the subspaces generating π_{sub} and π_{prin}, respectively. The next proposition is our starting point.

Proposition (Creutzig–Genra–Nakatsuka–S. '21$^+$)

For $\lambda \in \mathfrak{a}^*$, there exists $\check{\lambda} \in \check{\mathfrak{a}}^*$ such that the following functors

$$\Omega^+_\lambda(?) := \Omega_\lambda\big((?) \otimes V_{\mathbb{Z}}\big) : \mathscr{C}_{\mathsf{sub}} \to \mathscr{C}_{\mathsf{prin}},$$

$$\Omega^-_{\check{\lambda}}(?) := \Omega_{\check{\lambda}}\big((?) \otimes V_{\sqrt{-1}\mathbb{Z}}\big) : \mathscr{C}_{\mathsf{prin}} \to \mathscr{C}_{\mathsf{sub}}$$

are mutually quasi-inverse on appropriate full subcategories.

Cohomological Interpretation

Recall that **relative Lie algebra cohomology** plays an important role in connecting representation theory to geometric objects.

Its **semi-infinite** geometric analog is introduced by B. Feigin ('84) and Frenkel–Garland–Zuckerman ('86) for "string field theories"[16].

More recently, T. Creutzig and A. Linshaw ('20$^+$, '21$^+$) conjectured various $\mathcal{W}$-superalgebras are related via the **geometric Langlands kernels** and the **relative semi-infinite cohomology**.

In this work we prove their conjecture in the simplest case!!

[16] For a mathematical exposition, we refer the reader to [Voronov '93].

Geometric Langlands Kernel

For $\psi^{-1} + \psi_!^{-1} = 1$, the **geometric Langlands kernel**[17] is

$$A[\mathfrak{gl}_N, \psi] := \bigoplus_{\lambda \in P^+} V^{\psi - N}(\lambda) \otimes V^{\psi_! - N}(\lambda) \otimes V_{\sqrt{N}\mathbb{Z} + \frac{s(\lambda)}{\sqrt{N}}} \otimes \pi,$$

where P^+ is the set of dominant integral weights for $\mathfrak{sl}_N$, $V^k(\lambda)$ is the corresponding Weyl module, π is the Heisenberg vertex algebra generated by $\mathfrak{gl}_1$, and $s \colon P^+ \to P/Q \simeq \mathbb{Z}/N\mathbb{Z}$.

When $N = 1$, this is just the free field vertex superalgebra

$$\mathcal{K}_0 := A[\mathfrak{gl}_1, \psi] = V_{\mathbb{Z}} \otimes \pi,$$

which is independent of ψ.

[17] See [Creutzig–Gaiotto '20, Creutzig–Linshaw '20$^+$] for detail.

Relative Semi-infinite Cohomology

For $\lambda \in \mathbb{C}$, we have the following decomposition

$$\mathcal{K}_\lambda := V_{\mathbb{Z}} \otimes \pi_\lambda = \bigoplus_{\mu} \pi^{\dagger}_{\mathsf{sub},\lambda+\mu} \otimes \pi_{\mathsf{prin},\check{\lambda}+\check{\mu}},$$

where $\pi^{\dagger}_{\mathsf{sub}}$ has the negative level opposite to π_{sub}.

Therefore the relative semi-infinite complex[18]

$$C_\lambda(\widehat{\mathfrak{a}}, \mathfrak{a}, ?) := \left((?) \otimes \mathcal{K}_\lambda \otimes \Lambda^{\frac{\infty}{2}}_{\mathsf{rel}}\right)^{\mathfrak{a}}$$

carries a level-zero $\widehat{\mathfrak{a}}$-action and one can construct the **relative semi-infinite cohomology functor** [Frenkel–Garland–Zuckerman '86]

$$H^+_\lambda(?) := H^0\big(C_\lambda(\widehat{\mathfrak{a}}, \mathfrak{a}, ?), \mathrm{d}_{\mathsf{rel}}\big) \colon \mathscr{C}_{\mathsf{sub}} \to \mathscr{C}_{\mathsf{prin}}.$$

[18] $\Lambda^{\frac{\infty}{2}}_{\mathsf{rel}}$ is isomorphic to the symplectic fermion vertex superalgebra of rank one.

Coset = Cohomology [1/2]

Our first main result is as follows:

Main Result A (Creutzig–Genra–Nakatsuka–S. '21$^+$)

For any $\lambda \in \mathfrak{a}^*$, we have a natural isomorphism

$$\Omega^+_\lambda(?) \simeq H^+_\lambda(?) \colon \mathscr{C}_{\mathsf{sub}} \to \mathscr{C}_{\mathsf{prin}}$$

of linear functors and a similar result for $\Omega^-_{\check{\lambda}}(?)$ as well.

For example, if we pick an object M of $\mathscr{C}$ such that

$$M = \bigoplus_{\mu} \Omega_{\lambda+\mu}(M) \otimes \pi_{\mathsf{sub},\lambda+\mu},$$

then the relative semi-infinite complex $C_\lambda(\widehat{\mathfrak{a}}, \mathfrak{a}, M)$ is given by

$$\bigoplus_{\mu} \Omega_{\lambda+\mu}(M) \otimes \boldsymbol{\pi}_{\mathbf{sub},\boldsymbol{\lambda+\mu}} \otimes \boldsymbol{\pi}^{\dagger}_{\mathbf{sub},\boldsymbol{\lambda+\mu}} \otimes \pi_{\mathsf{prin},\check{\lambda}+\check{\mu}} \otimes \boldsymbol{\Lambda}^{\frac{\infty}{2}}_{\mathbf{rel}}.$$

Coset $=$ Cohomology [2/2]

Our first main result is as follows:

Main Result A (Creutzig–Genra–Nakatsuka–S. '21$^+$)

For any $\lambda \in \mathfrak{a}^*$, we have a natural isomorphism

$$\Omega^+_\lambda(?) \simeq H^+_\lambda(?) \colon \mathscr{C}_{\text{sub}} \to \mathscr{C}_{\text{prin}}$$

of linear functors and a similar result for $\Omega^-_{\check{\lambda}}(?)$ as well.

By using the following isomorphism [Frenkel–Garland–Zuckerman '86]

$$H^i\left(\pi_{\text{sub},\lambda+\mu} \otimes \pi^\dagger_{\text{sub},\lambda+\mu} \otimes \Lambda^{\frac{\infty}{2}}_{\text{rel}}, \mathrm{d}_{\text{rel}}\right) \simeq \delta_{i,0}\mathbb{C},$$

we obtain the corresponding relative semi-infinite cohomology

$$H^+_\lambda(M) \simeq \bigoplus_{\nu} \Omega_{\lambda+\mu}(M) \otimes \pi_{\text{prin},\check{\lambda}+\check{\mu}} \overset{\textbf{FS}}{\simeq} \Omega^+_\lambda(M).$$

Compatibility with Fusion Product

Let Q denote the $\mathfrak{a}$-weight set of $\mathcal{W}^\ell(\mathfrak{sl}_n, f_{\text{sub}})$ and

$$M_i = \bigoplus_{\mu \in Q} \Omega_{\lambda_i+\mu}(M_i) \otimes \pi_{\text{sub},\lambda_i+\mu} \in \mathrm{Ob}(\mathscr{C}_{\text{sub}}) \quad (\lambda_i \in \mathfrak{a}^*)$$

for $i \in \{1,2\}$. Then our second main result is as follows:

Main Result B (Creutzig–Genra–Nakatsuka–S. '21$^+$)

The fusion product $M_1 \boxtimes M_2$ exists in a certain full subcategory of $\mathscr{C}_{\text{sub}}$ if and only if $H^+_{\lambda_1}(M_1) \boxtimes H^+_{\lambda_2}(M_2)$ exists in the corresponding full subcategory of $\mathscr{C}_{\text{prin}}$. Moreover, we have a natural isomorphism

$$H^+_{\lambda_1}(M_1) \boxtimes H^+_{\lambda_2}(M_2) \simeq H^+_{\lambda_1+\lambda_2}(M_1 \boxtimes M_2).$$

Lastly, we apply this result to two interesting cases!!

1 Quantum Symmetry from Vertex Algebras

2 Duality in Principal $\mathcal{W}$-algebras

3 Beyond Principal $\mathcal{W}$-algebras

4 Main Results

5 Examples: C_2-cofinite/non-C_2-cofinite Cases

C_2-cofinite Case

Let $\mathcal{W}_\ell(\mathfrak{g}, f)$ denote the **simple quotient** of $\mathcal{W}^\ell(\mathfrak{g}, f)$ and so on.

Theorem 5.1 (cf. Creutzig–Linshaw '20$^+$ for $r \geq 3$)

When $\ell = -n + \frac{n+r}{n-1}$ and $(n+r, n-1) = 1$, we have

$$\mathrm{Com}\big(\pi_{\mathsf{sub}}, \mathcal{W}_\ell(\mathfrak{g}, f)\big) \simeq \mathrm{Com}\big(\pi_{\mathsf{prin}}, \mathcal{W}_{\check{\ell}}(\check{\mathfrak{g}})\big) \simeq \mathcal{W}_{\ell_!}(\mathfrak{g}_!),$$

where $\mathfrak{g}_! = \mathfrak{sl}_r$ and $(\ell + h^\vee)^{-1} + (\ell_! + h_!^\vee)^{-1} = 1$.

Theorem 5.2 (Creutzig–Genra–Nakatsuka '21)

For ℓ as above, there is a chain of **simple current**[a] **extensions**

$$\big(\mathcal{W}_{\ell_!}(\mathfrak{g}_!) \otimes V_{\sqrt{(n+r)r}\mathbb{Z}}\big) \otimes V_{\sqrt{n(n+r)}\mathbb{Z}} \subseteq \mathcal{W}_{\check{\ell}}(\check{\mathfrak{g}}) \otimes V_{\sqrt{n(n+r)}\mathbb{Z}} \subsetneq \mathcal{W}_\ell(\mathfrak{g}, f) \otimes V_{\mathbb{Z}}.$$

In particular, $\mathcal{W}_{\check{\ell}}(\check{\mathfrak{g}})$ is C_2-cofinite and rational.

[a] Simple invertible objects in V-mod are referred to as **simple currents** of V.

Fusion Product of $\mathcal{W}_{\check{\ell}}(\check{\mathfrak{g}})$-modules

Finally, our last main result is as follows:

Main Result C (Creutzig–Genra–Nakatsuka–S. '21$^+$)

For $(n,r) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ with $(n+r, n-1) = 1$, the semisimple monoidal structure of

$$\mathcal{W}_{\check{\ell}}(\check{\mathfrak{g}})\text{-mod} = \mathcal{W}_{-(n-1)+\frac{n-1}{n+r}}(\mathfrak{sl}_{1|n})\text{-mod} = \mathscr{C}_{\text{prin}}$$

can be explicitly described in terms of that of

$$\mathcal{W}_{\ell_!}(\mathfrak{g}_!)\text{-mod} = \mathcal{W}_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r)\text{-mod}, \tag{1}$$

$$\mathcal{W}_{\ell}(\mathfrak{g}, f)\text{-mod} = \mathcal{W}_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{\text{sub}})\text{-mod} = \mathscr{C}_{\text{sub}}. \tag{2}$$

Note that the structure of (1) is determined by Frenkel–Kac–Wakimoto ('92) and that of (2) for even n is by Arakawa–van Ekeren ('19$^+$). We extend the latter result to all n by using the previous simple current extensions.

Non-C_2-cofinite Case (Work in Progress)

Even if the C_2-cofiniteness fails, we expect that a braided monoidal structure may exist on a category of appropriate modules.

In fact, at least when $\ell = -n + \frac{n}{n+1}, -n + \frac{n+1}{n}$, or generic,

$$\mathrm{Com}\big(\pi_{\text{sub}}, \mathcal{W}_{\ell}(\mathfrak{sl}_n, f_{\text{sub}})\big) \simeq \mathrm{Com}\big(\pi_{\text{prin}}, \mathcal{W}_{\check{\ell}}(\mathfrak{sl}_{1|n})\big)$$

contains a simple Virasoro VOA $\mathcal{V}$ and we expect the following:

Strategy by Induction Method (cf. Creutzig–McRae–Yang '21)

Let $(\mathcal{W}, \pi)$ denote $(\mathcal{W}_{\ell}(\mathfrak{sl}_n, f_{\text{sub}}), \pi_{\text{sub}})$ or $(\mathcal{W}_{\check{\ell}}(\mathfrak{sl}_{1|n}), \pi_{\text{prin}})$. Then the fusion product $M_1 \boxtimes M_2$ of $\mathcal{W}$-modules **may exist** when M_i for $i \in \{1,2\}$ is an appropriate sum of **C_1-cofinite** $\mathcal{V} \otimes \pi$-submodules.

Future Directions [1/2]

Since there is a conjectural relationship[19] between

$$\mathcal{W}_k(\mathfrak{gl}_{m|n}) \overset{?}{\leftrightsquigarrow} U_{q_1}(\mathfrak{gl}_{m|n}) \otimes U_{q_2}(\mathfrak{gl}_m) \otimes U_{q_3}(\mathfrak{gl}_n)$$

for appropriate $(k; q_1, q_2, q_3)$, it seems natural to expect that

$$\mathscr{C}_{\text{prin}} = \mathcal{W}_{-(n-1)+\frac{n-1}{n+r}}(\mathfrak{sl}_{1|n})\text{-mod}$$

is related with the **semisimplified** category of finite-dimensional modules for a **relevant quantum supergroup at root of unity**.

[19]When $m = 0$, the right-hand side corresponds to the **modular double** of $U_q(\mathfrak{gl}_n)$. See [Bershtein–Feigin–Merzon '18] for detail (cf. [Cheng–Kwon–Lam '08]).

Future Directions [2/2]

For example, the non-C_2-cofinite subregular $\mathcal{W}$-algebra

$$\mathcal{B}_{n+1} := \mathcal{W}_{-n+\frac{n}{n+1}}(\mathfrak{sl}_n, f_{\text{sub}})$$

corresponds to the (A_1, A_{2n-1}) **Argyres–Douglas theory**[20] via the $2\text{d}/4\text{d}$ correspondence [Adamović–Creutzig–Genra–Yang '21].

In this context, the Feigin–Semikhatov duality can be regarded as a special case[21] of the $\mathfrak{S}_3$**-triality** in Y-algebras [Gaiotto–Rapčák '19].

We expect that the cohomological approach is efficient as well in extending our result to more general cases (work in progress).

[20]From this viewpoint, we may regard $\mathcal{B}_2$ as the free bosonic $\beta\gamma$-system.

[21]Our case is related to $Y_{n,1,0}[\Psi]$ presented in [Gaiotto–Rapčák '19].