Approximate Optimality Conditions in Fractional Semi-Infinite Multiobjective Optimization[1]

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Abstract. This paper provides some new results on weak approximate solutions in fractional multiobjective optimization problems. Specifically, we establish necessary optimality conditions of Fritz–John type for a local weakly ϵ-efficient solution in fuzzy form and, by using limiting constraint qualification, we provide necessary optimality conditions of Karush–Kuhn–Tucker type for a weakly ϵ-quasi-efficient solution. To this purpose advanced tools of variational analysis and generalized differentiation are used.

1 Introduction

In this paper, we are interested in a fractional semi-infinite multiobjective optimization problem, which admires the following mathematical form:

$$\begin{array}{lll}\text{(FSMP)} & \min\limits_{\mathbb{R}^m_+} & f(x)\colon=\left(\dfrac{p_1(x)}{q_1(x)},\dots,\dfrac{p_m(x)}{q_m(x)}\right) \\ & \text{s.t} & x\in F\colon=\{x\in S \mid g_t(x)\leqq 0, t\in T\},\end{array}$$

where p_k, q_k, $k\in M=\{1,\dots,m\}$ and g_t, $t\in T$ (possibly infinite index set) are locally Lipschitz on $\mathbb{R}^n$, S is a nonempty closed subset of $\mathbb{R}^n$, and $\mathbb{R}^n$ is the Euclidean space of dimension n. For the sake of convenience, we assume further that $q_k(x)>0$, $k=1,\dots,m$ for all $x\in S$, and that $p_k(\bar{x})\leqq 0$, $k=1,\dots,m$ for the reference point $\bar{x}\in S$.

Recently, optimization programming problems with fractional objective functions have been investigated intensively by many researchers (see e.g. [1, 2, 7, 9] and references therein). However, to the best of our knowledge, there are not so many papers dealing with approximate solutions to such class of optimization problems. Moreover, in classical approach, establishing optimality conditions in a fractional optimization problem requires concavity\convexity assumptions. However, its objective function is generally not a convex one. Hence using the extremal principle and other advanced techniques of a variational analysis and generalized differentiation to establish optimality conditions seems to be suitable for nonconvex\nonsmooth fractional multiobjective optimization problems.

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We investigate necessary optimality conditions of Karush–Kuhn–Tucker (KKT) type for local weakly ϵ-efficient (in fuzzy form) and weakly ϵ-quasi-efficient solutions of a fractional semi-infinite multiobjective optimization problem. Finally, we provide some remark about further research.

2 Preliminaries

Let us recall some notations and preliminary results which will be used throughout this paper; see e.g., [4, 10].

The nonnegative orthant of $\mathbb{R}^n$ is defined by $\mathbb{R}^n_+ : = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_i \geqq 0,\ i = 1, \dots, n\}$. The bracket $\langle \cdot, \cdot \rangle$ stands for inner product of given vectors in $\mathbb{R}^n$. The interior, and the closure of S are denoted, respectively, by $\operatorname{int} S$, $\operatorname{cl} S$. In this setting, the *polar* cone of a set $S \subset \mathbb{R}^n$ is defined by

$$S^\circ : = \{x^* \in \mathbb{R}^n \mid \langle x^*, x \rangle \leqq 0,\ \forall x \in S\}. \tag{2.1}$$

Given a multifuction $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with values $G(x) \subset \mathbb{R}^m$ in the collection of all the subsets of $\mathbb{R}^m$ (and similarly, of course, in infinite dimensions). The limiting construction

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} G(x) : = \Big\{ y \in \mathbb{R}^m \mid \exists\ x_k \to \bar{x},\ y_k \to y \\ \text{with } y_k \in G(x_k) \text{ for all } k \in \mathbb{N} : = \{1, 2, \dots\} \Big\}$$

is known as the Painlevé-Kuratowski upper/outer limit of G at $\bar{x}$.

A set $F \subset \mathbb{R}^n$ is called closed around $\bar{x} \in F$ if there is a neighborhood U of $\bar{x}$ such that $F \cap \operatorname{cl} U$ is closed. F is said to be locally closed if F is closed around x for every $x \in F$. We assume that sets under consideration are locally closed.

Given $\bar{x} \in F$, define the collection of Fréchet/regular normal cone to F at $\bar{x}$ by

$$\widehat{N}(\bar{x}; F) = \widehat{N}_F(\bar{x}) : = \left\{ v \in \mathbb{R}^n \;\middle|\; \limsup_{x \xrightarrow{F} \bar{x}} \frac{\langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leqq 0 \right\}, \tag{2.2}$$

where $x \xrightarrow{F} \bar{x}$ means that $x \to \bar{x}$ with $x \in F$. If $x \notin F$, we put $\widehat{N}(x; F) : = \emptyset$.

The Mordukhovich/limiting normal cone $N(\bar{x}; F)$ to F at $\bar{x} \in F \subset \mathbb{R}^n$ is obtained from regular normal cones by taking the sequential Painlevé-Kurotowski upper limits as

$$N(\bar{x}; F) := \operatorname*{Lim\,sup}_{x \xrightarrow{F} \bar{x}} \widehat{N}(x; F). \tag{2.3}$$

If $\bar{x} \notin F$, we put $N(\bar{x}; F) : = \emptyset$.

For an extended real-valued function $\phi : \mathbb{R}^n \to \overline{\mathbb{R}} :=[-\infty, \infty]$ its domain is defined by

$$\operatorname{dom} \phi :=\{x \in \mathbb{R}^n \mid \phi(x) < \infty\},$$

and its epigraph is defined by

$$\operatorname{epi} \phi :=\{(x, \mu) \in \mathbb{R}^n \times \mathbb{R} \mid \mu \geqq \phi(x)\}.$$

Let $\phi \colon \mathbb{R}^n \to \bar{\mathbb{R}}$ be finite at $\bar{x} \in \operatorname{dom} \phi$. Then the collection of basic subgradients, or the (basic/Mordukhovich/limiting) subdifferential, of ϕ at $\bar{x}$ is defined by

$$\partial\phi(\bar{x}) :=\{v \in \mathbb{R}^n \mid (v, -1) \in N((\bar{x}, \phi(\bar{x})); \operatorname{epi} \phi)\}. \tag{2.4}$$

Considering the indicator function $\delta(\cdot; F)$ defined by

$$\delta(\cdot; F) = \begin{cases} 0, & \text{if } x \in F, \\ +\infty, & \text{otherwise,} \end{cases}$$

we have a relation between the Mordukhovich normal cone and the limiting subdifferential of the indicator function as follows [8, Proposition 1.19]:

$$N(\bar{x}; F) = \partial\delta(\bar{x}; F), \qquad \forall \bar{x} \in F. \tag{2.5}$$

The generalized Fermat's rule is formulated as follows [8, Proposition 1.30]: Let $\phi \colon \mathbb{R}^n \to \bar{\mathbb{R}}$ be finite at $\bar{x}$. If $\bar{x}$ is a local minimizer of ϕ, then

$$0 \in \widehat{\partial}\phi(\bar{x}) \text{ and } 0 \in \partial\phi(\bar{x}). \tag{2.6}$$

For establishing optimality conditions, the following lemmas which are related to the Mordukhovich/limiting subdifferential calculus are very useful.

Lemma 2.1 *[8, Corollary 2.21 and Theorem 4.10(ii)]*

(i) Let $\phi_i \colon \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2, \ldots, m$, $m \geqq 2$ be lower semicontinuous around $\bar{x} \in \mathbb{R}^n$, and let all but one of these functions be Lipschitz continuous around $\bar{x}$. Then
$\partial(\phi_1 + \phi_2 + \ldots + \phi_m)(\bar{x}) \subset \partial\phi_1(\bar{x}) + \partial\phi_2(\bar{x}) + \ldots + \partial\phi_m(\bar{x})$.

(ii) Let $\phi_i \colon \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2, \ldots, m$, $m \geqq 2$ be lower semicontinuous around $\bar{x}$ for $i \in I_{\max}(\bar{x})$ and be upper semicontinuous at $\bar{x}$ for $i \notin I_{\max}(\bar{x})$, suppose that each ϕ_i, $i = 1, \ldots, m$, is Lipschitz continuous around $\bar{x}$. Then we have the inclusion
$$\partial(\max \phi_i)(\bar{x}) \subset \bigcup \left\{ \partial\Bigg(\sum_{i \in I_{\max}(\bar{x})} \lambda_i \phi_i \Bigg)(\bar{x}) \;\Bigg|\; (\lambda_1, ..., \lambda_m) \in \Lambda(\bar{x}) \right\},$$
where the equality holds and the maximum functions are lower regular at

$\bar{x}$ if each ϕ_i is lower regular at this point and sets $I_{\max}(\bar{x})$ and $\Lambda(\bar{x})$ are defined as follows:

$$I_{\max}(\bar{x})\colon = \{i \in \{1, \ldots, m\} \mid \phi_i(\bar{x}) = (\max \phi_i)(\bar{x})\},$$

$$\Lambda(\bar{x})\colon = \{(\lambda_1, \ldots, \lambda_m) \mid \lambda_i \geqq 0,\ \sum_{i=1}^{m} \lambda_i = 1,$$
$$\lambda_i\Big(\phi_i(\bar{x}) - (\max \phi_i)(\bar{x})\Big) = 0\}.$$

Lemma 2.2 *[8, Corollary 4.8] Let $\phi_i\colon \mathbb{R}^n \to \bar{\mathbb{R}}$ for $i = 1, 2$ be Lipschitz continuous around $\bar{x}$ with $\phi_2(\bar{x}) \neq 0$. Then we have*

$$\partial\left(\frac{\phi_1}{\phi_2}\right)(\bar{x}) \subset \frac{\partial(\phi_2(\bar{x})\phi_1)(\bar{x}) - \partial(\phi_1(\bar{x})\phi_2)(\bar{x})}{[\phi_2(\bar{x})]^2}. \tag{2.7}$$

Linear space for semi-infinite programming is denoted by

$$\mathbb{R}_+^{(T)} := \{\mu = (\mu_t)_{t\in T} \mid \mu_t \geqq 0 \text{ for all } t \in T$$
$$\text{but only finitely many } \mu_t \neq 0\},$$

and set of active multipliers at $\bar{x} \in S$ is defined by

$$A(\bar{x}) := \{\mu \in \mathbb{R}_+^{(T)} \mid \mu_t g_t(\bar{x}) = 0,\ \forall t \in T\}. \tag{2.8}$$

Finally, we recall some version of Ekeland Variational Principle which originally was given in [5].

Lemma 2.3 (Ekeland Variational Principle) *[8, Theorem 2.12] Let $\phi\colon \mathbb{R}^n \to \bar{\mathbb{R}}$ be a lower semicontinuous function bounded from below, $\epsilon > 0$ and $x_0 \in \mathbb{R}^n$ be given such that $\phi(x_0) \leqq \inf \phi(x) + \epsilon$. Then for every $\lambda > 0$, there is $\bar{x} \in \mathbb{R}^n$ such that $\|\bar{x} - x_0\| \leqq \lambda$, $\phi(\bar{x}) \leqq \phi(x_0)$, and*

$$\phi(\bar{x}) < \phi(x) + \frac{\epsilon}{\lambda}\|x - \bar{x}\| \text{ whenever } x \neq \bar{x}.$$

3 Optimality Conditions

In this section we establish approximate necessary conditions for fractional semi-infinite multiobjective optimization problems.

Definition 3.1 *Let ϵ be in $\mathbb{R}_+^m \backslash \{0\}$. A feasible point $\bar{x} \in F$ is said to be a local weakly ϵ-efficient solution for* (FSMP), *if there exists a neighborhood U of $\bar{x}$ and there is no other $x \in U \cap F$ such that*

$$\frac{p_k(x)}{q_k(x)} < \frac{p_k(\bar{x})}{q_k(\bar{x})} - \epsilon_k,\ \forall\ k \in M.$$

Definition 3.2 *Let ϵ be in $\mathbb{R}^m_+ \backslash \{0\}$. A feasible point $\bar{x} \in F$ is said to be a weakly ϵ-quasi-efficient solution for* (FSMP), *if there exists no other $x \in F$ such that*

$$\frac{p_k(x)}{q_k(x)} + \epsilon_k \|x - \bar{x}\| < \frac{p_k(\bar{x})}{q_k(\bar{x})}, \ \forall \ k \in M.$$

First, we provide a Fritz-John type approximate necessary optimality condition for a local weakly ϵ-efficient solution of (FSMP) with the help of a real-valued function ψ described in [6]. The following theorem was motivated by Chuong and Kim [2].

Theorem 3.1 *Let $\bar{x}$ be a local weakly ϵ-efficient solution of* (FSMP). *Then for any $v > 0$, there exist $x_v \in F$ and $\beta_k \geqq 0$, $k \in M$ and $\mu_t \geqq 0$, $t \in T(\mu)$ with $\sum_{k \in M} \beta_k + \sum_{t \in T(\mu)} \mu_t = 1$, such that $\|x_v - \bar{x}\| \leqq v$ and*

$$\begin{aligned}
&0 \in \sum_{k \in M} \lambda_k \left(\partial p_k(x_v) - \frac{p_k(x_v)}{q_k(x_v)} \partial q_k(x_v) \right) + \sum_{t \in T(\mu)} \mu_t \partial g_t(x_v) \\
&\quad + \frac{\max_{k \in M}\{\epsilon_k\}}{v} \mathbb{B} + N(x_v; S), \\
&\quad \beta_k \left[\frac{p_k(x_v)}{q_k(x_v)} - \frac{p_k(\bar{x})}{q_k(\bar{x})} + \epsilon_k - \psi(x_v) \right] = 0, \ k \in M, \\
&\quad \mu_t [g_t(x_v) - \psi(x_v)] = 0, \ t \in T(\mu), \\
&\quad \text{with } \lambda_k = \frac{\beta_k}{q_k(x_v)}, \ k \in M, \text{ and} \\
&\quad \psi(x) : = \max_{k \in M, t \in T(\mu)} \left\{ \frac{p_k(x)}{q_k(x)} - \frac{p_k(\bar{x})}{q_k(\bar{x})} + \epsilon_k, g_t(x) \right\}, \ x \in \mathbb{R}^n.
\end{aligned} \tag{3.9}$$

Remark 3.1 *In theorem 3.1, if x_v satisfies $N(x_v; F) \subset \bigcup_{\mu \in A(x_v)} \left[\sum_{t \in T} \mu_t \partial g_t(x_v) \right] + N(x_v; S)$, then there exist $\mu \in A(x_v)$ defined by in (2.8) and $\beta \in \mathbb{R}^m_+ \backslash \{0\}$ with $\sum_{k \in M} \beta_k = 1$, such that conditions (3.9) can be changed to KKT type conditions* *as follows:*

$$
\begin{aligned}
&0 \in \sum_{k\in M} \lambda_k \left(\partial p_k(x_v) - \frac{p_k(x_v)}{q_k(x_v)} \partial q_k(x_v) \right) + \sum_{t\in T} \mu_t \partial g_t(x_v) \\
&\quad + \frac{\max_{k\in M}\{\epsilon_k\}}{v} \mathbb{B} + N(x_v; S), \\
&\beta_k \left[\frac{p_k(x_v)}{q_k(x_v)} - \frac{p_k(\bar{x})}{q_k(\bar{x})} + \epsilon_k - \psi(x_v) \right] = 0, \;\; k \in M, \qquad (3.10) \\
&\text{with } \lambda_k = \frac{\beta_k}{q_k(x_v)}, k \in M, \text{ and} \\
&\text{the function } \psi(x) \colon = \max_{k\in M} \left\{ \frac{p_k(x)}{q_k(x)} - \frac{p_k(\bar{x})}{q_k(\bar{x})} + \epsilon_k \right\}, \;\; x \in \mathbb{R}^n.
\end{aligned}
$$

It should be mentioned again that functions ψ for approximate necessary optimality conditions of Fritz-John type and KKT type are slightly different.

Definition 3.3 *[1] Let $\bar{x} \in F$. We say that the limiting constraint qualification (LCQ) is satisfied at $\bar{x}$ iff*

$$
N(\bar{x}; F) \subset \bigcup_{\mu \in A(\bar{x})} \left[\sum_{t\in T} \mu_t \partial g_t(\bar{x}) \right] + N(\bar{x}; S). \qquad (3.11)
$$

The forthcoming theorem presents necessary condition of (KKT) type for weakly ϵ-quasi-efficient solutions for (FSMP).

Theorem 3.2 *Let the (LCQ) be satisfied at $\bar{x} \in F$. If $\bar{x}$ is a weakly ϵ-quasi-efficient solution of* (FSMP), *then there exist $\beta \in \mathbb{R}^m_+ \setminus \{0\}$ with $\sum_{k\in M} \beta_k = 1$ and $\mu \in A(\bar{x})$ defined (2.8) such that*

$$
\begin{aligned}
0 \in &\sum_{k\in M} \lambda_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{t\in T} \mu_t \partial g_t(\bar{x}) \\
&+ \sum_{k\in M} \beta_k \epsilon_k \mathbb{B} + N(\bar{x}; S),
\end{aligned} \qquad (3.12)
$$

where $\lambda_k = \dfrac{\beta_k}{q_k(\bar{x})}$, $k \in M$.

Further Research

Sufficient conditions for weakly ϵ-quasi-efficient solutions for (FSMP) can be derived with the help of generalized convex functions introduced in [2] and assumption on stationary point of denominator. Duality relations between the primal and Wolfe type dual problems will be discussed in our further research.

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